

# A Cobweb Model with Local Externalities

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## Abstract

This paper considers an extension of the standard cobweb model in a market with local externalities. In contrast with the standard cobweb model, firms must forecast both prices and local quantities; we develop new constructive stability and existence conditions for equilibria with positive outputs. We find evidence of clusters of firms whose output behavior is correlated as equilibrium is reached. We also show that an appropriately defined “representative agent model” with a global externality exhibits the same mean or second order properties of aggregate output as the more complex model with local externalities.

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*JEL classification:* C62; D62; E60; E30

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## 1. Introduction

Local interactions and their consequences on the economic behavior have recently been the subject of unremitting interest in various aspects of economics. These interactions feature the possibility that, among other things, an agent takes heed from the choices other agents make in their neighborhood. This property has found numerous successful applications for explaining economic as well as social behavior. In the context of social behavior examples include differences in community crime rates (Gleaser, Sacerdote and Scheinkman (1996)), high school performance (Steinberg et al. (1996)), spread of religion (Becker and Murphy (2001)), social segregation (Schelling (1971)). And in the context of economic behavior examples include price movements in financial markets (Follmer, Horst and Kirman (2005), Horst (2005)), spread of new technology (Ellison and Fudenberg (1993)), economic growth and income distribution (Durlauf (1993, 1996 a,b)), and macroeconomic coordination failure (Cooper and John (1988)). Other noticeable applications of social interaction which do not involve neighborhood effects are expectation formation and asset price volatility (Brock (1993)) and price dynamics with costly learning (Brock and Hommes (1997)).

Concurrent with this evolution, there has also been an expansion in the literature that investigates, both empirically and theoretically, the importance of external economies in various economic context. For example in the growth model of Lucas (1988) firms have constant returns to scale but experience externalities based on human capital accumulation. Similarly, Romer (1986) shows that the use of ‘knowledge’ as a factor of production can lead to increasing returns in that the spillover of knowledge between firms can act as an externality assuming that there is some freedom in the flow of information between firms. Caballero and Lyons (1992, 1990) show in a cross-industry empirical analysis for US and European data that external economies play an important role. At the core of these ideas lies Marshall’s concept that increasing returns may be external to a firm, which otherwise may be operating competitively, but internal to the *industry*; an idea most relevant for our work. Indeed, studies we have cited are limited to the aggregate and average effects of external economies and ignore the firm-to-firm effect, i.e. local effects or externalities.

Consider this limitation from another useful angle. The rational expectations model of Lucas (1972) considers a world in which agents need to separate local and global information. In such a setting, firms observe global variables such as price and aggregate quantities lagged one period but there are also certain local effects. Now consider a simple modification of this picture in which firms also observe the quantity decisions of firms similar to them and are affected by them; again the idea of inter-firm interactions harks back. For example, in the auto industry, profits of firms will depend on output of firms in a related industry such as the tire industry and hence there are particularly strong incentives for executives in the auto industry to collect accurate information on outputs in the tire industry and vice versa. These considerations are also key in factor markets such as labour markets; the profit of a firm will depend on wages paid to workers at similar firms for the very reason that this affects the wage that this firms has to set.

We hence develop a model in which firms have both global and local interactions. The firms are price-takers –behavior which is consistent with the external economies literature but also capturing the notion of global interaction. Firms also enjoy externalities with other firms closest to them in the form of output

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externalities –as in the social interaction models and also capturing our local interactions. More specifically, firms are located on a lattice and profits depend on the outputs of nearby firms as well as global variables such as the price. While in a rational expectations equilibrium, all firms have the same price expectations, the strong local effects in the profit function present some interesting and nontrivial questions about convergence to rational expectations equilibrium. We investigate the properties of naive or myopic adjustment processes within the context of the cobweb model (Ezekiel (1937) and Waugh (1964)). To complete our investigation, we also discuss the goodness of the approximation of our equilibria to that of a representative agent model.

In this paper we restrict ourselves to homogenous, symmetric interaction structures and analyze our results within the context of myopic behavior. It is both realistic and interesting to restrict the interactions of firms as in a world of a large number of firms it is reasonable to expect that firms may interact more with some firms than others. This may be due to location, complementarity in nature of output or simply the costs of considering interactions. We focus on myopic behavior for analytical tractability although there exists superior expectation formation processes such as the adaptively rational equilibrium of Brock and Hommes (1997). The simple example we examine displays many important features that have been observed in models of social interaction and also cobweb dynamics but have not been applied in the particular setup we present.

At this stage we would like to consider some related work. The works by Hommes (1994, 1998), Brock and Hommes (1997) and Hommes and Goeree (2000) are of particular interest as these develop cobweb dynamics under a variety of expectation formation schemes as well as nonlinearities in the demand-supply system. One may view this paper as generalizing the cobweb model by: a) allowing for a system of firms and b) by letting these firms interact with one another through the externality effects. The results we obtain relax conditions for stability and convergence with myopic expectations. Another important contribution is that positive local interactions increase the stability margin, which is a fresh result in the cobweb literature, but well established in a pure social interaction context (see for example Durlauf (1997), Brock and Durlauf (2001) and the literature therein). Furthermore, Another important facet of our results is our analysis and derivation of aggregate statistical properties of the resulting equilibrium. Our model predicts both correlations of aggregate output over time and a correlation structure across the space of firms.

## 2. The Economic Environment

We consider a lattice  $\mathcal{L}$  and define the set of neighbors of a firm located at lattice site  $i$  as  $(i)$ . For example in the two dimensional lattice is shown in Fig. (1), the white flag is the location of a firm  $i$  and the grey flags identify the locations of her nearest neighbors. An important economic question is how the cobweb model and its equilibrium properties depend on the type and the dimension of the lattice; in particular, it is necessary for there to be some range and types of lattices which produce the same qualitative results for there to be useful theoretical predictions obtained from models of this type. Whether this will be the case depends on the type of the profit function. Clearly, when there are no interactions and all firms are independent, the type of lattice is irrelevant. Since a rough scale invariance appears to be observed in economic activity (c.f. Granger (1966) or Barsky and Miron (1989)) it may be that interactions between small groups of agents have much the same properties as equilibrium for the whole economy. If this is the case, each firm's profit function will depend only on that of several other firms or neighbors. Aggregate behavior can still be quite correlated and complicated if all firms are linked to different firms. Indeed, we show this in the context of a homogenous and symmetric interaction structure.

We define the expected profit function as:

$$\Pi_i^e = P_i^e \phi_i - a_i \phi_i - \frac{b_i}{2} \phi_i^2 + \sum_{j \in (i)} \eta_{i,j} \phi_j \phi_i \quad (1)$$

where  $a_i, b_i > 0$ ,  $\Pi_i^e$  is expected profits of firm  $i$ ,  $P_i^e$  is firm  $i$ 's expectation of the market price,  $\phi_i$  is the output of firm  $i$  and  $\eta_{i,j}$  are interaction parameters which capture the strength of local relationships between the output of firm  $j$  and the profits of firm  $i$ . Eq. (1) is exactly the same as in (Arifovic,1994) with the exception of the last term which captures the effects of externalities.

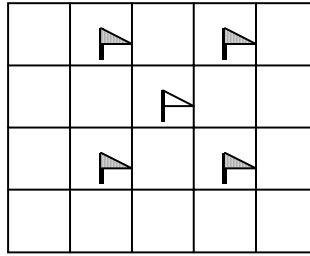


Figure 1: A Two Dimension Square Lattice.

If the  $\eta_{i,j}$  are all positive and we fix prices or at least price expectations momentarily, a positive shock to a nearby firm's production process will lead to higher output and hence a positive effect on profits and output. In such a setup, the behavior of firms will be positively correlated. In the opposite case where  $\eta_{i,j}$  are negative and price expectations or prices are fixed, we expect the behavior of firms to be negatively correlated. This particular model is related to models of interacting particle systems in physics such as the Ising model. Models of interacting particle systems in economics which have been developed in Durlauf (1993) and Follmer (1974). The case where  $\eta_{i,j}$  are positive are analogous to ferromagnetic field theories and the case where  $\eta_{i,j}$  are negative correspond to antiferromagnetic field theories. In economic models, the roughly corresponding terms of *strategic complements* and *strategic substitutes* have been developed in (Geanakoplos, Klemperer, Bulow, 1985) and developed further in terms of their macroeconomic implications by Cooper and John(1988)<sup>1</sup>. A further useful interpretation of  $\eta_{i,j}$  is that it captures the cost-savings or the external-economies firm  $i$  makes when it produces more output in response to the increase in output of  $j$  associated firms. It is worth noting that the fall in the marginal cost of firm  $i$  when the output of related  $j$  firms increases by a unit is captured by the sum of  $\eta_{i,j}$ s.

The difference between the approach in the strategic complements/substitutes literature and here is that such approaches do not generally consider the local nature of interactions and instead focus on a dependence on average behavior. Assuming average dependence may be in some sense an approximation for the more complex case of local interactions, it is worthwhile to investigate the quality of that approximation.

We define the market demand curve as:

$$P = c - d \sum_{k=1}^n \phi_k \quad (2)$$

where  $n$  is the number of firms in the economy and  $d \geq 0$ . The market demand curve Eq. (2) is chosen because it is simple and facilitates comparison with results in the literature. In a more complex setting, the demand curve might come, for instance, from an overlapping generations model in which case price would depend in addition on interest rates and labor income. We make this point to illustrate that results from a model such as we are considering may also be useful for general equilibrium analysis. In such circumstances, it may also turn out to be useful to allow for heterogeneity on the consumer side as well; such issues are clearly beyond the scope of this paper.

The first order conditions for the firm's maximization problem are:

$$P_i^e = a_i + b_i \phi_i - \sum_j \eta_{i,j} \phi_j. \quad (3)$$

Rational expectations equilibrium equates the expected price with the actual market clearing price and

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<sup>1</sup>Examples of macroeconomic models with strategic complementarities and thus Keynesian multiplier features are: (Diamond, 1982), (Haart, 1982), (Heller, 1986) and (Haltiwinger and Waldman, 1985). Models with strategic complements/substitutes are of course ubiquitous in industrial organization.

hence implies:

$$c - d \sum_{k=1}^n \phi_k = a_i + b_i \phi_i - \sum_j \eta_{i,j} \phi_j \quad (4)$$

for all  $i$ . We define  $\phi$  to be the vector containing each of the  $\phi_i$  and we let  $\Omega$  be a matrix with elements:

$$\Omega_{ij} = \begin{cases} b_i + d & \text{if } i = j \\ -\eta_{i,j} + d & \text{if } i \neq j \\ d & \text{otherwise.} \end{cases}$$

We further let  $\mathbf{a}$  be a column vector containing each of the  $a_i$  and let  $\mathbf{c}$  be a column vector with elements  $c$ . Rational expectations equilibrium occurs where:

$$\phi = \Omega^{-1}(\mathbf{c} - \mathbf{a}). \quad (5)$$

For a rational expectations equilibrium to exist, the vector of output,  $\phi$ , must be nonnegative and prices have to be positive. From the demand equation (2), the firms will be able to charge a positive price as long as the aggregate output does not surpass the threshold:

$$\sum_{i=1}^N \phi_i < \frac{c}{d}. \quad (6)$$

Given that  $c \geq a_i \forall i$ , the properties of the matrix  $\Omega$  will determine whether outputs are positive. The problem simplifies quite a bit in the case where demand is constant (e.g.,  $d = 0$ ) and there are positive spillovers  $\eta_{i,j} > 0$  so we will focus on this case first. In this case, the positivity of  $\phi$  can be verified by noting that  $\Omega_{ij}$  is closely related to an input-output matrix. In the next result, the market price is always positive because of the assumption that  $d = 0$ .

**Proposition 1** *A rational expectations equilibrium will exist with positive outputs assuming  $d = 0$  when  $\Omega^{-1}$  satisfies the Hawkins-Simons condition that all its principal minors are positive.*

**Proof.** To prove this recall the following equation from input-output analysis (c.f. (Takayama, 1985), p. 359):

$$A\mathbf{x} + \mathbf{q} = \mathbf{x} \quad (7)$$

where  $A$  is a matrix of input-output coefficients,  $\mathbf{x}$  is a production vector and  $\mathbf{q}$  is a consumption vector. Eq. (7) can be rewritten as:

$$(I - A)\mathbf{x} = \mathbf{q}. \quad (8)$$

We now transform our problem (e.g., Equation (5)) into a formulation that closely resembles the input-output analysis. Let  $d = 0$  and a column vector  $\mathbf{x}$  with elements  $x_i = \phi_i b_i$ . We define:

$$\Omega_{ij} = \begin{cases} \frac{\eta_{i,j}}{b_j} & \text{if } j \in (i) \\ 0 & \text{otherwise.} \end{cases} \quad (9)$$

Then, our problem looks as follows

$$\begin{bmatrix} 1 & \frac{-\eta_{1,2}}{b_2} & \frac{-\eta_{1,3}}{b_3} & \dots & \dots & \frac{-\eta_{1,j}}{b_j} \\ \frac{-\eta_{2,1}}{b_1} & 1 & \frac{-\eta_{2,3}}{b_3} & \dots & \dots & \frac{-\eta_{2,j}}{b_j} \\ \frac{-\eta_{3,1}}{b_1} & \frac{-\eta_{3,2}}{b_2} & 1 & \dots & \dots & \frac{-\eta_{3,j}}{b_j} \\ \vdots & \vdots & \vdots & \dots & \dots & \vdots \\ \frac{-\eta_{i,1}}{b_1} & \frac{-\eta_{i,2}}{b_2} & \frac{-\eta_{i,3}}{b_3} & \dots & \dots & 1 \end{bmatrix} \begin{bmatrix} \phi_1 b_1 \\ \phi_2 b_2 \\ \phi_3 b_3 \\ \vdots \\ \vdots \\ \phi_i b_i \end{bmatrix} = \begin{bmatrix} c - a_1 \\ c - a_2 \\ c - a_3 \\ \vdots \\ \vdots \\ c - a_i \end{bmatrix}$$

or

$$\mathbf{\Omega}\mathbf{x} = (\mathbf{c} - \mathbf{a})$$

The matrix above closely resembles the input-output Equation (8) where the vector  $\mathbf{x}$  contains the elements  $\phi_i b_i$ ,  $\mathbf{\Omega} = \mathbf{I} - \mathbf{A}$  and  $\mathbf{q} = (\mathbf{c} - \mathbf{a})$ . Given that  $b_i > 0$ , the positivity of  $\phi$  is verified by using the Hawkins-Simon sufficient conditions from the input-output theory (c.f. (Takayama, 1985), ch.5) that the principal minors of matrix  $\mathbf{\Omega}$  are positive. ■

While the Hawkins-Simon condition holds, the interpretation may be significantly different than that of input-output theory. For instance, consider a model with two firms. The Hawkins-Simon condition is that:<sup>2</sup>

$$\det \begin{pmatrix} 1 & -\frac{\eta_{1,2}}{b_2} \\ -\frac{\eta_{2,1}}{b_1} & 1 \end{pmatrix} > 0 \quad (10)$$

This condition can be written as:

$$b_1 b_2 > \eta_{1,2} \eta_{2,1} \quad (11)$$

For simplicity, we choose the normalization  $b_2 = 1$ . Then this condition says that the direct effect on costs is greater than the indirect effects of externalities in reducing costs. If the indirect effects are larger, then it is impossible for an equilibrium to exist. If this condition does not hold one can imagine a situation where a firm, having realized the benefits from externalities, produces just a little more marginal output which will then induce, through its externality effect, the remaining firms to produce more which then in turn creates further stimulus for our firm to produce more and so on.

The more general case where  $d \neq 0$  is considerably more difficult. Defining  $\mathbf{1}$  to be a vector of 1's, price positivity from Equation (6) requires that aggregate output does not exceed the threshold:

$$\mathbf{1}'\phi = \mathbf{1}'\mathbf{\Omega}^{-1}(\mathbf{c} - \mathbf{a}) < \frac{c}{d}. \quad (12)$$

and output positivity requires:

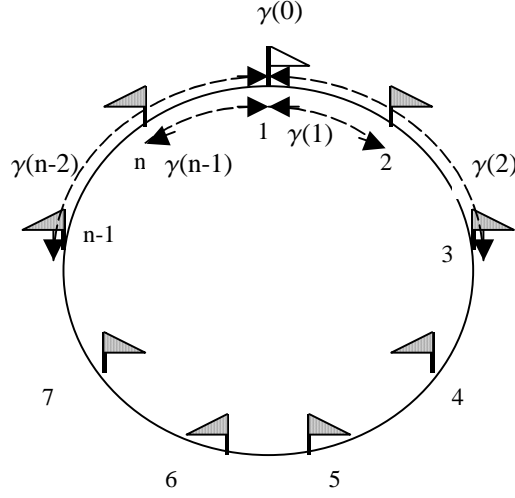
$$\phi = \mathbf{\Omega}^{-1}(\mathbf{c} - \mathbf{a}) > 0. \quad (13)$$

To proceed further, we shall make the assumptions that the matrix  $\mathbf{\Omega}$  is a symmetric circulant matrix and that the parameters  $a_i$ ,  $b_i$ , and  $\eta_{i,j}$  are identical across firms. An  $n \times n$  symmetric circulant matrix is formed from  $n$  vector by cyclically permuting the entries. In other words, we assume (c.f. (Brockwell and Davis (1987), p. 131):

$$\mathbf{\Omega} = \begin{pmatrix} \gamma(0) & \gamma(1) & \gamma(2) & \gamma(3) & \cdots & \gamma(2) & \gamma(1) \\ \gamma(1) & \gamma(0) & \gamma(1) & \gamma(2) & \cdots & \gamma(3) & \gamma(2) \\ \gamma(2) & \gamma(1) & \gamma(0) & \gamma(1) & \cdots & \gamma(4) & \gamma(3) \\ \gamma(3) & \gamma(2) & \gamma(1) & \gamma(0) & \cdots & \gamma(5) & \gamma(4) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ \gamma(1) & \gamma(2) & \gamma(3) & \gamma(4) & \cdots & \gamma(1) & \gamma(0) \end{pmatrix}. \quad (14)$$

The economic interpretation of this matrix structure is that externalities affect firms symmetrically; periodicity is imposed so that the location of firms relative to the boundaries does not matter economically and what really matters is the distance separating one firm relative to one another (see Figure 2. An explanation of this figure is in the example below). Although the matrix structure is restrictive, it does represent a baseline case of perfect symmetry and it is also particularly tractable; indeed, as the number of firms goes to infinity, this matrix is equivalent to the autocovariance matrix of stationary time series analysis if it is diagonally dominant and hence positive definite. Our assumption about the matrix structure enables the development of general results about multidimensional externalities.

<sup>2</sup>The condition that the first principal minor be positive is automatically met.

Figure 2: N firms located on a circle with  $\gamma(\cdot)$  externalities.**Example**

As a simple example of  $\Omega$ , we could set  $\gamma(n) = d$  for  $n = 2 \dots \frac{n}{2}$  and  $\gamma(1) = -\eta + d$ ,  $\gamma(0) = b + d$ . In this case, the effects of externalities are restricted to nearest neighbors. In Figure 2 we have flagged  $n$  firms around a circle. Firm's 1 output is influenced by all its neighbors from either direction and  $\gamma(\cdot)$  quantifies that externality. Thus,  $\gamma(0)$  is firm one's own externality-effect,  $\gamma(1)$  captures the externality-effect of the second firm on firm one till we reach  $\gamma(n-1)$  which is the effect of the  $n^{\text{th}}$  firm on firm one. Restricting the externalities of firm one to its closest neighbors (i.e firms 2 and  $n$ ) and assuming that the externality is symmetric and equal to  $\eta$  we get that  $\gamma(1) = \gamma(n-1) = -\eta + d$  and  $\gamma(n) = d$ . This process then repeats itself when we turn our attention to the second and the third firm until we reach firm  $n$ .

Under these circumstances the matrix  $\Omega$  for  $n$  firms takes the form:

$$\Omega = \begin{pmatrix} b+d & -\eta+d & d & d & \cdots & d & -\eta+d \\ -\eta+d & b+d & -\eta+d & d & \cdots & d & d \\ d & -\eta+d & b+d & -\eta+d & \cdots & d & d \\ d & d & -\eta+d & b+d & \cdots & d & d \\ \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ -\eta+d & d & d & d & \cdots & -\eta+d & b+d \end{pmatrix}. \quad (15)$$

Each row of  $\Omega$  corresponds to a firm and each column corresponds to the effect of another firm. The last element in the first row is nonzero because the last firm is considered by periodicity as the nearest neighbor to the first firm as in Figure 2.

**Proposition 2** *Assuming the matrix  $\Omega$  is symmetric circulant and the cost parameters  $a_i$  and  $b_i$  are uniform across firms, equilibrium will exist with positive prices and outputs if:*

$$0 < \left[ \gamma(0) + 2 \sum_{j=1}^{[(n-1)/2]} \gamma(j) + \left( \left[ \frac{n}{2} \right] - \left[ \frac{n-1}{2} \right] \right) \gamma\left(\frac{n}{2}\right) \right]^{-1} < \frac{c}{nd(c-a)} \quad (16)$$

where the notation  $[b]$  refers to the integer part of  $b$ .

**Proof.** See the appendix. ■

This result shows that it is possible to solve for a rational expectations equilibrium merely by examining the row sum of the matrix  $\Omega^{-1}$ . For instance consider the model with five firms in the context of Fig 2. Theorem 2 then says that for outputs and price to be positive we need:

$$0 < \frac{1}{b + 5d - (\eta_{1,2} + \eta_{1,5})} < \frac{c}{5d(c - a)}$$

For this inequality to hold, the key is that  $b + 5d > \eta_{1,2} + \eta_{1,5}$ . Intuitively, this condition states that total marginal benefits, captured by the interrelationships between firms' output, not to exceed the total marginal costs – the rate at which marginal costs rises as more output is produced. However, because of the way the boundary conditions are imposed, it is only applicable to systems with one dimensional externalities and in many circumstances, one would expect the interrelationships to be higher dimensional as in Figure (1).

We now proceed to extend our results on externalities where firm interact with one another through multiple dimensions. By multiple dimensions, we mean the multiple tensor product of externality matrices. For instance, for a two dimensional model with  $n$  firms in the  $x$  direction and  $m$  firms in the  $y$  direction, we define  $m \times m$  matrix  $\Omega_1$  and a  $n \times n$  matrix  $\Omega_2$ . Then:

$$\Omega = \Omega_1 \otimes \Omega_2. \tag{17}$$

**Proposition 3** *For a two dimensional model with matrix  $\Omega = \Omega_1 \otimes \Omega_2$  where  $\Omega_1$  and  $\Omega_2$  are symmetric circulant matrices, aggregate output is equal to the number of firms in the economy times  $(c - a)$  divided by the first eigenvalues of the circulant matrices of both dimensions.*

**Proof.** In this case, using tensor product identities and assuming that  $P_i$  is a Fourier matrix and  $D_i$  is a diagonal matrix with the eigenvalues of  $\Omega_i$  as its elements where  $i = 1, 2$  refers to the dimension, we have that:

$$\Omega^{-1} = (P_1 \otimes P_2) (D_1 \otimes D_2)^{-1} (P_1 \otimes P_2)^{-1}$$

Therefore, summing across firms we have:

$$\begin{aligned} \mathbf{1}'\Omega^{-1}(\mathbf{c} - \mathbf{a}) &= \mathbf{1}'(P_1 \otimes P_2) (D_1 \otimes D_2)^{-1} (P_1 \otimes P_2)^{-1} (\mathbf{c} - \mathbf{a}) \\ &= \mathbf{1}'(P_1 \otimes P_2) (D_1^{-1} \otimes D_2^{-1}) (P_1' \otimes P_2') (\mathbf{c} - \mathbf{a}) \\ &= \mathbf{1}'(P_1 D_1^{-1} \otimes P_2 D_2^{-1}) (P_1' \otimes P_2') (\mathbf{c} - \mathbf{a}) \\ &= \mathbf{1}'(P_1 D_1^{-1} P_1 \otimes P_2 D_2^{-1} P_2) (\mathbf{c} - \mathbf{a}) \end{aligned}$$

The second step follows from tensor product identities and the fact that the matrices  $P_1$  and  $P_2$  are orthogonal. The other steps rely only on tensor product identities. From the last line, the sums over the frequencies  $\omega_{1,m}$  and  $\omega_{2,m}$  separate; by the properties of the Fourier  $P_i$  matrices which have columns given by Eq. (50) - (52), each sum results in  $\frac{1}{\lambda_o(i)}$  where  $i = 1, 2$  refers to the dimension, it follows that aggregate output is:

$$\mathbf{1}'\Omega^{-1}(\mathbf{c} - \mathbf{a}) = mn \frac{c - a}{\lambda_0(1)\lambda_0(2)}, \tag{18}$$

and therefore the aggregate output is proportional to the inverse of the row sum of the  $\Omega$  matrix. ■

Since this result provides an explicit formula for outputs, it is straightforward to determine whether these outputs are positive and whether the corresponding equilibrium prices are positive.

### 2.1. A Computed Example

We consider as an example economy a 2 dimensional economy with only nearest neighbor externalities (such as in Figure 1). We suppose that  $a = 70$ ,  $d = 0.01$ ,  $b = 2.0$  and  $c = 80$ . Furthermore, the nearest neighbor interaction coefficient is  $\eta = 0.10$ . We assume firms are located on a uniform lattice such as Fig. 1 where there are  $X$  firms in the  $x$  direction and  $Y$  firms in the  $y$  direction. In this case, since every firm is affected by four other firms, the row sum of the matrix is  $-4 \times \eta + b + XY \times d$ . If  $X = Y = 4$ , then the row sum is  $-0.40 + 2.0 + 16 \times 0.01 = 1.76$ . Aggregate output is then  $16 \times (c - a)/1.76 = 160/1.76$ . The market price is  $80 - 0.01 \times 160/1.76 \approx 79.09$ .

This simple example illustrates that even though we have constructed a complex economy with heterogeneous agents and local externalities, equilibrium outputs and prices can effectively be computed on the back of an envelope; this makes equilibria particularly easy to construct and analyze. One may exercise caution on our simple calculations in that it involves quantities that are not typically observable. The result we obtain is in many ways in line with work on local interactions. We impose homogeneity of externality, a separation of local and global externality and concavity of the objective function. In the context of large systems with random social interaction in the utility functions, the important work by Horst and Scheinkman (2005) find that similar restrictions have to be respected to achieve a unique equilibria.

## 3. Convergence

In this section, we will utilize the insights of the previous section to examine the stability of the cobweb adjustment process with local externalities. We recall (see Chiang, 1988 or Takayama, 1985) that with a cobweb adjustment process producers (indexed by  $i$ ) set expected prices equal to the realized market price in the previous period, i.e., myopic expectations:

$$P_{t;i}^e = P_{t-1}. \quad (19)$$

Given this expectations formation mechanism, the evolution of output is governed by the difference equation:

$$c - d \sum_{i=1}^N \phi_{i,t-1} = a_i + b_i \phi_{i,t} - \sum_{j \in (i)} \eta_{i,j} \phi_{j,t;i}^e, \quad (20)$$

where  $\phi_{j,t;i}^e$  is the firm  $i$  forecast of the output of firm  $j$  at time  $t$ . We assume that firms forecast quantities produced by other firms on the basis of *naive expectation* so that:

$$\phi_{t,j;i}^e = \phi_{t-1,j} \quad (21)$$

where  $\phi_{t,j;i}^e$  is the forecast by firm  $i$  of firm  $j$  output. Then Eq. (20) becomes:

$$c - d \sum_{i=1}^N \phi_{i,t-1} = a_i + b_i \phi_{i,t} - \sum_{j \in (i)} \eta_{i,j} \phi_{j,t-1}. \quad (22)$$

To simplify notation, we define  $\Xi$  to be a matrix with all elements equal to  $d$  and a matrix  $\Theta$  with elements:

$$\Theta_{ij} = \begin{cases} 0 & \text{if } i = j \\ -\eta_{i,j} & \text{if } j \in (i) \\ 0 & \text{otherwise} \end{cases}$$

and a diagonal matrix  $B$  with diagonal elements  $B_{ii} = b_i$ .

Then, Eq. (22) becomes a first-order linear difference equation:

$$\mathbf{c} - \Xi \phi_{t-1} = \mathbf{a} + B \phi_t + \Theta \phi_{t-1} \quad (23)$$

or:

$$\phi_t + B^{-1}(\Xi + \Theta)\phi_{t-1} = B^{-1}[\mathbf{c} - \mathbf{a}] \quad (24)$$

Thus, stability of the system will depend on whether the eigenvalues of the matrix  $W$ :

$$W = B^{-1}(\Xi + \Theta) \quad (25)$$

are of modulus less than unity. This task is simplified by assuming that  $b_i$  is uniform across firms since we get:

$$W = (1/b)[\Xi + \Theta].$$

Given this simplification, the stability of the system will then depend on whether or not the eigenvalues of the terms in square brackets in the equation above i.e.:

$$W_2 = (\Xi + \Theta) \quad (26)$$

are less than  $b$  in modulus.

The special case with no externalities, so  $\eta_{i,j} = 0$ , corresponds to a cobweb model of quantity dynamics with naive price expectations. In that case the quantity behaviour is described by the following difference equation,

$$\phi_t = \frac{(\mathbf{c} - \mathbf{a})}{\mathbf{b}} - \frac{d}{\mathbf{b}} \sum_{i=1}^N \phi_{i,t-1}. \quad (27)$$

In this case without externalities, monotonic demand and supply curves and where there are  $X$  firms in the  $x$  direction and  $Y$  firms in the  $y$  direction all producing symmetric quantities, the equilibrium quantities are locally stable if only and if

$$-1 < \frac{XY \times d}{\mathbf{b}} < 1. \quad (28)$$

Intuitively, the stability condition states that the ratio of the derivatives of the demand and supply curves with respect to output have to be smaller than unity (in absolute terms)<sup>3</sup>. Imagine  $0 < XY \times d/\mathbf{b} < 1$ , the condition implies that a fixed point  $\phi^*$  is reached on an oscillating path as  $t \rightarrow +\infty$ . Alternatively, if  $XY \times d/\mathbf{b} > 1$ , then  $\phi^*$  is unstable in that  $\phi_t$  diverges away from the fixed point in an oscillatory fashion as  $t \rightarrow +\infty$ .

What happens when  $\eta_{i,j} \neq 0$ ? In the case of linear demand and supply curves our next result establishes that when the matrix  $W_2$  is circulant, implying that externalities affects firms uniformly ( $\eta_{i,j} = \eta$ ), a sufficient condition for stability is that the sum of the absolute values of the row elements of the matrix  $W_2$  is less than  $b$ . The necessary condition is that magnitude of the maximum of the Fourier transform of the first row is less than  $b$ . For these conditions, we also find that positive interactions improve the stability margin whereas negative interaction have the opposite affect.

**Proposition 4** *Assuming  $W_2$  has a circulant form Eq. (14), the cobweb dynamics are stable: (1) if the sum of the absolute values of the row elements of the matrix  $W_2$  is less than  $b$ , and (2) only if the maximum of Fourier transform of the first row of the matrix is less than  $b$ .*

**Proof.** The eigenvalues of  $W_2$  are given by the expressions ((Brockwell and Davis (1987)), p. 128-133):

$$\lambda_0 = \sum_{|j| \leq [\frac{N}{2}]} \gamma(h) \quad (29)$$

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<sup>3</sup>Notice the similarity of this condition with that of the textbook cobweb model with naive expectations without externalities where traditionally the focus is on dynamics of prices (and not output). See for example Hommes (1994, page 317).

$$\lambda_j = \sum_{|j| \leq \lfloor \frac{n}{2} \rfloor} \gamma(h) e^{-i\omega_j h} \quad j = 1 \dots \left\lfloor \frac{n}{2} \right\rfloor \quad (30)$$

$$\lambda_{n-j} = \lambda_j \quad j = 1 \dots \left\lfloor \frac{n-1}{2} \right\rfloor \quad (31)$$

where  $\omega_j = \frac{2\pi j}{n}$  and  $n$  is the number of elements in a row of the matrix.

For (1), we note  $\lambda_0$  (the first eigenvalue) is simply the sum of the first row of  $W_2$  and this sum should not exceed  $b$ . This condition is also important for Proposition two. For (2), we note that the  $\lambda_j$  above are just the Fourier transform of the first row of the matrix  $W_2$ . Because  $|e^{-i\omega_j h}| < 1$ , the result follows<sup>4</sup>. Intuitively, these conditions generalize the stability conditions without externality in Eq.(28). Indeed, now the slope of the aggregate demand function *net* of externalities must not exceed that of the supply function. So long as the conditions (1) and (2) are satisfied the system is stable. However, one can notice from the composition of the eigenvalues (the row sums of  $W_2$ ) that positive local externalities reduce the value of the characteristic roots (in absolute terms) of each firm for a given slope of the supply function  $b$ . Such positive interactions then improve the stability margin (i.e., the distance between the critical bound at which instability occurs (i.e. unity) and the current value of the characteristic root) and hence make the conditions for local stability less stringent than the corresponding stability condition Eq. (28) of the cobweb model. However, when the interactions are negative our stability margin is reduced as they increase the value (in absolute terms) of the characteristic roots given  $b$ . The following example further clarifies this point. ■

### 3.1. Computational Example

In Sec. 2.1 above, we considered an example economy with only 2 dimensional nearest neighbor externalities (such as in Figure 1). Suppose that  $a = 70$ ,  $d = 0.01$ ,  $b = 2.0$ ,  $c = 80$ ,  $\eta = 0.50$ . The absolute row sum of the matrix  $W_2$  is  $XYd - 4\eta$  where  $X$  is the number of firms in the  $X$  direction and  $Y$  is the number of firms in the  $Y$  direction (the factor 4 is due to 4 nearest neighbors as shown in Fig. 1). The stability condition is

$$-1 < \frac{XY \times d - 4\eta}{b} < 1 \quad (32)$$

If  $X = Y = 4$ , the absolute row sum is  $0.16 - 2 = -0.84 < |b| = 2$ . Thus, the computed example in Sec. 2.1 is a stable equilibrium. Note that our example automatically satisfies Proposition 2 as well. However, a strong positive interaction may also lead to the violation of the critical bound  $-1$ , as we demonstrate in the next section.

Now, consider the identical strength of interaction as before but only that the externality is negative  $\eta = -0.50$ . In this case the absolute row sum is  $0.16 + 2 = 2.16 > b$ . Clearly, this computed example is unstable because the ratio of derivatives of net demand and supply functions is greater than the critical bound  $+1$ . Furthermore, the necessary condition is more easily violated compared with the corresponding situation with positive interactions.

Finally, comparing Eq. (28) with Eq. (32) it is clear that positive interaction that satisfy proposition 4 make stability conditions less stringent meanwhile negative interactions have the opposite affect.

### 3.2. A Non-Linear Cobweb Model with Externalities

A natural question that arises is the extent to which the results change for nonlinear demand and supply functions. While it is beyond the scope of this paper to explore this complex problem in detail, in this section we explore the case of a nonlinear S-shaped supply curve where output is bounded and there is a linear demand curve which are frequently used in the literature (see for instance: Hommes (1991a, 1991b, 1994, 1998)). An intuition for the S-shaped supply curve where output increases sluggishly at both ends

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<sup>4</sup>Some computer programs may use different normalizations for the Fourier transform than in Eq. (29), Eq. (30) and Eq. (31) such as division by  $\sqrt{2\pi}$ . As a result, care must be exercised in using (2) in practice.

is that at low prices producers face sunk costs while at high prices they face capacity constraints. In our model such a supply curve can be derived from an expected profit maximization problem where total costs are of polynomial degree 4 (also see Hommes 1998, p342):

$$\Pi_i^e = P_i^e \phi_i - a_i \phi_i - \frac{b_i}{2} \phi_i^2 - \frac{e_i}{3} \phi_i^3 - \frac{f_i}{4} \phi_i^4 + \sum_{j \in (i)} \eta_{i,j} \phi_j \phi_i. \quad (33)$$

The supply curve then takes the form

$$P^e = a + b_i \phi_i + e_i \phi_i^2 + f_i \phi_i^3 - \sum_j \eta_{i,j} \phi_j^e \quad (34)$$

where the key difference between Eq. (34) and literature is the interaction terms. The demand curve remains as in Eq. (2).

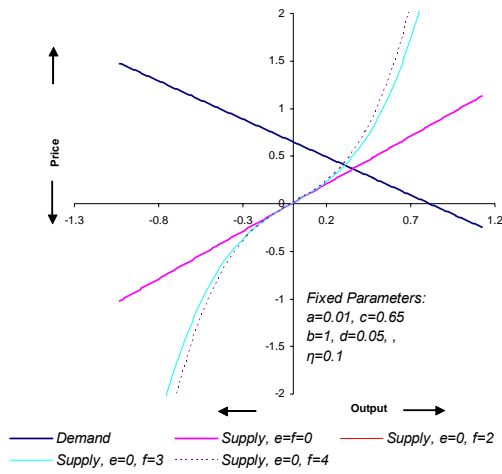


Fig 3a. The S-Shaped supply for various  $f$ .

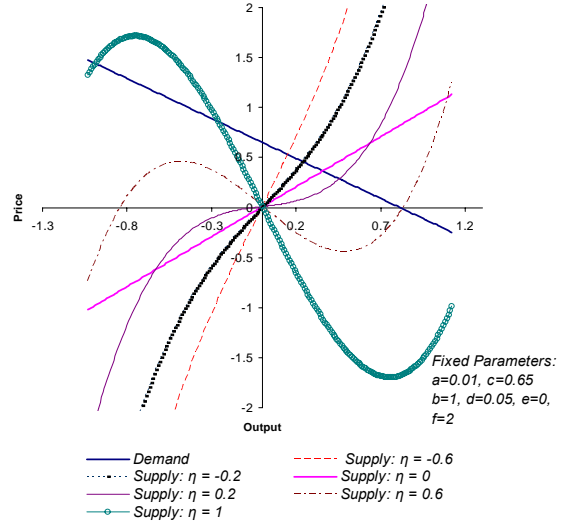


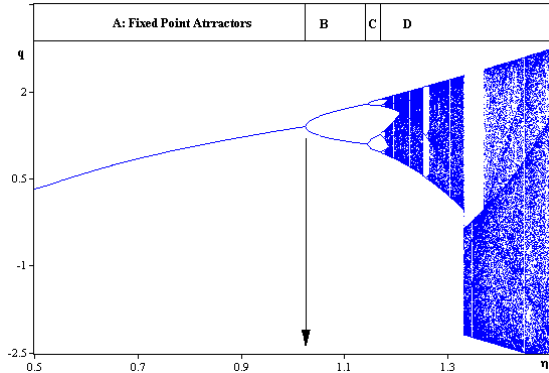
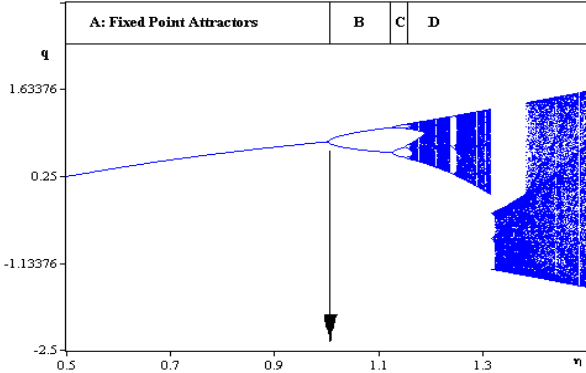
Fig 3b. The S-Shaped Supply for various  $\eta$ .

In Figs. 3a & b we plot the supply function of firm  $i$  assuming uniform externalities to the closest neighbor (2-dimension lattice with 4 firms) and  $\phi_j^e = \phi_i$ . For numerical simulations in this section we fix the slope of the demand curve  $d = 0.05$  and  $c = 0.65$ . We assume that there are 4 firms each in both  $X$  and  $Y$  directions. Furthermore, we find it convenient to set  $e = 0$ ,  $f > 0$  in all our plots and obtain a monotonic S-shaped supply curve where output is bounded. The objective of these plots is to investigate the effects of externalities and nonlinearities on the stability conditions. In other words, we investigate the stability of the system when parameters  $f$ , which tunes the steepness of the supply curve, and the externality parameter  $\eta$  are increased.

In Fig. 3a, as  $f$  is raised, the supply curve becomes steeper and instability become more likely since the slope of supply curve becomes smaller relative to the demand curve and assuming that  $\eta = 0.1$ . Therefore, nonlinearities in the supply curve make instability more likely; a result that is consistent with previous studies such as Hommes (1994, 1998).

In Fig. 3b we fix the steepness parameters  $e = 0$ ,  $f = 2$  and vary the interaction term so that  $-0.6 \leq \eta \leq 0.6$ . In this case as interaction terms become positive the slope of the S-shaped supply curve increases relative to  $d$ ; making stability more likely. Therefore, positive interactions help in the stability of a nonlinear setup. However, strong positive interactions may also generate a non-monotonic supply function, such as  $\eta = 1$  in Fig. 3b, leading to chaotic dynamics as we see below.

Next, we analyze further how the behaviour of the quantities using Eqs.(2), (21), (34) and naive price expectations reacts as the interaction terms change for different S-shapes of the supply curve<sup>5</sup>. For numerical simulations we assume a 2-dimension lattice with four firms. We fix the slope of the demand curve  $d = 0.01$  and  $c = 0.65$ . We assume that there are 4 firms each in both  $X$  and  $Y$  directions. Interactions are uniform and have an initial value  $\eta = 0.1$ . We concentrate on this simple system as a large number of firms in our context of uniform firms and symmetric interaction will only affect the results quantitatively.

Fig 4a:  $e = 0, f = 1, 0.5 < \eta < 1.5$ .Fig 4b:  $e = 0, f = 3, 0.5 < \eta < 1.5$ 

In plots 4a & b we show *examples* of how various values of  $f$  affect the dynamics of our system. The bifurcation diagram 4a is instructive. It suggests that for  $\eta$  between 0.5 and 1.02,  $q$  has a single attractive orbit (zone A) assuming  $e = 0$  and  $f = 1$ . As  $\eta$  increases past 1.01, that attractive orbit bifurcates into an attractive orbit of period 2 (zone B). As  $\eta$  continues to increase, further bifurcations take place. In zone C an attractive orbit of period 4. Past the point 1.2, chaotic behaviour seems to reign in the dark region. However, there are windows of calm within the chaos. In diagram 4b, we have  $f = 3$  and find that although the features of the dynamic are same there are two differences: (a) chaotic behaviour occurs for lower values of  $\eta$  and (b) chaos occurs for a narrower range of values of  $q$ . The former is because higher values of  $f$  increase the gradient of the supply curve relative to the demand curve (as in Fig. 3a) and the latter is due to the binding (or further compressing) of the supply curve as  $f$  increases. The mechanics for the instability in 4a & b can be understood by considering Eq. (34) together with the plot 3b. Indeed, assuming naive expectations in prices and neighbouring production, higher positive values of  $\eta$  generate non-monotonocities in the supply function (for example consider the case of  $\eta = 1$  in plot 3b) which are the cause for the bifurcation paths to chaotic behaviour in plots 4a & b. Hence, on the one hand, positive interactions reduce the cobweb fluctuations and increase the stability margin with respect to the critical bound  $+1$ . On the other hand, strong enough positive interaction may also cause instability as the critical bound  $-1$  may be encroached.

What happens in the case of negative interactions in our example? For values  $\eta > -0.42$  the system is stable since the critical bound  $+1$  from Eq. (32) is not violated. Crossing this limit, the supply function is positively sloped, as is suggested in plot 3b, but the stability condition is violated since the ratio of derivatives of demand net of interactions and supply functions is greater than the critical bound  $+1$ . The qualitative dynamics in this case are thus similar to that of an unstable cobweb model with naive expectation and monotonic demand and supply schedules (see for example Hommes (1994, 1998), Chiarella (1988), Jensen and Urban (1984), Artstein (1983)). Indeed, numerous simulations for our examples, not reported here, indicate that  $q$  diverges and explodes.

<sup>5</sup>We use the E&F Chaos package for our bifurcation diagrams available at [www.feb.uva.nl/cendef](http://www.feb.uva.nl/cendef). For documentation on E&F Chaos see Diks, Hommes, Panchenko and Weide (2006).

#### 4. Statistical Dynamics

In this section, we augment our main model to include stochastic shocks so as to investigate the statistical properties of aggregates in the approach to equilibrium. Statistical dynamics help us understand the speed of approach to equilibrium as well as how correlated behaviors are over time and over spatial coordinates. The approach in this section is also used to develop the steady state representative agent analysis in the following section.

We modify Eq.(24) to:

$$B^{-1}(\Xi + \Theta)\phi_{t-1} + \phi_t = B^{-1}[\mathbf{c} - \mathbf{a}] + \epsilon_t \quad (35)$$

where  $\epsilon_t$  is a random shock. To analyze the statistical dynamics of this model, we note that Eq. (35) is a vector autoregression model so that it would be straightforward to compute its autocorrelations and its spectral density matrix. However, such calculations would provide little insight into the spatial statistics of output.

##### 4.1. One Dimensional Externalities

To proceed further, we rewrite Eq. (35) as a difference equation in both spatial and temporal variables. We have the revised equation:

$$b_j^{-1}d\phi_{j,t-1} - b_j^{-1}\sum_{i \in (j)} \eta_{ij}\phi_{i,t-1} + \phi_{j,t} = b_j^{-1}(c - a_j) + \epsilon_{j,t}. \quad (36)$$

where  $j$  is the spatial index. As before, we assume homogeneity so that  $b_j = b$ ,  $a_j = a$ .

To proceed further, we assume that the lattice is one-dimensional with periodic boundary conditions and that the externalities are homogeneous nearest neighbor effects with strength  $\eta$ . Taking a Fourier transform with respect to spatial variables, we obtain a difference equation in time for the spatial Fourier ordinates:

$$b^{-1}d\hat{\phi}_{k,t-1} - b^{-1}\eta[e^{-ik} + e^{ik}]\hat{\phi}_{k,t-1} + \hat{\phi}_{k,t} = b^{-1}(c - a)\delta_k + \hat{\epsilon}_{k,t}. \quad (37)$$

Long-range correlations are captured by low-frequency spatial behavior. To examine this behavior, we solve for the steady state which is:

$$\hat{\phi}_{k,*} = \frac{b^{-1}(c - a) + \hat{\epsilon}_k}{b^{-1}d - b^{-1}\eta[e^{-ik} + e^{ik}] + 1}. \quad (38)$$

To find the steady state spectrum, we subtract off the mean and square the result and take expectations:

$$S_\phi(k) = \frac{S_\epsilon(k)}{|b^{-1}d - b^{-1}\eta[e^{-ik} + e^{ik}] + 1|^2}. \quad (39)$$

As  $k \rightarrow 0$ , the leading order terms in the squared expression in the denominator are quadratic in  $k$ :<sup>6</sup>

$$S_\phi(k) \approx \frac{S_\epsilon(k)}{|b^{-1}d + b^{-1}\eta[-1 + 2k^2] + 1|^2}. \quad (40)$$

To interpret, Eq. (40) we suppose that  $\epsilon_{j,t}$  is a spatially uncorrelated Gaussian noise process. In this case, far from equilibrium outputs are uncorrelated but as equilibrium is approached, output exhibits a decaying spectrum so that outputs of firms are positively correlated and the range of correlations far exceeds that of the nearest neighborhood effects.

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<sup>6</sup>The denominator in Eq. (39) is positive by the stability conditions.

#### 4.2. Two Dimensional Externalities

The results in the subsection extend readily to multiple dimensional externalities. To illustrate this, we analyze two dimensional externalities in this section. We construct the revised equation

$$b_{x,y}^{-1}d\phi_{x,y,t-1} - b_{x,y}^{-1} \sum_{(\tilde{x},\tilde{y}) \in (x,y)} \eta_{x,y;\tilde{x},\tilde{y}}\phi_{\tilde{x},\tilde{y},t-1} + \phi_{x,y,t} = b_{x,y}^{-1}(c - a_{x,y}) + \epsilon_{x,y,t}. \quad (41)$$

where  $x$  and  $y$  are the spatial coordinates in two dimensions. We again assume homogeneity so that  $b_{x,y} = b$  and  $a_{x,y}$ , the externalities are homogeneous nearest neighbor, and that the lattice is periodic. We take a two dimensional Fourier transform with respect to the spatial variables  $x$  and  $y$ . We have:

$$b^{-1}d\hat{\phi}_{k_1,k_2,t-1} - b^{-1}\eta\hat{\phi}_{k_1,k_2,t-1} [e^{ik_1} + e^{ik_2} + e^{-ik_1} + e^{-ik_2}] + \hat{\phi}_{k_1,k_2,t} = b^{-1}(c - a) + \hat{\epsilon}_{k_1,k_2,t}. \quad (42)$$

Solving for a steady state, subtracting the mean and taking expectations with respect to the random variable  $\hat{\epsilon}$  we have:

$$S_\phi(k_1, k_2) = \frac{S_\epsilon(k_1, k_2)}{|b^{-1}d - b^{-1}\eta[e^{-ik_1} + e^{ik_1} + e^{-ik_2} + e^{ik_2}] + 1|^2}, \quad (43)$$

which is equivalent to Eq. (39) as  $k_1, k_2 \rightarrow 0$  with the change of variables  $k = \sqrt{k_1^2 + k_2^2}$  (c.f. Eq. 40).

We find that as the spatial frequency variables go to zero, the steady state spatial spectrum of output exhibits a negative slope so that steady state behavior is dominated by long-run correlations relative to the initial distribution of output.

### 5. The Representative Agent Model

We now provide an interpretation of the representative agent model with global externalities as an approximation to a more complex model with local externalities. In a setting with externalities, we define a representative agent model as a statistical model which replaces the local external effect with its average. This is a reasonable assumption because the representative agent model assumes that all agents behave identically so that without this averaging, there would be no representative agent. In addition, this is the model which is indeed used by a large number of papers on externalities and increasing returns including Diamond (1982) and Caballero and Lyons (1992). This model is also analogous to the ‘‘mean field theory’’ of statistical physics which has been previously used in Brock (1992) and Aoki (1995). Since a representative agent model is much simpler to solve than a complex system of interacting economic agents, it is useful to examine the accuracy of this approximation. To keep the analysis simple, we assume as above that the externalities and cost parameters are homogeneous.

**Proposition 5** *A representative agent approximation produces an exact prediction for the level of steady state output.*

**Proof.** By Proposition. 2, outputs are proportional to the inverse of the row sum of the  $\Omega$  matrix. However, the representative agent approximation by construction holds constant the row sum so output is unchanged.

■

The representative agent model may, however, potentially be correct about mean behavior and wrong about aggregate dynamics. The results in Sec. 4.1 and 4.2 apply to the spectrum of local outputs, not to aggregate dynamics. To say something about aggregate dynamics, we need an additional result.

**Proposition 6** *Consider a macroeconomic time series  $y(t)$  which is the aggregate over spatial coordinates of a statistically homogeneous observable  $f(x, t)$  on a spatially periodic lattice  $\mathbf{L}^d$ . The temporal correlations of  $y(t)$  are uniquely determined by zero frequency spatial behavior.*

**Proof.** Let  $f(x, t)$  be an observable such as the demeaned aggregate output of firms of a given type  $x \in \mathbf{L}^d$ , then the aggregate output of all firms  $y(t)$  is defined as:

$$y(t) = \sum_{|x| \leq \mathbf{N}} f(x, t) \quad (44)$$

where  $\mathbf{N}$  is a vector of arbitrary length. The covariance function of  $y(t)$  is thus defined as:

$$F(s) = \text{cov}(y(t), y(t+s)) = \sum_{|x| \leq \mathbf{N}, |x'| \leq \mathbf{N}} \text{cov}(f(x, t), f(x', t+s)) \quad (45)$$

The covariance function is independent of time  $t$  as we have assumed that  $f(x, t)$  is statistically homogeneous in the variables  $x$  and time  $t$ . Taking the Fourier transform of the covariance function of  $y$ ,  $F(s)$ , with respect to the variable  $s$  yields the frequency spectrum of  $y(t)$ :

$$\begin{aligned} \hat{F}(\omega) &= \sum_s \sum_{|x| \leq \mathbf{N}} \sum_{|x'| \leq \mathbf{N}} \text{cov}(f(x, t), f(x', t+s)) e^{-2\pi i \omega s} \\ &= \sum_s \sum_{|x| \leq \mathbf{N}} C(x, s) e^{-2\pi i \omega s} \\ &= \sum_{k, x} \hat{C}(k, \omega) e^{2\pi i k \cdot x} \\ &= \sum_k \delta^{(d)}(k) \hat{C}(k, \omega) = \hat{C}(0, \omega) \end{aligned} \quad (46)$$

where:

$$C(x, s) = \left( \prod_{j=1}^d 2N_j + 1 \right) \text{cov}(f(y, t), f(y+x, t+s)). \quad (47)$$

Thus, if the spectra of  $f(x, s)$  are of the form  $\hat{C}(k, \omega)$ , then the time spectra of the aggregates is simply  $\hat{C}(0, \omega)$ . This means that even if length and time scales are coupled, spectral behavior of economic aggregates only depend on time correlations and characteristic dynamics at zero frequency (e.g., over the longest possible spatial horizons). ■

This result means that spectral behavior of local aggregates and more microeconomic scales will naturally depend on interaction of length and time scales. However, the specific dynamic statistical interactions of different industries on short and intermediate length scales would not seem to matter at all for the spectral behavior of economic aggregates in time.

Therefore, we need only examine the steady state spectral formulas in Sec. 4.1 and Sec. 4.2 to determine the accuracy of the representative agent approximation. We focus on Eq. (37) as there is little qualitative difference in the second order behavior of one and two dimensional externalities in our model. Assuming there are an odd number  $N$  of firms and a representative agent approximation, Eq. (37) becomes:<sup>7</sup>

$$b^{-1} d \hat{\phi}_{k, t-1} - b^{-1} \frac{2\eta}{n} \sum_{|m| < \lfloor \frac{n}{2} \rfloor} e^{imk} \hat{\phi}_{k, t-1} + \hat{\phi}_{k, t} = b^{-1} (c - a) \delta_k + \hat{\epsilon}_{k, t}. \quad (48)$$

We note that when  $k = 0$ , Eq. (48) is equivalent to Eq. (37) which implies that the representative agent approximation is exact for the statistics of steady state aggregate output. This result is in line with much of the existing literature. Our result is in line with Föllmer (1972) where a unique equilibrium is achievable as long as microeconomic specification deliver a unique macro phase. In Föllmer's case microeconomic

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<sup>7</sup>The result also applies if  $N$  is even.

specification referred to homogenous Markov economies which short range interactions. In our case the microeconomic specification is that of symmetric interactions that lead to a unique macro-aggregation. Another recent related paper is Brock and Durlauf (2001) that shows in a model of social interaction with positive externalities which are spatially homogenous that multiple *but* stable average behavior may emerge when individuals experience both private and social utilities from their decisions. Binder and Pesaran (2001) show in a consumption life-cycle setup that the behavior of selfish individuals and the behavior of those who like to conform may overlap in the presence of social interactions under some restrictive conditions. This paper extends such results justifying representative agent models of individuals to firms. The results we obtain have interesting implications for the statistical theory of markets of Duncan Foley (1994, 1999) which stresses the importance of interactions between agents in order to achieve an equilibrium probability distribution of transaction prices in the market. Indeed, unlike in Walras laws, where the fictional auctioneer makes sure that trade takes place at *only* equilibrium prices, in Foley's work competition between traders ensures a mean price along with a distribution. In this paper we have shown that when there are firms with symmetric interaction, and assume that such firms represent a cluster in the economy, then a unique set of equilibrium output and hence prices emerge both at the micro and macro level.

Although our approximations are exact, the representative-firm model we derive is open to criticism along the lines of the seminal work of Kirman (1992) and further developed by Hartely (1997) and Forni and Lippi (1997 & 1999) even though much of these criticisms are directed towards representative consumer models. First, the extent to which the representative-firm proxied here actually represents all the firms deserves rethinking. Indeed, the our approximation has a specification-bias in that we have affixed ideas in the context of homogenous and symmetric interactions between firms. An approximation that takes into consideration heterogenous externalities may differ from the one achieved here. However, such aggregation might in fact be unsolvable, as is recognized on Kirman's and Hartely's works. Eq. (9) alludes to the difficulty of obtaining the conditions for stability unless some restrictions are imposed (as we do in our Eq. (15)). Second, the approximation is susceptible to the Lucas-Critique in that deep parameters (i.e., the local interactions in our case) are policy invariant (see Hartely (1997, p 40-47)). For instance, it is possible that during times of high anticipated economic activity, through a relaxed monetary policy, it may pay-off more for firms to collect information on related firms and exploit the externality affect studied in the paper. Under such circumstances the individual effect as well as the resulting aggregate effects may differ by policy regime. Third, by allowing for more heterogeneity than one proposed in this paper we should not expect the dynamic statistical properties of aggregate output to share those properties of individual firm's output. However, as we have seen that aggregation is much simpler when common-idiosyncratic components (i.e., symmetric interactions) are allowed in the micromodel; see Forni and Lippi (1997, Chap 7). Nonetheless, the direct presence of mildly interacting-firms (since interactions are restrictive) in our model is a small *but* positive step towards building a truly macro-microfounded cobweb model with externalities.

## 6. Conclusion

In this paper we take a step towards extending a cobweb model of quantity dynamics that incorporates local externalities. This extension requires agents to make both price and local quantity forecasts based on behavior in the previous period. We analyze stability and the spatial distribution of output in context of symmetric or uniform externalities. We find that the cobweb dynamics with linear demand and supply functions with naive expectations in quantities and prices reach a stable equilibrium when the gradient of aggregate demand net of externalities does not exceed that of the supply function. For this condition, positive interactions tend to extend the stability margin whereas negative interactions tend to have the opposite effect. This result is also found to be robust to reasonable S-shapes of the nonlinear supply function. Furthermore, under a statistical definition of a representative agent model, we also show that, in certain circumstances, it is exact for the mean and correlations of aggregate statistics for the more complex model with local externalities. Since local externalities are likely to be important in practice, this paper provides a justification for the use of representative agent models in macroeconomic modelling with symmetric externalities. This is to be expected as our model has linear quadratic costs and hence produces linear decision rules which aggregate effectively and in more complex settings we would expect some deviation of the representative agent model

approximation as argued in much of the related literature. There are two natural extensions of our model. The first relates to introducing a complex and heterogeneous forms of interactions between firms so that some firms positively affect another while others have the opposite affect. The second amounts to analyzing the model in the context of a heterogeneous belief systems of the form present in Brock and Hommes (1997) Goeree and Hommes (2000). On both counts we expect the dynamic properties of the model to change and be of a far more complex nature than one presented here. This is because the less uniformly firms interact with one another, the less likely there is to be a unique outcome. On the application side, the ideas in the paper can be useful for various fields of which we would like to highlight two. The recent surge in interest in issues of the coordination of monetary policy between countries, where central banks affects one another but the degree of interaction differs by the size of the country and where output as well as aggregate price forecasts of each country matter, are of particular relevance here (see for example Canzoneri, Cumby and Diba (2005) and the literature therein). Finally, there is also the possibility of studying the type of equilibria and coordination-failures that would emerge in the Cooper and John (1988) economy where agent's payoffs are affected by various types of interactions and agent's form various types of expectations on the behaviour of the neighbouring agents.

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**Appendix : Proof of Theorem two.**

**Proof.** Since the parameters  $a_i$  and  $b_i$  are assumed constant across firms, we can set  $a_i = a$ ,  $b_i = b$ . We now need to invert our matrix  $\Omega$  in order find a solution to the Equation (5). It is well known<sup>8</sup> that for any symmetric matrix,  $A$ , there exists a matrix  $F$  such that  $F^{-1}AF = I$  satisfying

$$F^{-1}AF = \Lambda = \text{diag}(\lambda_0, \lambda_1, \dots, \lambda_n)$$

where  $\lambda_i$ ,  $i = 0, 1, \dots, n$ , are the eigenvalues of  $A$ . This transformation known as diagonalization is required for our matrix  $\Omega$ .

The circulant matrix,  $\Omega$ , is generally<sup>9</sup> diagonalized by the  $n \times n$  Fourier matrix  $F^* = [f_{ij}]$  where

$$f_{ij} = \frac{1}{\sqrt{n}} \omega^{(i-1)(j-1)}, \quad i, j = 1, 2, \dots, n$$

and  $\omega = \exp(\frac{-2\pi i}{n}) = \cos(\frac{2\pi}{n}) - \mathbf{i} \sin(\frac{2\pi}{n})$ ,  $\mathbf{i} = \sqrt{-1}$ . For example, by a Fourier matrix of order  $n$  we shall mean

$$\frac{1}{\sqrt{n}} \begin{bmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & \omega & \omega^2 & \dots & \omega^{n-1} \\ 1 & \omega^2 & \omega^4 & \dots & \omega^{2(n-1)} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & \omega^{n-1} & \omega^{2(n-1)} & \dots & \omega^{(n-1)(n-1)} \end{bmatrix} \quad (49)$$

It can be easily established that  $F$  and its conjugate transpose  $F^*$  are symmetric. It can also be verified that the Fourier matrix is unitary i.e.,  $F^* = F^{-1}$  (see, Davis, 1994).

Using the Fourier Matrix to diagonalise our circulant matrix  $\Omega$  we get the roots,

$$F^* \Omega F = \Lambda = \text{diag}[\lambda_0, \lambda_1, \dots, \lambda_{n-1}]$$

$$\lambda_j = \begin{cases} \sum_{h=-(n-1)/2}^{(n-1)/2} \gamma(h) e^{-i2\pi h j/n}, & n \text{ odd} \\ \sum_{h=1-n/2}^{n/2} \gamma(h) e^{-i2\pi h j/n}, & n \text{ even} \end{cases}, \quad j = 0, 1, \dots, n-1.$$

where the periodicity property implies that  $e^{-2\pi j(n-h)/n} = e^{i2\pi j h/n}$ . The column vectors of  $F^*$  are the universal set of (right) eigenvectors of  $\Omega$ . This is so, because generally a matrix that is used for diagonalization is formed by putting side by side the eigenvectors of the matrix that is being diagonalized. The only issue with the set of eigenvectors we have is that for our application it is preferable to work with real vectors rather than complex ones. Consider the case when  $n$  is odd (see Fuller, 1976, p 136), the eigenvalue can also be written as

$$\lambda_j = \sum_{h=-(n-1)/2}^{(n-1)/2} \gamma(h) \cos \frac{2\pi}{n} h j, \quad j = 0, 1, \dots, n-1$$

where it is clear that the eigenvalue  $\lambda_j = \lambda_{n-j}$  ( $j = 1, 2, \dots, [n-1]/2$ ). We can use the eigenvectors  $e'_j$  and  $e'_{n-j}$  associated with these repeated roots in order to find a pair of vectors which are orthonormal. These new vectors lie in the same vector space as our original eigenvectors and are given by

$$\frac{1}{\sqrt{2}}(e'_n + e'_{n-j}) = v'_{2j-1} = \sqrt{\frac{2}{n}} (1 \quad \cos(\omega_j) \quad \cos(2\omega_j) \quad \dots \quad \cos((n-1)\omega_j))$$

$$\frac{1}{\sqrt{2}}i(e'_{n-j} - e'_j) = v'_{2j} = \sqrt{\frac{2}{n}} (0 \quad \sin(\omega_j) \quad \sin(2\omega_j) \quad \dots \quad \sin((n-1)\omega_j)) \quad (50)$$

where  $\omega_j = \frac{2\pi j}{n}$ . This pattern also prevails for the eigenvectors of circulant matrix  $\Omega$  when  $n$  is even. However, the eigenvector  $e_{n-1}$  is:

$$e'_{n-1} = \sqrt{\frac{1}{n}} (1 \quad \cos(\omega_{n/2}) \quad \cos(2\omega_{n/2}) \quad \dots \quad \cos((n-1)\omega_{n/2})) \quad (51)$$

<sup>8</sup> See, for example, Barnett (1992, p149).

<sup>9</sup> See Davis (1994) or Barnett (1992).

Note that when  $n$  is odd or even the eigenvector associated with  $\lambda_0$  is:

$$e'_0 = \sqrt{\frac{1}{n}} (1 \ 1 \ \dots \ 1) \quad (52)$$

However when  $n$  is even, in addition to the multiple roots there is a vector  $\frac{1}{\sqrt{n}}(1, -1, \dots, -1)$  associated with the  $j = \frac{n}{2}$  root.

To summarize, we have shown that the circulant matrix,  $\Omega$ , is diagonalized by the Fourier matrix  $F$ . This also means that  $\Omega = F\Lambda F^*$  where  $\Lambda$  is the diagonal matrix of eigenvectors. Next, we use this diagonalization to derive the inverse of  $\Omega$  in order to solve for outputs in Equation (5).

The inverse of the circulant matrix  $\Omega$  is given by:

$$\Omega^{-1} = F^* \Lambda^{-1} F.$$

This holds as Moore-Penrose conditions for generalized inverses are immediately verifiable. Namely that

- 1)  $\Omega\Omega^{-1}\Omega = \Omega$ ,
- 2)  $\Omega^{-1}\Omega\Omega^{-1} = \Omega^{-1}$ ,
- 3)  $(\Omega\Omega)^* = \Omega\Omega^{-1}$ ,
- 4)  $(\Omega^{-1}\Omega)^* = \Omega^{-1}\Omega$ .

The elements of the  $j$ th row and  $k$ th column in the inverse matrix,  $\Omega^{-1}$ , can be represented as:

$$\Omega_{jk}^{-1} = \frac{1}{n}\lambda_0^{-1} + \sum_{m=1}^{n-1} \lambda_m^{-1} e_{m;j} e_{m;k} \quad (53)$$

Summing elements of the inverse across columns  $k$  (and using trigonometric identities) yields:

$$\phi = \Omega^{-1}(\mathbf{c} - \mathbf{a}) = \lambda_0^{-1}(c - a) \quad (54)$$

Summing across firms yields:

$$\mathbf{1}'\Omega^{-1}(c - a) = \sum_{i=1}^N \phi_i = n\lambda_0^{-1}(c - a) \quad (55)$$

For nonnegative prices:

$$\sum_{i=1}^n \phi_i < \frac{c}{d}. \quad (56)$$

so that:

$$\lambda_0^{-1} < \frac{c}{nd(c - a)}. \quad (57)$$

Using the definition of  $e_0$  above and the fact that:

$$\lambda_0 = \sqrt{n} \sum_{j=0}^{n-1} \Omega_{1,j+1} e_{0j} = \gamma(0) + 2 \sum_{j=1}^{[(n-1)/2]} \gamma(j) + \left( \left[ \frac{n}{2} \right] - \left[ \frac{n-1}{2} \right] \right) \gamma\left(\frac{n}{2}\right)$$

(the last term is nonzero only for even  $n$ ), the result follows. ■

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