

Linear-Quadratic Approximation, External Habit and Targeting Rules

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Abstract

We examine the linear-quadratic approximation of non-linear dynamic stochastic optimization problems. A discrete-time version of Magill (1977a) is generalized to models with forward-looking variables paying special attention to second-order conditions. This is the ‘large distortions’ case in the literature. We apply the approach to monetary policy in a DSGE model with external habit in consumption. We then develop a condition for ‘target-implementability’, a concept related to ‘targeting rules’. Finally, we extend the approach to a comparison between cooperative and non-cooperative equilibria in a two-country model and show that the ‘small distortions’ approximation is inappropriate for this exercise.

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1 Introduction

This paper is about macroeconomic optimization problems in general with a particular application to welfare-optimal monetary policy. The title may sound like a smorgasbord of disparate topics, but in fact, as we will demonstrate, the correct linear-quadratic (LQ) approximation of a nonlinear optimization problem, external habit in DSGE models and the idea of general targeting monetary rules as proposed by Svensson(2003, 2005) are all related.

LQ approximations to nonlinear dynamic optimization problems in macroeconomics are widely used for a number of reasons. First, for LQ problems the characterization of time-consistent and commitment equilibria for a single policy maker, and even more so for many interacting policymakers, are well understood. Second, the certainty equivalence property results in optimal rules that are robust in the sense that they are independent of the variance-covariance matrix of additive disturbances. Third, policy can be decomposed into deterministic and stochastic components. This is a very convenient property since it enables the stochastic stabilization component to be pursued using simple Taylor-type feedback rules rather than the exceedingly complex optimal counterpart. Fourth, the stability of the system is conveniently summarized in terms of eigenvalues. Finally for sufficiently simple models, LQ approximation allows analytical rather than numerical solution.

But what is the correct procedure for replacing a stochastic nonlinear optimization problem with a LQ approximation? As pointed out by Judd (1998), some common methods employed by economists have produced poor approximations which fail to consistently incorporate all relevant second-order terms and thus open up the possibility of spurious results. These pitfalls are also very neatly exposed in Kim and Kim (2003) and Kim and Kim (2006).

We make four main contributions to the optimal macroeconomic policy literature. First, we follow Judd (1998), pages 507-509, in drawing attention to a general Hamiltonian framework for approximating a nonlinear problem by an LQ one due to Magill (1977a), who appears to be the first to have applied it in the economics literature.¹ This paper is the precursor to a recent literature led by Michael Woodford that considers an LQ approximation to the Ramsey problem in the context of DSGE models.² Whereas Judd's emphasis is on the perturbation approach which focuses

¹See also, Magill (1977b).

²See Woodford (2003), Benigno and Woodford (2003, 2005), Altissimo *et al.* (2005), Benigno and Woodford (2007) for one-country models and Benigno and Benigno (2006) for a two-country generalization. The 'large distortions' case of Benigno and Woodford (2003) and Benigno and Benigno (2006) uses the method of undetermined coefficients, but their more recent work uses

on computing derivatives of the nonlinear optimization problem our paper is about replacing various nonlinear problems in a one-country and two-country context with ones that are LQ. It should be noted that the Judd second-order perturbation and Hamiltonian approaches generate the same LQ approximation. We develop the Magill framework in presenting a discrete-time version of his results generalized to rational expectations models with forward-looking variables. These results include second-order necessary conditions for non-concave intertemporal problems which are rarely discussed in the literature and, to our knowledge, have not been published anywhere for forward-looking systems. We remedy this by deriving these conditions, and later on explain how they relate to the non-optimality of zero inflation for certain parameter combinations in a New Keynesian setting.

Second, we apply the Hamiltonian approach to a fairly standard New Keynesian DSGE model with external habit in consumption. We show that in such a model the natural rate of output can exceed the efficient rate if habit is sufficiently strong and this feature has important implications for the form of the LQ approximation.

Third, we introduce the concept of target-implementability which requires the quadratic approximation of the welfare to be expressed in terms of targets or ‘bliss points’ for linear combinations of macroeconomic variables. Such a property fits in with the notion of targeting rules proposed by Svensson (2003, 2005). We then derive sufficient conditions for the LQ approximation to have this property in the vicinity of a zero-inflation steady state.

Finally, we extend the Hamiltonian approach to a comparison between cooperative and non-cooperative equilibria in a two-bloc model. In the simple model of Clarida *et al.* (2002) without habit in consumption, we show that the commonly used procedures for LQ approximation assuming either an efficient zero-inflation steady state (achieved by a subsidy) or ‘small distortions’ are inappropriate unless one bloc is very small, or the instrument in the Nash equilibrium is assumed to be output. If both blocs are large and the instrument is taken to be inflation (or inflation targets perfectly achieved) then the Hamiltonian approach, which we extend to the two-country non-cooperative equilibrium, must be employed.

The rest of the paper is organized as follows. Section 2 sets out the results of a discrete-time version of the continuous-time results of Magill (1977a). In Theorems 1 to 3 we extend his results to models with forward-looking variables. In Section 3 we introduce the general class of DSGE models to be studied. We examine the

what amounts to the Hamiltonian approach of Magill which involves less algebraic manipulation and provides a more convenient algorithm suitable for numerical computation. The working paper version of this paper, Levine *et al.* (2007a), demonstrates the equivalence of these two approaches.

social planner’s problem where optimization is only subject to resource constraints. We then proceed to the Ramsey problem where, in addition, the policy-maker faces constraints in the form of decentralized decisions by households and firms, given the instrument at her disposal. The LQ approximation to the Ramsey problem is analyzed for the ‘efficient case’ (where subsidies eliminate distortions in the steady state) and the ‘small distortions case’ where such subsidies are not available.

Section 4 applies the general results of preceding sections to welfare-maximizing monetary policy in a fairly standard New Keynesian framework, with external habit in consumption. We derive the corresponding LQ approximation to the policy-maker’s problem, and briefly comment on its representation when there is no habit. In addition we compare the quadratic expansions in the efficient case and the large distortions case. We then define and discuss the notion of *target-implementability*; this is essentially about the conduct of monetary policy by the setting of targets in the loss function by the monetary authority, as advocated by Svensson, when it engages in stabilization policy. We obtain sufficient conditions for the target implementability of the LQ approximation to the Ramsey problem in the vicinity of a zero-inflation steady state. Finally, to demonstrate that second-order conditions for a maximum do matter, we provide an example for which the necessary and sufficient second-order conditions for optimality are not satisfied although the zero-inflation steady state satisfies the necessary first-order conditions.

Section 5 compares numerical solutions to the Ramsey problem using small and large distortions LQ approximations, with the exact nonlinear solution for the model of section 4. In Section 6, we extend our work to the two-bloc case. We demonstrate the inadequacy of the ‘small distortions’ approach, and derive an appropriate LQ approximation for the ‘large distortions’ case, which we compare with Benigno and Benigno (2006). Section 7 provides some concluding remarks.

2 The LQ Approximation to a General Dynamic Optimization Problem

In this section we provide the mathematical background to LQ approximations that provide accurate first-order approximations to deviations from the optimal solution to a deterministic dynamic optimization (or optimal control) problem. We firstly state the requirements for an LQ approximation to be accurate to first order, and then explain how it may be obtained using the Lagrangian of the problem. We apply this to a very simple economic model with a rudimentary output/inflation tradeoff.

The reason for so doing is to demonstrate that the method of Benigno and Woodford (2003, 2005) as used in this model is equivalent to implementing a Lagrangian (or more strictly a Hamiltonian) approach to the objective function.

2.1 Necessary and Sufficient Conditions for an Accurate LQ Approximation:

Suppose we have a deterministic dynamic optimization problem expressed in the form³

$$\max \sum_{t=0}^{\infty} \beta^t U(X_{t-1}, W_t) \quad s.t. \quad X_t = f(X_{t-1}, W_t) \quad (1)$$

given initial and possibly transversality conditions, which has a steady state solution \bar{X}, \bar{W} for the states X_t and the policies W_t . Define $x_t = X_t - \bar{X}$ and $w_t = W_t - \bar{W}$ as representing the first-order approximation to deviations of states and policies from their steady states. Suppose we write a candidate first-order approximation to the problem as

$$\max \sum_{t=0}^{\infty} \beta^t \left(\begin{bmatrix} x_{t-1}^T & w_t^T \end{bmatrix} Q \begin{bmatrix} x_{t-1} \\ w_t \end{bmatrix} - 2 \begin{bmatrix} x_{t-1}^T & w_t^T \end{bmatrix} b \right) \quad s.t. \quad x_t = Ax_{t-1} + Bw_t + c \quad (2)$$

Then a necessary condition for this to be a first-order accurate solution is that $b = 0$, $c = 0$ i.e. the objective function must be purely quadratic and the dynamics purely linear in deviations. The reasons for this are clear; firstly, suppose that the system starts in the steady state; it could only remain in the steady state $x_t = 0$, $w_t = 0$ if $c = 0$. Secondly, if $b \neq 0$ then there is a bliss-point $Q^{-1}b \neq 0$ which would be desirable, so that the solution to the problem starting at the steady state would not remain at the steady state.

The implications of these conditions are rather serious given the manner in which many LQ approximations were conducted within the economics profession up until a few years ago. Often, the approach used would often not even involve finding the steady state of the optimal solution to the problem. Even were that the case, the next error to be committed would be to use merely a Taylor series approximation

³An alternative representation of the problem is $U(X_t, W_t)$ and $E_t[X_{t+1}] = f(X_t, W_t)$ where X_t includes forward-looking non-predetermined variables and $E_t[X_{t+1}] = X_{t+1}$ for the deterministic problem where perfect foresight applies. Whichever one uses, it is easy to switch from one to the other by a simple re-definition. Magill (1977a) adopted a continuous-time model without forward-looking variables. We present a discrete-time version with forward-looking variables. As we demonstrate in the paper, although the inclusion of forward-looking variables significantly alters the nature of the optimization problem, these changes only affect the boundary conditions and not the steady state of the optimum which is all we require for LQ approximation.

to the constraint function $f(X, W)$, whose first-order expansion cannot of course be guaranteed to equal zero.

However there exists a very straightforward approach to finding the appropriate approximation. Define the problem using a Lagrangian L

$$L = \sum_{t=0}^{\infty} \beta^t [U(X_{t-1}, W_t) - \lambda_t^T (X_t - f(X_{t-1}, W_t))] \quad (3)$$

so that a *necessary* condition for the solution to (1) is that the Lagrangian is stationary at all $\{X_s\}$, $\{W_s\}$ i.e.

$$U_W + \lambda_t^T f_W = 0 \quad U_X - \frac{1}{\beta} \lambda_{t-1}^T + \lambda_t^T f_X = 0 \quad (4)$$

These necessary conditions for an optimum do not imply that there is an asymptotic steady state to (4). However for the purposes of this paper, let us assume that this is the case, so that a steady state $\bar{\lambda}$ for the Lagrange multipliers exists as well. Now define the Hamiltonian $H_t = U(X_{t-1}, W_t) + \bar{\lambda}^T f(X_{t-1}, W_t)$. The following is the discrete time version of Magill (1977a):

Theorem 1: If a steady state solution $(\bar{X}, \bar{W}, \bar{\lambda})$ to the optimization problem (1) exists, then for any small initial perturbation x_0 about \bar{X} , the solution to the problem

$$\max \frac{1}{2} \sum_{t=0}^{\infty} \beta^t \begin{bmatrix} x_{t-1} & w_t \end{bmatrix} \begin{bmatrix} H_{XX} & H_{XW} \\ H_{WX} & H_{WW} \end{bmatrix} \begin{bmatrix} x_{t-1} \\ w_t \end{bmatrix} \quad s.t. \quad x_t = f_X x_{t-1} + f_W w_t$$

where H_{XX} , etc denote second-order derivatives evaluated at (\bar{X}, \bar{W}) , has the same stability properties as the solution to (1).

Judd (1998), (page 506) thus identifies this as the LQ approximation to the problem (1). The reason why this result holds is because the derivatives of the Lagrangian with respect to X_t and W_t are zero when evaluated at $(\bar{X}, \bar{W}, \bar{\lambda})$. By definition, $\sum_{t=0}^{\infty} \beta^t U(X_{t-1}, W_t) = \sum_{t=0}^{\infty} \beta^t [U(X_{t-1}, W_t) - \bar{\lambda}^T (X_t - f(X_{t-1}, W_t))]$, and the first-order term of the Taylor series expansion of the latter expression is zero. Although we deal directly with forward-looking systems in Theorem 3(b) below, we note that the above theorem applies both to backward-looking engineering-type systems and to rational expectations (RE) systems, in that approximation is about the long run of the optimum. However in the case of RE, the conventional optimum

is obtained as a time-inconsistent solution, but the LQ approximation can also be used to obtain the timeless perspective optimum.⁴

For the result of theorem to hold $(\bar{X}, \bar{W}, \bar{\lambda})$ must satisfy (4). These, it should be stressed, are necessary but not sufficient conditions for a local maximum. A standard sufficient condition for optimality is that the functions $f(X, W)$ and $U(X, W)$ are concave, but this is rarely satisfied in examples from economics. A more useful sufficient condition is the following:

Theorem 2: A sufficient condition for for the steady state of (4) to be a local maximum is that the matrix of second derivatives of H in (2.1) is negative semi-definite⁵.

This condition is easy to check, but in the event that it does not hold, the following discrete time version of the sufficient conditions for an optimum in Magill (1977a) is applicable when the constraints and/or the welfare function are non-concave. It is based on iterating on the quadratic approximation to the value function. Part (a) below is a standard result, and relates to the fact that one requires a second-order condition to be met for the policy variables. Part (b), which is the main theoretical result of this paper, extends part (a) to the case when there are forward-looking variables.

Theorem 3:

(a) Case with no forward-looking variables: A necessary and sufficient condition for the solution (4) to the dynamic optimization problem (1) to be a local maximum is that $\beta f_W^T P_t f_W + H_{WW}$ is negative definite for all t , where the matrices $f_X, f_W, H_{XX}, H_{XW}, H_{WW}$ are all evaluated along the solution path and P_t satisfies

⁴Benigno and Woodford (2007), in their quadratification of the Lagrangian, emphasise the 'timeless perspective' which imposes initial conditions on the *ex ante* optimal rule that ensures a self-consistency (though not Kydland-Prescott 'time consistency') in the form of commitment. This policy is then used as a benchmark in the relative evaluation of other policy rules. The LQ approximation then requires some modification to take into account another initial precommitment constraint. If, as is common in the literature, one adopts a *conditional* welfare loss measure, starting at the zero-inflation steady state then the timeless perspective is not relevant. For an *unconditional* welfare loss measure averaging over all possible initial states using the distribution of states calculated under the optimal commitment policy the timeless perspective becomes an issue. However, for a discount factor close to unity, the differences between the measures are of second order.

⁵A simple example of a problem for which a maximum exists, but for which this sufficient condition does not hold is: $\max x^2 - y^2$ such that $y = ax + b$. It is easy to see that the stationary point is a maximum when $|a| > 1$.

the backwards Riccati equation given by:

$$P_{t-1} = \beta f_X^T P_t f_X - (\beta f_X^T P_t f_W + H_{XW})(\beta f_W^T P_t f_W + H_{WW})^{-1}(\beta f_W^T P_t f_X + H_{WX}) + H_{XX} \quad (5)$$

and the value function of small perturbations x_t about the path of the optimal solution dynamic optimization problem is given by $\frac{1}{2}x_t^T P_t x_t$.

(b) Case with forward and backward-looking variables: Consider a rational expectations system, where we order X_t as predetermined followed by non-predetermined variables, so that the latter dynamic constraints involve forward-looking expectations. Suppose that there is a long-run steady state solution to the first-order conditions. Then a further necessary and sufficient condition for this to be a maximum is that the bottom right-hand corner P_{22} of the the steady-state Riccati matrix P is negative definite.⁶

Proof. See Appendix A.

As mentioned above we assume the existence of a steady state solution to (4) given by $[\bar{X}, \bar{W}, \bar{\lambda}]$, since we are interested in approximations about the latter. Hence the matrices in (5) (apart from P_t) are constant. Thus this theorem provides a means of checking whether a candidate solution to (4) actually is optimal. Note that the perturbed system is in standard linear-quadratic format, which is the basis for this result.

We also note that Magill (1977a)'s result easily extends to the stochastic case as well. Thus if the dynamic equations are written as $X_t = f(X_{t-1}, W_t, \varepsilon_t)$, where the ε_t have mean zero and are independently normally distributed then any perturbations

⁶This is equivalent to Proposition 2 of Benigno and Woodford (2007), although the latter also require the further sufficient condition that the system under control be stable. The latter is not a mathematical sufficient condition, since welfare is bounded provided that the modulus of all eigenvalues is less than $\beta^{-\frac{1}{2}}$; it is a well-known result in LQ optimal control theory that this will always be the case, but stability is a natural requirement under the assumption of rational expectations. Benigno and Woodford (2007) provide another motivation for (b) in the case when there is a hyperplane of forward-looking variables that is unaffected by the policy instrument; although not stated in the theorem, we tacitly assume that the deterministic system is controllable, so in effect ignore this possibility. We also note that in an earlier version of Benigno and Woodford (2007) there is reference to a book that contains a purportedly equivalent frequency domain result to (a), but from Trentelman and Rapisarda (2001) one can see that the book is in error.

about the deterministic solution are solutions to the problem

$$\begin{aligned} \max E_0 \sum_{t=0}^{\infty} \beta^t \begin{bmatrix} x_{t-1} & w_t & \varepsilon_t \end{bmatrix} \begin{bmatrix} H_{XX} & H_{XW} & H_{X\varepsilon} \\ H_{WX} & H_{WW} & H_{W\varepsilon} \\ H_{\varepsilon X} & H_{\varepsilon W} & H_{\varepsilon\varepsilon} \end{bmatrix} \begin{bmatrix} x_{t-1} \\ w_t \\ \varepsilon_t \end{bmatrix} \\ \text{s.t. } x_t = f_X x_{t-1} + f_W w_t + f_\varepsilon \varepsilon_t \end{aligned} \quad (6)$$

3 The Social Planner's and Ramsey Problems

In this section we introduce the class of DSGE models to be studied. We assume a set of consumers, each with given endowments, whose objective is to maximize an intertemporal utility function. Typically this will incorporate consumption and leisure, but we shall state the objectives in a general fashion, so that they can incorporate habit as well. Thus the objective is for individual i to maximize an expected utility function of the form

$$E_0 \sum_{t=0}^{\infty} \beta^t u(W_{t-1}, W_{it}) \quad (7)$$

where the vector W_{it} represents individual i 's choices e.g. consumption and labour supply. This utility function may also incorporate habit or catching-up, and may therefore also be dependent on aggregate or average choices made in the previous period W_{t-1} .

We also assume that any resource constraints sum to a set of aggregate resource constraints. One can then define the social planner's problem in terms of the representative individual as that of

$$\max E_0 \sum_{t=0}^{\infty} \beta^t U(X_{t-1}, W_t) \quad \text{s.t. } X_t = f(X_{t-1}, W_t, \varepsilon_t) \quad (8)$$

where the set of constraints in this problem now represent the set of (possibly intertemporal) resource constraints and exogenous processes describing the environment. These might include a dynamic equation for capital accumulation, and also capital utilization as in Smets and Wouters (2003).⁷

⁷Although there appear to be significant differences in the functions u of (7) and U of (8), these are merely cosmetic. If we incorporate W_t as a subset of the state X_t , i.e. $X_t = [X_{1t}^T \ X_{2t}^T]^T$, where X_{1t} represents the resource constraints, and $X_{2t} = W_t$, then the presence of X_{t-1} in U is merely a generalization of including W_{t-1} in U .

3.1 Characterization of the Efficient Level

Ultimately we are going to approximate the nonlinear stochastic optimization problem in the vicinity of a suitably chosen deterministic steady state. We therefore focus on the deterministic optimization problem. Defining the Lagrangian

$$\sum_{t=0}^{\infty} \beta^t [U(X_{t-1}, W_t) - \lambda_t^T (X_t - f(X_{t-1}, W_t))] \quad (9)$$

the following first order conditions provide the necessary conditions for the solution:

$$U_W(X_{t-1}, W_t) + \lambda_t^T f_W(X_{t-1}, W_t) = 0 \quad U_X(X_{t-1}, W_t) + \lambda_t^T f_X(X_{t-1}, W_{t+1}) - \frac{1}{\beta} \lambda_{t-1}^T = 0 \quad (10)$$

The steady state of the deterministic social planner's problem, the efficient level (denoted by *), is then given by

$$\begin{aligned} X^* &= f(X^*, W^*) & U_W(X^*, W^*) + \lambda^{*T} f_W(X^*, W^*) &= 0 \\ U_X(X^*, W^*) + \lambda^{*T} f_X(X^*, W^*) - \frac{1}{\beta} \lambda^{*T} &= 0 \end{aligned} \quad (11)$$

3.2 The Flexible-Price Solution and the Ramsey Problem

The difference between the efficient solution and that of the competitive or flexible-price solution is due to the externalities of habit and of firm and labour market power. As we shall see below for a particular example, the externality due to consumption habit works in the opposite direction to the externalities that produce the mark-ups in prices and wages. In principle it is possible to set a proportional tax (or subsidy) in the flexible-price case that yields a 'natural' level of output exactly equal to the efficient level of output of the social planner.

Thus typically in economic models of this type we would assume monopolistic competition by firms. This leads to mark-up pricing, and creates a wedge between the level of output under competition - the natural rate - and the level of output that could be achieved by a social planner - the efficient level. This wedge may be exacerbated if we assume that there is labour market power as well. The latter is not incorporated by Benigno and Woodford (2005), but is common in most other New Keynesian models e.g. Clarida *et al.* (2002). We also assume that costs for firms are continuous, which rules out state-dependent $S - s$ policies; we do this because such policies cannot be easily aggregated. Initially we ignore the stochastic problem because the deterministic problem is sufficient to set up the LQ approximation.

Thus far we have only discussed the efficient and flexible-price levels of output. A more general model takes into account the fact that neither wages nor prices are completely flexible. As a consequence, we must discuss the case where a policymaker is required to maximize average welfare, in this case by choosing the optimal path for inflation. This is a particular case of the Ramsey problem.

Without going into details for the moment, the Ramsey problem differs from the social planner's problem in that there is decentralized decision-making that has to be taken into account by the Ramsey planner. One can incorporate this in a general way into the model by expanding the state space to take account of the additional dynamic behaviour of the system. From the point of view of the Ramsey policymaker, the problem must be rewritten as one of maximizing

$$\sum_{t=0}^{\infty} \beta^t V(X_{t-1}, Z_{t-1}, W_t) \quad (12)$$

In the New Keynesian model that we study below, the only difference between the functions U and V is that there is an effect of price dispersion, where the latter is one of the components of the new set of variables Z_t . Price dispersion affects the disutility of labour. We assume that the decentralized decisions can somehow be aggregated, so that the constraints that must be satisfied by the Ramsey policymaker constitute both the resource constraints and the additional implementation constraints⁸ (typically associated with price and wage-setting, but excluding the intertemporal wealth constraint):

$$X_t = f(X_{t-1}, W_t) \quad Z_t = g(Z_{t-1}, X_{t-1}, W_t; \tau) \quad (13)$$

It is important to appreciate that the implementation constraints associated with Z_t represent individuals' and firms' decisions, and may involve future expectations. We take the approach that the Ramsey policymaker has a reputation for precommitment, so that we can take expectations of the future as always being fulfilled, and therefore regard all equations as backward looking. Suppose in addition that all factor prices are fixed so that inflation is 0 i.e. the appropriate elements of the vector Z are set equal to 0; we then obtain a solution to the 'natural' rate by solving for the steady state $\bar{X} = f(\bar{X}, \bar{W})$, $\bar{Z} = g(\bar{Z}, \bar{X}, \bar{W}; \tau)$. This is also known as the flexible price equilibrium. An important consideration is that the natural rate will be dependent on the tax/subsidy rate τ .

⁸This terminology is now widely used e.g. Khan *et al.* (2003), Kim *et al.* (2006b).

3.3 LQ Approximation of the Ramsey Problem: Efficient Case

Woodford (2003) now points out a key result for LQ-approximation. If at all possible, the aim of the Ramsey policymaker is to stabilize the economy about the efficient level of output. Let us assume therefore that the proportional tax/subsidy is set at exactly the level at which the flexible price equilibrium achieves the efficient level of output. This implies that there exists a value τ^* such that the efficient rate, coupled with zero inflation, is a solution to $Z^* = g(X^*, Z^*, W^*; \tau^*)$.

The main result of this section is dependent on the ability (a) to expand the utility function about the steady state efficient solution without the presence of linear terms and (b) to expand the constraints about the steady state efficient solution without the presence of constant terms.

Theorem 4:

The stabilization problem for the Ramsey policymaker can be approximately expressed as a quadratic expansion of the welfare function about the efficient level, provided that the Taylor series of $\tilde{V} = V(X_{t-1}, Z_{t-1}, W_t) - U(X_{t-1}, W_t)$ about the efficient level has no first-order terms.

Proof: See Appendix C.

The implication of this proof is that the welfare function cannot always be approximated as a constant plus quadratic terms, centred on the efficient rate, once the resource constraints have been incorporated. There are two conditions that must be satisfied for this approximation to be valid. Firstly the condition on \tilde{V} above needs to be checked; secondly the implementation constraints must incorporate a tax/subsidy rate such that their steady-state solution is characterized by zero inflation and the efficient level of output.

If the tax/subsidy rate is inconsistent with the above, then there is a distortion relative to the efficient case, and it is then necessary to assess whether this is a large or a small distortion.

3.4 The Small Distortion Case

Suppose that the tax/subsidy is insufficient to eliminate the inefficiency, but that the latter is small. There are then two approaches to obtain an approximation to the LQ approximation. The first is take deviations about the inefficient steady state. This will, as we have seen above, produce an approximation to the welfare that contains

a constant term (the steady-state welfare in the efficient case), and a quadratic term. The error in the approximation is then in the dynamic equation describing individual decisions. In this case, the efficient level Z^* is not consistent with the steady state because $\tau \neq \tau^*$, which means that the linearized approximation of the dynamic equation in for Z_t will contain a term $Z^* - g(X^*, Z^*, W^*; \tau)$; if this is small, it may be ignored.

The alternative is to take deviations about the natural rate, as done by Woodford (2003), Appendix E. Define the non-zero inflation natural rate as $(\bar{X}, \bar{Z}, \bar{W})$, which will be dependent on τ . The dynamic equations in deviation form then no longer contain a constant, but the linear terms in the welfare approximation (B.1) are now of the form:

$$\begin{aligned}
& (U_X(\bar{X}, \bar{W}) - \frac{1}{\beta}\lambda^{*T} + \lambda^{*T} f_X(\bar{X}, \bar{W}))x_{t-1} + (U_W(\bar{X}, \bar{W}) + \lambda^{*T} f_W(\bar{X}, \bar{W}))w_t \\
\cong & (H_X + (\bar{X} - X^*)^T H_{XX} + (\bar{W} - W^*)^T H_{WX})x_{t-1} \\
& + (H_W + (\bar{X} - X^*)^T H_{XW} + (\bar{W} - W^*)^T H_{WW})w_t \\
= & ((\bar{X} - X^*)^T H_{XX} + (\bar{W} - W^*)^T H_{WX})x_{t-1} \\
& + ((\bar{X} - X^*)^T H_{XW} + (\bar{W} - W^*)^T H_{WW})w_t
\end{aligned} \tag{14}$$

Thus the linear terms can be ignored provided that $\bar{X} - X^*$ and $\bar{W} - W^*$ are small.

To summarize then, there are two ways of assessing whether 'small distortions' are indeed small, and which relate directly to the necessary conditions examined in Section 2. This is done either by (1) evaluating the effect on the constant in the aggregated decentralized equations or (2) the effect on the first-order terms in the Hamiltonian. A further method of assessing the limitations of the small distortion case is discussed in the next section, by comparing the weights on the quadratic terms of the LQ welfare approximation for the efficient and the non-efficient case. This provides an arguably more direct assessment of the error in the approximation; this is because it is less easy to assess the impact of the errors described above.

4 LQ Approximation of Optimal Monetary Policy in a DSGE Model

We now turn to the main model of the paper and to optimal monetary policy. We initially investigate the large distortions approximation and towards the end of this section, we study both the efficient and the small distortions case.

The standard New Keynesian model ascribes a fixed probability in each period of changing prices (and wages). This leads to dynamic equations for the overall price index, and in turn this leads in the Woodford (2003) case to different choices of labour supply by individuals, and in the Clarida *et al.* (2002) case to each individual providing the same quantity of labour. In the former, the policymaker takes the average of the utility function, which for small variance of shocks is approximately the same as flexible-price level of the utility function, but with an additional effect from the spread of prices. In the latter, although labour supply is the same for each worker, it is dependent on the spread of demand for each good; this in turn leads to the utility function differing from the flexible price utility function by a term dependent on the spread of prices and wages.

The model is of a cashless economy with external habit in consumption. Consumers of type i maximize the intemporal trade-off between consumption C_{it} - taking into account a desire to consume at a level similar to that of last period's average consumption C_{t-1} - and leisure. The latter is accounted for by penalising working time N_{it} .

Unlike Clarida *et al.* (2000) we do not incorporate a proportional tax (or subsidy) into the model in order to ensure that the steady state, or natural rate, of output is at the efficient level. Instead we use the methodology of Section 2 to obtain a quadratic approximation to the welfare when the natural rate differs from the efficient rate. This is an issue also addressed by Benigno and Woodford (2005) using the less direct methods outlined in the example of Section 2.

We can summarize the model in a concise form as:

Household Utility:

$$\Omega_0 = E_0 \left[\sum_{t=0}^{\infty} \beta^t \left[\frac{(C_{it} - h_C C_{t-1})^{1-\sigma}}{1-\sigma} - \kappa \frac{N_{it}^{1+\phi}}{1+\phi} \right] \right] \quad (15)$$

Household Behaviour:

The first-order conditions for households are as follows:

$$1 = \beta E_t \left[D_{t,t+1} \left(\frac{C_{it+1} - h_C C_t}{C_{it} - h_C C_{t-1}} \right)^{-\sigma} \frac{1}{\Pi_{t+1}} \right] \quad (16)$$

$$\frac{W_{it}}{P_t} = \frac{\kappa}{(1 - \frac{1}{\eta})} N_{it}^{\phi} (C_{it} - h_C C_{t-1})^{\sigma} \quad (17)$$

where $D_{t,t+1}$ is the stochastic discount factor on holdings of one-period bonds, and

the gross inflation rate Π_t is given by

$$\Pi_t \equiv \frac{P_t}{P_{t-1}}$$

All consumers can trade in a complete set of state contingent bonds, and therefore engage in complete risk-sharing, so that (16) represents the Keynes-Ramsey intertemporal first-order condition for consumption across all consumers, taking habit into account. Equation (17) equates relative marginal utilities of consumption and leisure to the real wage. W_{it}, P_t are measures of the nominal wage of the i th agent and of price respectively. (17) also incorporates market power of individual consumers, who are all distinct from the point of view of production skills and face a demand curve

$$N_{it} = \left(\frac{W_{it}}{W_t} \right)^{-\eta} N_t$$

where W_t and N_t denote aggregates, $N_t \equiv \sum_i N_{it}^{\frac{\eta-1}{\eta}}$. Aggregate output Y_t is similarly defined by aggregating over all labour inputs.

There is market-clearing in wages, and in this set-up all agents set the same wage and work the same number of hours. Thus (17) holds when i is deleted, so for this setup there is no need to aggregate W_t, N_t .

Firms:

Unlike workers, firms only reset prices in any given period with probability $1 - \xi$. Thus the optimal price P_t^0 for any firm that sets its price at t must take into account any future periods during which the price remains unchanged.⁹

The first-order condition for profit-maximization for the j th firm over the duration of the optimal price not being reset takes into account the elasticity of substitution ζ between goods, which provides firms with monopolistic power. It is given by

$$P_t^0 E_t \left[\sum_{k=0}^{\infty} \xi^k D_{t,t+k} Y_{t+k}(j) \right] = \frac{\kappa}{(1 - 1/\zeta)} E_t \left[\sum_{k=0}^{\infty} \xi^k D_{t,t+k} P_{t+k} MC_{t+k} Y_{t+k}(j) \right]$$

where marginal cost is given by the real product wage $MC_t = \frac{W_t}{A_t P_t}$ and the stochastic

⁹It is easy to show that if there is planned indexation to the overall price index as well i.e. the future price at time $t + k$ is given by $P_t^0 (P_{t+k-1}/P_t)^\gamma$ then all the results presented here are the same when Π_t is replaced by Π_t/Π_{t-1}^γ .

discount factor $D_{t,t+k}$ is given by

$$D_{t,t+k} = \beta^k \left(\frac{C_{t+k} - h_C C_{t+k-1}}{C_t - h_C C_{t-1}} \right)^{-\sigma} \frac{P_t}{P_{t+k}}$$

The first-order condition (18) is cumbersome to manipulate. However it is possible to express this price-setting rule in terms of *difference equations* that are far easier to manipulate. To do this first note that

$$Y_{t+k}(j) = \left(\frac{P_t^0}{P_{t+k}} \right)^{-\zeta} Y_{t+k}$$

and multiplying both sides of (18) by $(\frac{P_t^0}{P_t})^\zeta (C_t - h_C C_{t-1})^{-\sigma}$ and in addition noting that $P_{t+k}/P_t = \Pi_{t+k} \dots \Pi_{t+1}$, the firms' staggered price setting can be succinctly described by

$$Q_t = \Lambda_t / H_t$$

where we have defined variables Q_t , H_t and Λ_t by

$$\begin{aligned} Q_t &\equiv P_t^0 / P_t \\ H_t - \xi \beta E_t[\Pi_{t+1}^{\zeta-1} H_{t+1}] &= Y_t (C_t - h_C C_{t-1})^{-\sigma} \\ \Lambda_t - \xi \beta E_t[\Pi_{t+1}^\zeta \Lambda_{t+1}] &= \frac{\kappa}{(1 - 1/\zeta)(1 - 1/\eta) A_t} Y_t N_t^\phi \end{aligned}$$

Price index inflation given by

$$1 = \xi \Pi_t^{\zeta-1} + (1 - \xi) Q_t^{1-\zeta}$$

The production function at the firm level is defined as

$$Y_t(j) = A_t N_t(j)$$

where A_t represents a common technology shock and $N_t(j)$ is an aggregate of differentiated labour chosen by firm j . Then aggregate output across firms, some of whom can re-optimize prices at time t , is given by $Y_t = A_t \sum_j N_t(j) = A_t N_t$.

4.1 Price Dispersion and the Costs of Inflation

Here we discuss the effects of inflation on the dispersion of prices due to firms' behaviour discussed above, and the implications for total employment. These dispersion effects will lead to costs of inflation, as we shall see later.

The impact of price dispersion arises from labour input being the same for each individual, but dependent on demand for each good:

$$N_t = \sum_j N_t(j) = \frac{Y_t}{A_t} \sum_j \frac{Y_t(j)}{Y_t} = \frac{Y_t}{A_t} \sum_j \left(\frac{P_t(j)}{P_t} \right)^{-\zeta}$$

Now define the effect of price dispersion on labour demand as $D_t = \sum_j \left(\frac{P_t(j)}{P_t} \right)^{-\zeta}$. Assuming that the number of firms is large, we can use the law of large numbers to obtain the following dynamic relationship:

$$D_t = \xi \Pi_t^\zeta D_{t-1} + (1 - \xi) Q_t^{-\zeta} \quad (18)$$

4.2 The Ramsey Problem

For simplicity, we ignore government spending, so that $Y_t = C_t$. As a consequence of the price dispersion result above, the deterministic Ramsey problem for a policy maker (with commitment) is characterized by choosing a trajectory for inflation to maximize

$$\Omega_0 = \sum_{t=0}^{\infty} \beta^t \left[\frac{(Y_t - Z_t)^{1-\sigma}}{1-\sigma} - \frac{\kappa}{1+\phi} N_t^{1+\phi} D_t^{1+\phi} \right] \quad (19)$$

subject to the constraints of the previous section. Note that the dynamics of the term D_t^ϕ contains only second-order terms, and therefore satisfies Theorem 2. Then had we included an optimal subsidy rate, we would have been able to expand the utility function about the efficient rate. We can now write the Lagrangian for the policymaker's optimal control problem as follows:

$$\begin{aligned} L = & \Omega_0 + \sum_{t=0}^{\infty} \beta^t [\lambda_{1t}(Z_{t+1} - h_C Y_t) + \lambda_{2t}(1 - \xi \Pi_t^{\zeta-1} - (1 - \xi) Q_t^{1-\zeta}) \\ & + \lambda_{3t}(Q_t H_t - \Lambda_t) + \lambda_{4t}(H_t - \xi \beta \Pi_{t+1}^{\zeta-1} H_{t+1} - Y_t (Y_t - Z_t)^{-\sigma}) \\ & + \lambda_{5t}(\Lambda_t - \xi \beta \Pi_{t+1}^\zeta \Lambda_{t+1} - \frac{\kappa}{\alpha} N_t^{1+\phi} D_t^\phi) + \lambda_{6t}(D_t - \xi \Pi_t^\zeta D_{t-1} - (1 - \xi) Q_t^{-\zeta})] \end{aligned}$$

where we define $\alpha = (1 - 1/\zeta)(1 - 1/\eta)$. We now obtain:

Result 1

The zero-inflation¹⁰ steady-state Ramsey values are given by

$$\begin{aligned} \Pi = Q = 1 \quad \Lambda = H = \frac{Y^{1-\sigma}(1-h_C)^{-\sigma}}{1-\beta\xi} \quad D = 1 \quad Y = AN \quad (1-h_C)^{-\sigma} = \frac{\kappa Y^{\phi+\sigma}}{\alpha A^{1+\phi}} \\ \lambda_5 = \frac{1-\beta h_C - \alpha}{\frac{\sigma(1-h_C\beta)}{1-h_C} + \phi} = -\lambda_4 \quad \lambda_3 = (1-\xi)\lambda_5 \quad \lambda_2 = \frac{H\lambda_5 + \zeta\lambda_6}{(1-\zeta)} \\ \lambda_6 = \frac{Y^{1-\sigma}(1-h_C)^{-\sigma} \left(\frac{\alpha\sigma}{1-h_C} + \phi \right) (1-h_C\beta)}{1-\beta\xi \frac{\sigma(1-h_C\beta)}{1-h_C} + \phi} \end{aligned} \quad (20)$$

Proof. See Appendix D.

Now that we have the steady-state values of the Lagrange multipliers, we are in a position to apply Theorem 1. We first linearize the relationships between the variables, and then obtain the quadratic approximation of the Lagrangian.

4.3 Linearization of Dynamics

We linearize about a zero-inflation steady state and later examine the second-order conditions for this to be appropriate for our LQ approximation. Define $h_t, \lambda_t, q_t, \pi_t$ as deviations of $H_t, \Lambda_t, Q_t, \Pi_t$ from their steady state values. In addition define $y_t = (Y_t - Y)/Y \approx \log Y_t/Y$, $a_t = (A_t - A)/A$ and define $z_t = (Z_t - Z)/Y$.¹¹

Linearization of the constraints yields

$$\begin{aligned} Hq_t = \lambda_t - h_t; \quad \xi\pi_t = (1-\xi)q_t; \quad z_{t+1} = h_C y_t \\ h_t - \beta\xi(\zeta - 1)H\pi_{t+1} - \beta\xi h_{t+1} = Y^{1-\sigma}(1-h_C)^{-\sigma} \left(y_t - \frac{\sigma}{1-h_C}(y_t - z_t) \right) \end{aligned} \quad (21)$$

$$\lambda_t - \beta\xi\zeta\Lambda\pi_{t+1} - \beta\xi\lambda_{t+1} = \frac{\kappa(1+\phi)Y^{1+\phi}}{\alpha A^{1+\phi}}(y_t - a_t) \quad (22)$$

Now subtract (21) from (22). Noting that $\Lambda = H$, and substituting from (21) yields a Phillips curve relationship of the form:

$$\pi_t = \beta\pi_{t+1} + \frac{(1-\xi)(1-\beta\xi)}{\xi} \left(\phi y_t + \frac{\sigma}{1-h_C}(y_t - z_t) - (1+\phi)a_t \right) \quad (23)$$

¹⁰Later, in section 5, we consider the conditions for the Ramsey problem to have a zero-inflation steady state.

¹¹We linearize and later quadratify in levels and proportional differences, rather than adopt log-linearization. There are subtle differences between these practices discussed in Fernandez-Villaverde and Rubio-Ramirez (2006).

Note that linearization of the dispersion term around zero inflation is irrelevant, since it reduces to $d_t - \xi d_{t-1} = 0$.¹² Also note that a_t can be a stochastic process turning the optimization problem into one that is stochastic.

4.4 The Accurate LQ Approximation

At this point we apply the result of Section 2, in order to obtain a quadratic approximation to the period t value of the Hamiltonian. Ignoring the steady state value of the latter, the remaining terms are given by:

$$\begin{aligned}
& -\frac{1}{2}(Y - Z)^{-\sigma-1}Y^2\sigma(y_t - z_t)^2 - \frac{1}{2}\kappa\phi N^{1+\phi}y_t^2 - \lambda_5\frac{\kappa}{2\alpha}\phi(1 + \phi)N^{1+\phi}y_t^2 \\
& + \kappa(1 + \phi)N^{1+\phi}y_t a_t + \lambda_5\frac{\kappa}{\alpha}(1 + \phi)^2N^{1+\phi}y_t a_t \\
& - \lambda_5\sigma Y^2(Y - Z)^{-\sigma-1}(y_t - z_t)y_t + \frac{1}{2}\lambda_5\sigma(\sigma + 1)Y^3(Y - Z)^{-\sigma-2}(y_t - z_t)^2 \\
& - \frac{\xi}{2}\pi_t^2((\zeta - 1)(\zeta - 2)\lambda_2\Pi^{\zeta-3} + (\zeta - 1)(\zeta - 2)\Pi^{\zeta-3}H\lambda_4 + \zeta\lambda_5(\zeta - 1)\Pi^{\zeta-2}\Lambda \\
& + \lambda_6\zeta(\zeta - 1)\Pi^{\zeta-2}D) - \xi\pi_t\lambda_t\zeta\lambda_5\Pi^{\zeta-1} - \xi\pi_t h_t(\zeta - 1)\lambda_4\Pi^{\zeta-2} \\
& + \frac{1}{2}q_t^2(\lambda_2(1 - \xi)(1 - \zeta)\zeta Q^{-1-\zeta} + \lambda_6(1 - \xi)(1 + \zeta)\zeta Q^{-2-\zeta}) + q_t h_t \lambda_3 \quad (24)
\end{aligned}$$

After eliminating h_t, λ_t, q_t using (21), and substituting the steady state values above, we finally arrive at the *correct quadratic approximation to the nonlinear Ramsey problem* as the maximization of $E_0[\sum_{t=0}^{\infty}\beta^t U_t]$ with respect to $\{\pi_t\}$, subject to (23) where

$$\begin{aligned}
U_t = & -\frac{\kappa}{2\alpha}N^{1+\phi}\left[\frac{\sigma}{1 - h_C}(y_t - h_C y_{t-1})^2 + \phi(\alpha + \lambda_5(1 + \phi))y_t^2 \right. \\
& - 2(1 + \phi)(\alpha + \lambda_5(1 + \phi))y_t a_t + 2\lambda_5\frac{\sigma}{1 - h_C}(y_t - h_C y_{t-1})y_t \\
& \left. - \lambda_5\frac{\sigma(\sigma + 1)}{(1 - h_C)^2}(y_t - h_C y_{t-1})^2 + \frac{\xi\zeta}{(1 - \xi)(1 - \beta\xi)}(\alpha + (1 + \phi)\lambda_5)\pi_t^2\right] \quad (25)
\end{aligned}$$

4.5 Summary of the General Procedure

We summarize the general Hamiltonian procedure by providing the following step-by-step recipe the practitioner should follow¹³:

¹²As Kim *et al.* (2006a) show, this feature follows from the particular choice of variables with respect to which we applied the Taylor series approximation. If we had chosen a different normalization and linearized with respect to $\sqrt{\log D_t/D}$ instead, we would have a bifurcation problem.

¹³MATLAB software to implement this procedure for both the one-country and two-country cases is available at: www.econ.surrey.ac.uk/people/rpierse/papers.html. Our one-country code is similar to that reported in Altissimo *et al.* (2005).

1. Set out the deterministic nonlinear problem for the Ramsey Problem, to maximize the representative agents utility subject to nonlinear dynamic constraints.
2. Write down the Lagrangian for the problem.
3. Calculate the first order conditions. We do not require the initial conditions for an optimum since we ultimately only need the steady-state of the Ramsey problem.
4. Calculate the steady state of the first-order conditions. The terminal condition implied by this procedure is that the system converges to this steady state.
5. Calculate a second-order Taylor series approximation, about the steady state, of the Hamiltonian associated with the Lagrangian in 2.
6. Calculate a first-order Taylor series approximation, about the steady state, of the first-order conditions and the original constraints.
7. Use 4. to eliminate the steady-state Lagrangian multipliers in 5. By appropriate elimination both the Hamiltonian and the constraints can be expressed in minimal form. This then gives us the accurate LQ approximation of the original nonlinear optimization problem in the form of a minimal linear state-space representation of the constraints and a quadratic form of the utility expressed in terms of the states.

4.6 The Social Planner's Problem

The social planner can be regarded as maximizing (15) viewing all agents as identical, and so can set $C_{it} = C_t, N_{it} = N_t$, subject to the constraint $C_t = Y_t = A_t N_t$. The social planner chooses a trajectory for output which satisfies the first-order condition

$$[C_t - h_C C_{t-1}]^{-\sigma} - h_C \beta [C_{t+1} - h_C C_t]^{-\sigma} = \kappa \frac{Y_t^\phi}{A_t^{1+\phi}} \quad (26)$$

The efficient steady-state level of output $Y_{t+1} = Y_t = Y_{t-1} = Y^e$, say, is therefore given by

$$(Y^e)^{\phi+\sigma} = \frac{(1 - h_C \beta) A^{1+\phi}}{\kappa (1 - h_C)^\sigma} \quad (27)$$

We can now examine the inefficiency of the zero-inflation steady state. From the

first-order conditions for the real wage and prices this is given by $Y = \bar{Y}$ where

$$(\bar{Y})^{\phi+\sigma} = \frac{\left(1 - \frac{1}{\zeta}\right) \left(1 - \frac{1}{\eta}\right) A^{1+\phi}}{\kappa(1 - h_C)^\sigma} \quad (28)$$

It is easy to check that this is exactly the same steady-state level as that of the flexi-price economy where firms set prices optimally at every period. Comparing (27) and (28) we have the result first obtained by Choudhary and Levine (2006):

Result 2

The natural level of output, \bar{Y} , is below the efficient level, Y^e , if and only if

$$\alpha \equiv \left(1 - \frac{1}{\zeta}\right) \left(1 - \frac{1}{\eta}\right) < 1 - h_C\beta \quad (29)$$

In the case where there is no habit persistence in consumption, $h_C = 0$, then (29) always holds. In this case market power in the output and labour markets captured by the elasticities η, ζ respectively drive the natural rate of output below the efficient level. If habit persistence in consumption is sufficiently high, then (29) does not hold and the natural rate of output and employment are then too high compared with the efficient outcome and people are working too much. Why is this? In the efficient case, there is an incentive for the social planner to raise C_t relative to $h_C C_{t-1}$, but also a disincentive to raise C_t because of its effect on welfare in the next period. For decentralized consumers, there is no disincentive effect because each will ignore the effect of its current raised consumption in the next period. The greater is h_C the greater is the effect of the disincentive on the social planner.

Is there empirical support that (29) holds? Terms $\left(1 - \frac{1}{\zeta}\right)$ and $\left(1 - \frac{1}{\eta}\right)$ are the inverses of mark-ups over marginal costs in the output and labour markets respectively. A plausible upper bound on these mark-ups is 20% so $\alpha = \left(1 - \frac{1}{\zeta}\right) \left(1 - \frac{1}{\eta}\right) > \frac{1}{1.2^2}$. A condition on h_C for (29) to hold is therefore $h_C\beta < 0.306$. Most empirical estimates of habit in a quarterly model are in the range $h_C = [0.5, 0.9]$ which would see this condition not holding.¹⁴

¹⁴If we were to add a tax wedge T , then (29) becomes $(1 - T) \left(1 - \frac{1}{\zeta}\right) \left(1 - \frac{1}{\eta}\right) < 1 - h_C\beta$ which suggests that the condition would hold for a tax wedge $T > 0.5$ as observed in the euro-area. See Levine *et al.* (2007b).

4.7 The Efficient Case

For this particular example, we are able to illustrate Theorem 4 without needing directly to invoke a tax/subsidy and add a further algebraic burden. Suppose instead that the effect of habit is to directly offset the effect of the distortions due to η, ζ i.e. $1 - \alpha - \beta h_C = 0$, so that the value of output in the Ramsey problem (assuming zero inflation) is equal to the efficient level. We need to check that the sufficient condition on \tilde{V} of Theorem 4 holds in this case. One can see by inspection, that to first order about the efficient level of output and zero inflation, we have the following expansion:

$$\sum_{t=0}^{\infty} \beta^t \tilde{V}_t = \sum_{t=0}^{\infty} -\beta^t \frac{\kappa}{1+\phi} N_t^{1+\phi} (D_t^{1+\phi} - 1) \cong \sum_{t=0}^{\infty} -\beta^t \kappa N^{1+\phi} d_t \quad (30)$$

But to first order, it is easy to see that $d_t = \xi d_{t-1}$, so starting at $d_0 = 0$, it is clear that to first order, the value of (30) is zero.

It is now of interest to verify that expansion to second order in this case yields the same expansion as for the large distortions case. We first note that the second order expansion of D_t is given by

$$d_t = \xi d_{t-1} + \frac{1}{2} \frac{\xi \zeta}{1 - \xi} \pi_t^2 \quad (31)$$

from which it follows that in a second order expansion we can ignore d_t^2 . Furthermore, using (30) and (31), it follows that to second order about the efficient solution, we have

$$\begin{aligned} \sum_{t=0}^{\infty} \beta^t \left[\frac{(Y_t - Z_t)^{1-\sigma}}{1-\sigma} - \frac{\kappa}{1+\phi} N_t^{1+\phi} D_t^{1+\phi} \right] \cong \\ -\frac{1}{2} \sum_{t=0}^{\infty} \beta^t \left[\frac{\sigma (Y^e)^{1-\sigma}}{(1-h_C)^{1+\sigma}} (y_t - z_t)^2 + \kappa \phi (N^e)^{1+\phi} y_t^2 + \frac{\kappa \xi \zeta}{(1-\beta\xi)(1-\xi)} (N^e)^{1+\phi} \pi_t^2 \right] \end{aligned} \quad (32)$$

We now need to compare this with the expansion in the large distortions case, given by (25). Using (27) in (32), and using the fact that $\lambda_5 = 0$ (because $1 - \alpha - \beta h_C = 0$) in (25) for this case, it is easy to show that the quadratic expansions are identical.

4.8 The Small Distortion Case

The small distortion case assumes that the zero-inflation steady state about which we have linearized is approximately efficient. From result 2 this implies that $1 - \beta h_C - \alpha$ is small. We are now in a position to examine the nature of this ‘approximation to an approximation’ by examining the correctly quadratified single-period utility (25). From (20) we can see that this means that λ_5 must be small. An examination of (25) reveals that the small distortion case, which would omit all terms involving λ_5 , is valid only if $|\lambda_5(1 + \phi)| \ll \alpha$ or, using the definition of λ_5 , only if

$$(1 + \phi) \frac{|1 - \beta h_C - \alpha|}{\frac{\sigma(1 - h_C \beta)}{1 - h_C} + \phi} \ll \alpha \quad (33)$$

Typical estimated parameter values are $\sigma = 3$ (with this value or higher being confirmed within other contexts as well), $\phi = 1.3$. With h_C at the mid-point of the range of estimates at $h_C = 0.7$ this gives the left-hand-side of (33) as 0.22 and the right-hand side as 0.69. Neglected terms are therefore of the order of one third of those retained.

4.9 Second-Order Conditions and Target Implementability

We now return to issues concerning second-order conditions in Theorems 2 and 3. We first relate Theorem 2 to the concept of targeting rules. We then proceed to a numerical example where the zero-inflation steady state fails to satisfy conditions in Theorem 3. Thus the Ramsey steady state gives a non-zero inflation outcome.

Svensson (2003, 2005) suggests that real-world monetary policy is best viewed in terms of a “prescribed guide for monetary policy”. These would include “targeting rules” and “instrument rules”. The latter could consist of Taylor-type that prescribe the commitment of the monetary authority to change the nominal interest rate in response to changes in target macro-economic variables such as the output gap and past, current or expected future inflation rates. However on both normative and positive grounds he strongly argues for the former.¹⁵ A general targeting rule would specify the objectives to be achieved by for example setting out target variables, their targets and a loss function to be minimized. In the context of our quadratic approximations we can interpret these targets as ‘bliss points’, provided that the period t quadratic approximation achieves a maximum at these. This leads

¹⁵This paper does not engage with the targeting versus instrument rules debate (but see, for example, McCallum and Nelson (2004)).

to a particular form of targeting rule in the Svensson sense that we call ‘target-implementability’:

Definition: A period- t welfare function is **target-implementable** iff, in the vicinity of the steady state of the first-order conditions for a maximum, it can be written as a weighted sum of squares of linear terms, with all weights negative; that is a sum of the squares of deviations of target variables from their bliss points.

From Theorem 2 and the fact that a symmetric positive definite matrix Q can be written as $Q = X\Lambda X'$ where X is a matrix of eigenvectors and Λ is a diagonal matrix of real positive eigenvalues we now have:

Result 3

A necessary and sufficient condition for the solution to the Ramsey problem to be target-implementable is that, in the vicinity of the steady state of the first-order conditions for a maximum, the quadratic approximation to the Lagrangian is negative semi-definite.

Now consider the target-implementability of the welfare function in our LQ approximation to the Ramsey problem for our model. First consider the case without habit ($h_C = 0$). After some further effort (and subtracting an appropriate term in a_t^2), (25) then reduces to

$$\frac{-\kappa N^{1+\phi}}{2\alpha}(\phi + \sigma\alpha + 1 - \alpha) \left[\left(y_t - \frac{1+\phi}{\sigma+\phi} a_t \right)^2 + \frac{\zeta\xi}{(1-\xi)(1-\beta\xi)(\sigma+\phi)} \pi_t^2 \right] \quad (34)$$

This is clearly target-implementable with a stochastic output target $\frac{1+\phi}{\sigma+\phi} a_t$ and inflation target of zero (the steady state about which we have formed the LQ approximation). Since from Theorem 3 and result 2, the condition for target-implementability is a sufficient condition for the first-order conditions to define a local maximum, we can now also confirm that the zero inflation steady state (that we found to satisfy these first-order conditions for $h_C \geq 0$) is indeed appropriate. Our stochastic target is of course the flexi-price output so that (34) turns out to be micro-founded loss function popularized by Woodford (2003) that penalizes deviations of the output gap and inflation from zero.

Now consider the case $h_C > 0$. As for $h_C = 0$ we now need to demonstrate whether the natural rate as calculated, with zero inflation, is actually the steady state for the Ramsey problem. To check this, we need either to solve the correspond-

ing Riccati equation or to check the sufficient conditions of target-implementability. If the sufficient conditions of the latter are not satisfied, then checking the steady state Riccati matrix will not yield analytic results. This is because the equation governing it is highly nonlinear, and in addition the matrix is of dimension 2, so analytic solutions will not in general be found.

We therefore focus on target-implementability, and determine what conditions on the underlying parameters are required for (25) to be negative semi-definite. By inspecting this approximation we can see that apart from completing the square for the disturbance term a_t , it will be negative semi-definite provided that the terms in y_t and $(y_t - z_t)$ are negative semi-definite. This is equivalent to the requirement that we can write these terms as a weighted sum of y_t^2 and $(y_t - \nu z_t)^2$, for some ν .

In order to reduce the algebraic burden for checking target-implementability, we make the relatively innocuous approximation $\beta = 1$, since most quarterly models would assume a value of the order of 0.99. It turns out that even then, the conditions for target-implementability are rather messy, so instead we focus on a set of empirically innocuous sufficient conditions on the parameters.

Proposition

- (i) When (29) does not hold so that the natural rate is above the efficient rate, sufficient conditions for the Ramsey problem with habit in consumption to have a target-implementable zero inflation steady state are that $\sigma > 1$ and $\phi\sigma^2 > \phi + \sigma$.
- (ii) When (29) holds so that the natural rate is below the efficient rate, sufficient conditions for the zero-inflation steady state to be target-implementable are that $\alpha(1 + \sigma) > (1 - h_C)(1 - \phi)$ and $\phi^3 > \phi + \sigma$.

Proof: See Appendix E.

We note that for either case (i) or (ii), the single-period welfare loss may be rewritten

$$\begin{aligned}
& -\frac{\kappa}{2\alpha} N^{1+\phi} \left[a_1 (y_t - h_C y_{t-1} + \frac{a_2}{a_1} y_t)^2 + (a_3 - \frac{a_2^2}{a_1}) y_t^2 - 2(1 + \phi)(\alpha + \lambda_5(1 + \phi)) y_t a_t \right. \\
& \left. + \frac{\xi\zeta}{(1 - \xi)(1 - \beta\xi)} (\alpha + (1 + \phi)\lambda_5) \pi_t^2 \right] \\
& a_1 = \frac{\sigma}{1 - h_C} - \lambda_5 \frac{\sigma(\sigma + 1)}{(1 - h_C)^2} \quad a_2 = \lambda_5 \frac{\sigma}{1 - h_C} \quad a_3 = \phi(\alpha + \lambda_5(1 + \phi)) \quad (35)
\end{aligned}$$

with $a_1 > 0$, $a_3 - \frac{a_2^2}{a_1} > 0$. Thus at each period there is a bliss-point for inflation of 0, a bliss-point for output of $(1 + \phi)(\alpha + \lambda_5(1 + \phi)) / (a_3 - \frac{a_2^2}{a_1}) a_t$, and a bliss-point for

output growth $y_t - y_{t-1}$ in terms of last period's output given by $-\frac{(1+\frac{a_2}{a_1}-h_C)}{(1+\frac{a_2}{a_1})}y_{t-1}$.¹⁶ In the special case $h_C = 0$, there is no bliss-point for output growth, the coefficient on y_t^2 is $a_1 + 2a_2 + a_3$, so that the bliss-point for output is $\frac{1+\phi}{\sigma+\phi}a_t$ as in (34).

Using typical estimated parameter values discussed above, both of the sufficient conditions (i) and (ii) are easily satisfied. Necessary conditions are much more difficult to derive, as there is a wide range of parameter values for which the sufficient conditions of Theorem 3 are satisfied, even though the quadratic approximation is not negative definite. The following set of theoretically possible parameters however, yields a situation where the sufficient conditions of Theorem 3 are violated for some values of t : $\sigma < 0.6$, $\phi = 0.03$, $h_C = 0.75$, $\xi = 0.7$ and $\zeta = 5$, $\eta = 16$ (implying $\alpha = 0.75$). Thus we have the following:

Result 4: For the problem with habit, there are possible configurations of parameter values such that the solution to the Ramsey problem does not have a zero inflation steady-state equilibrium.

Although $\sigma < 0.6$ lies outside most, but not all estimates, this example is interesting because it serves as an example of why second-order conditions, routinely bypassed in the optimal dynamic policy literature, do matter. Benigno and Woodford (2007) discuss a case where randomized policy may be superior in a non-inflationary steady state. The conditions they derive for this correspond precisely to those which guarantee that the steady-state solution to the corresponding Riccati equation is either negative definite, or in which the negative definite solution is unstable. Only by implication do they suggest that the steady-state zero-inflation Ramsey solution must therefore be completely invalid.

5 Numerical Illustration

In this section we compare the exact solution to the Ramsey problem with two LQ approximations. The first of the latter is the accurate large approximation with the quadratic approximation given by (25). In the small distortions approximation we impose $\alpha \equiv (1 - \frac{1}{\zeta})(1 - \frac{1}{\eta}) = 1 - \beta h_C$ in (25) so that $\lambda_3 = \lambda_4 = \lambda_5 = 0$.

We consider three combinations of parameters: $\eta = \zeta = 6$ and $h_C = 0.6$; $\eta = \zeta = 3$ and $h_C = 0.1$; and $\eta = \zeta = 6$ and $h_C = 0.8$. Other parameter values are

¹⁶Note that $a_2 \geq 0$ iff $\lambda_5 \geq 0$. $\lambda_5 = 0$ for the efficient case; $\lambda_5 > 0$ if h_C is small and $\lambda_5 < 0$ as $h_C \rightarrow 1$.

$\sigma = 2$, $\phi = 1.7$ and $\beta = 0.99$. For these combinations from (27) and (28) we have that $\frac{Y}{Y_e} = 1.16$, 0.83 and 1.39 respectively. In the first case the Ramsey steady-state output is 16% above the efficient level, in the second 17% below and in the third 39% above. Thus we have two cases of moderate distortions in the steady state with output either too high or too low compared with the efficient level and one case of a very large steady state distortion.

Figures 1 and 2 compare the two approximations with the exact nonlinear solution to the Ramsey problem when the economy is subjected to a transient 1% technology shock at time $t = 1$. In the two cases of a moderate distortion, the accurate large distortions LQ approximation is remarkably close to the exact solution, whereas the small distortions approximation is substantially different for inflation, but reasonably close for output. For the very large distortion, the large distortions approximation is still accurate, but the small distortions approximation is seriously out for both inflation and output. Thus our simulations provide a clear numerical demonstration of the pitfalls of incorrect LQ approximation.¹⁷

6 Cooperation and Non-Cooperation in a Two-Bloc Model

We now investigate a two bloc version of our previous model as in Clarida *et al.* (2002). Their model does not have habit in consumption and here we retain this feature to keep the analysis tractable. The two blocs are of different sizes, in the ratio $(1 - \gamma) : \gamma$. Consumption preferences are Cobb-Douglas, and once we take the risk-sharing relationship into account, this results in terms-of-trade, S_t , defined as the relative price of imports to exports, being equal to relative output; i.e., $S_t = \frac{Y_t}{Y_t^*}$ where * denotes the foreign bloc. The resource constraint in the home bloc is then $C_t = kY_t S_t^{-\gamma}$ where $k = \gamma^\gamma(1 - \gamma)^{1-\gamma}$. By acting as a national monopolist the home planner can reduce home output, appreciate the terms of trade (i.e., reduce S_t) and increase domestic consumption. Analogous features apply to the foreign bloc.

¹⁷The solution procedure for the LQ Ramsey problem is described in Currie and Levine (1993). A comparison of alternative methods is to be found in Anderson *et al.* (1996). The solution procedure for the nonlinear Ramsey problem solves the nonlinear deterministic first order conditions in Appendix D using WinSolve (Pierse (2001)), a general nonlinear model solution program

6.1 The National Planners' and Social Planner's Problem

As for the single policymaker case we only need to consider the deterministic optimization problem. First consider a *non-cooperative game* between two national planners. In the absence of habit there is no structural dynamics so in each period the home planner maximizes

$$U_t = \frac{C_t^{1-\sigma}}{1-\sigma} - \frac{\kappa}{1+\phi} N_t^{1+\phi} \quad s.t. \quad C_t = k Y_t^{1-\gamma} Y^{*\gamma} \quad (36)$$

with respect to Y_t , with a corresponding problem for the foreign planner. It is easy to show that the non-cooperative steady-state solution is

$$\bar{Y}_{NC}^{\sigma+\phi} = \frac{A^{1+\phi}}{\kappa} (1-\gamma) k^{1-\sigma} \left(\frac{\gamma}{1-\gamma} \right)^{\frac{\gamma(1-\sigma)}{1+\phi}} \quad \bar{Y}_{NC}^{*\sigma+\phi} = \frac{A^{1+\phi}}{\kappa} \gamma k^{1-\sigma} \left(\frac{1-\gamma}{\gamma} \right)^{\frac{(1-\gamma)(1-\sigma)}{1+\phi}} \quad (37)$$

assuming $A = A^*$ in the steady state. By analogy with the single country case, we regard these levels of output as the nationally efficient levels under non-cooperation.

We now expand the single period utility in the home bloc as a Taylor series expansion up to second order terms about \bar{Y}_{NC} to obtain

$$\begin{aligned} & \frac{(k Y_t^{1-\gamma} Y_t^{*\gamma})^{1-\sigma}}{1-\sigma} - \frac{\kappa}{1+\phi} N_t^{1+\phi} \cong \frac{\Gamma_{NC}}{1-\sigma} - \frac{\kappa}{1+\phi} \bar{N}_{NC}^{1+\phi} \\ & + \left[(1-\gamma)\Gamma_{NC} - \kappa \bar{N}_{NC}^{1+\phi} \right] y_t + \gamma \Gamma_{NC} y_t^* \\ & - \frac{1}{2} \Gamma_{NC} \left[(1-\gamma)[(\gamma + \phi + (1-\gamma)\sigma)y_t^2] + 2\gamma(1-\gamma)(\sigma-1)y_t y_t^* + \gamma(1-\gamma + \gamma\sigma)y_t^{*2} \right. \\ & \left. - 2(1-\gamma)(1+\phi)y_t a_t + (\text{t.i.p})^{NC} \right] \end{aligned}$$

where $\Gamma_{NC} = (k \bar{Y}_{NC}^{1-\gamma} \bar{Y}_{NC}^{*\gamma})^{1-\sigma}$ and terms independent of policy (t.i.p.) consist of a term in a_t . Hence, using (37), we can write the deviation of utility about the steady state as

$$\begin{aligned} w_t^{NC} \equiv U_t^{NC} - \bar{U}_t^{NC} \cong & -\frac{1}{2} \Gamma_{NC} \left[(1-\gamma)(\gamma + \phi + (1-\gamma)\sigma)y_t^2 + 2\gamma(1-\gamma)(\sigma-1)y_t y_t^* \right. \\ & \left. + \gamma(1-\gamma + \gamma\sigma)y_t^{*2} - 2(1-\gamma)(1+\phi)y_t a_t - 2\gamma y_t^* \right] + (\text{t.i.p})^{NC} \end{aligned} \quad (38)$$

This result is striking; the use of the tax rate to shift the system to the efficient rate for the home country eliminates the first-order term in y_t , but does not eliminate

the term in y_t^* . The converse is of course true for the foreign LQ approximation. For an LQ approximation of a Nash game between national planners this is not a problem because for the home country, the terms in y^* , y^{*2} are the instruments of the foreign policymaker. Then (38) can be written as a quadratic expression in y_t , y_t^* and a_t plus terms independent of policy for the home planner to include the terms in y^* , y^{*2} . However for the Ramsey planner considered, in the next section, if we choose instruments normally associated with monetary policy, y_t^* *cannot* be included in t.i.p. and the condition for the LQ approximation for the efficient case set out in Theorem 4 does not hold.

So far our discussion has assumed that both blocs are large and interact in a strategic fashion. However if we consider the limit as the home country becomes very small in relation to the foreign bloc, then the small country can take terms in y^* as exogenously given even though the monetary instruments are not taken as output. The foreign bloc then regards itself as closed and the choice of instrument becomes irrelevant.¹⁸ For this particular set-up the efficient case approximation for the non-cooperative Ramsey game does apply.

Turning to the *cooperative* social planner's problem this is to maximize $(1 - \gamma)U_t + \gamma U_t^*$ subject to $C_t = C_t^* = kY_t^{1-\gamma}Y_t^{*\gamma}$. Again it is straightforward to show that the cooperative steady state solution is

$$\bar{Y}_C^{\sigma+\phi} = \bar{Y}_C^{*\sigma+\phi} = \frac{A^{1+\phi}}{\kappa} k^{1-\sigma} \quad (39)$$

and the cooperative quadratic approximation of the utility is given by

$$\begin{aligned} u_t^C &= -\frac{1}{2}\Gamma_C \left[(1-\gamma)(\gamma+\phi+(1-\gamma)\sigma)y_t^2 + 2\gamma(1-\gamma)(\sigma-1)y_t y_t^* \right. \\ &\quad \left. + \gamma(1-\gamma+\phi+\gamma\sigma)y_t^{*2} - 2(1+\phi)((1-\gamma)y_t a_t + \gamma y_t^* a_t^*) \right] \end{aligned} \quad (40)$$

where $\Gamma_C = \bar{Y}_C^{1-\gamma}\bar{Y}_C^{*\gamma})^{1-\sigma}$. Now the Taylor series expansion has successfully eliminate the linear terms in y_t and y_t^* and we have a suitable quadratic approximation of the utility. Two other features of (40) are worth noting: first $u_t^C \neq u_t^{NC}$ so there are some gains from cooperation. This may seem a trivial point, but later we show that in fact for this model with only technology shocks, an accurate large distortions approximation of the Ramsey problem with inflation rates as instruments results in $u_t^C = u_t^{NC}$ and subsequently no coordination gains. Second the quadratic approxi-

¹⁸See Batini *et al.* (2007) for this method of modelling a small open economy interacting with the rest of the world in a way that consistently treats the latter.

mation of the cooperative utility is not a simple size-weighted average of u_t^{NC} and u_t^{*NC} , as is often assumed in the literature.

6.2 Ramsey Planners

Ramsey planners in both non-cooperative and cooperative games have the same objectives as their social planning counterpart, but without the ability to plan consumption and labour supply paths. Instead they face a decentralized economy given by resource constraints, the market-sharing condition (which amounts to $C_t = C_t^*$ for this model) plus the price-setting behaviour of firms and the households' Euler equations. For the home bloc, the constraints therefore become

$$\begin{aligned} C_t = C_t^* &= kY_t^{1-\gamma}Y_t^{*\gamma}; & D_t &= \xi\Pi_t^\zeta D_{t-1} + (1-\xi)Q_t^{-\zeta} \\ 1 &= \xi\Pi_t^{\zeta-1} + (1-\xi)Q_t^{1-\zeta}; & H_t - \beta\xi[\Pi_{t+1}^{\zeta-1}H_{t+1}] &= C_t^{-\sigma}Y_t\left(\frac{Y_t^*}{Y_t}\right)^\gamma \\ Q_t H_t &= \Lambda_t; & \Lambda_t - \beta\xi[\Pi_{t+1}^\zeta\Lambda_{t+1}] &= \kappa\frac{1-\tau}{k\alpha}N_t^{1+\phi}D_t^\phi \end{aligned} \quad (41)$$

with analogous constraints for the foreign bloc. Note that Π_t in (41) is domestic price inflation. The Ramsey problem for the home country is then

$$\max \sum_{t=0}^{\infty} \beta^t \left[\frac{C_t^{1-\sigma}}{1-\sigma} - \frac{\kappa}{1+\phi} N_t^{1+\phi} D_t^{1+\phi} \right] \quad (42)$$

subject to the constraints (41), and analogous foreign constraints.

Much of the literature now follows Clarida *et al.* (2002) (see, for example, Pappa (2004)) and assumes that the Ramsey planners can use subsidy instruments, τ_t and τ_t^* to bring the zero-inflation steady state of the decentralized economies in line with the social optima. For the non-cooperative and cooperative games this require respectively, subsidies given by

$$\begin{aligned} (1 - 1/\eta)(1 - 1/\zeta) &= (1 - \tau^{NC})(1 - \gamma) = (1 - \tau^{*NC})\gamma \\ (1 - 1/\eta)(1 - 1/\zeta) &= (1 - \tau^C) = (1 - \tau^{*C}) \end{aligned} \quad (43)$$

The nature of the game is a two-stage process which we refer to as the *two-stage Ramsey game*. At stage 1 tax wedges are chosen so as to bring the steady state of the decentralized economy in line with the socially optimal allocation. In a non-cooperative game, each social planner's choice of consumption and leisure is a best response to the choice of the other; i.e., a Nash equilibrium in the individual blocs'

social optima. In the quadratic approximation (38), terms independent of policy (t.i.p), involve outcomes in the other bloc and are t.i.p only for the first stage of the game between national planners.

In the second stage the monetary instruments are used to achieve, as far as possible, the outcome of the first stage, but now there is staggered price-setting, inflation and therefore costs of inflation from price dispersion. The approximations to the Ramsey single-period utility functions are then given by (38) and (40) in the non-cooperative and cooperative cases respectively, with additional terms proportional to $\frac{\zeta\xi}{(1-\xi)(1-\beta\xi)}\pi_t^2$, as in the single country case discussed earlier.

The Nash equilibrium at this stage depends on the monetary instrument; in fact in Clarida *et al.* (2002) after expressing the LQ problem in terms of output gaps, the latter are the chosen instruments.¹⁹ This has the advantage that the efficient case LQ approximation set out above is valid. But monetary instruments are normally a choice from monetary growth paths; inflation targets with nominal interest rates subsequently chosen to exactly achieve these targets; or the nominal interest rates themselves. Furthermore, the Nash equilibria in nominal interest rates can be open-loop with the authorities responding to each other's interest rate paths. More appropriate in a stochastic environment with commitment are closed-loop Nash equilibria with each authority choosing their best response to each other's feedback commitment rule.

For any of these monetary policy games, the efficient case approximation is invalid for the non-cooperative equilibrium unless both Ramsey planners in the second stage were to be instructed by the national planners to adopt objectives that correspond to *their* quadratic approximation taking the other bloc's output as given. This game could be seen as an instrument-independent, but not goal-independent central bank instructed by a fiscal authority choosing subsidies. To summarize:

Theorem 5

The efficient case approximation for the two-stage Ramsey non-cooperative game only applies if at least one of the following conditions holds:

1. One bloc is very small.
2. Output (or the output gap) is the chosen instrument at stage two.
3. If a conventional monetary instrument such as inflation is chosen, then the

¹⁹This was not an issue for the single policymaker, but does arise in non-cooperative games, as the well-known Bertrand versus Cournot Nash equilibria in the oligopoly game clearly illustrates.

Ramsey planner at stage 2 aims to mimic the national planner and maximize her quadratic approximation of the household's utility function.

6.2.1 Small Distortions Approximation

Under what conditions does the small distortions approximations where subsidies are not available hold? In addition to the conditions in Theorem 5 we require that the distortions in the economy including taxes and subsidies, now outside the control of the Ramsey planner, are such that the latter happens to be close to τ^{NC} and τ^C for the non-cooperative and cooperative games respectively. But since $\tau^C > \tau^{NC}$ then the subsidy is lower than that for cooperation and (38) is *not* an apparent good 'small distortions' approximation for that game. A similar argument holds if the fiscal authorities have set their tax wedges close to τ^C . In short, u_t^{NC} and u_t^C cannot both be good 'small distortions' approximations in single-stage Ramsey game. Thus we have

Theorem 6

The small distortions approximations cannot be used to compare the non-cooperative and cooperative outcomes.

6.3 LQ Approximation using The Hamiltonian Approach

Suppose neither condition in Theorem 5 applies, or subsidies are not available to eliminate the distortions in the zero-inflation steady state. We are then left with a monetary authority, the Ramsey planner, with a conventional monetary instrument that is not output, attempting to stabilize a decentralized economy that has a large distortion in one of the cooperative or non-cooperative equilibria.

We now apply the Hamiltonian approach to obtain an LQ approximation for both cooperation and non-cooperation that is not subject to the problems highlighted in Theorem 5. Before doing so, we need to carefully choose the appropriate solution concept for the equilibrium of the noncooperative game. Firstly there is the issue of which instrument to use; here we have the choice of the inflation rate or the output gap, each one acting as a proxy for the true instrument, which is the interest rate. We choose the inflation rate, following Benigno and Benigno (2006). Secondly, there is the choice of equilibrium concept given the instrument, which we choose to be *open-loop Nash*. This means that the home country chooses its inflation rate subject to the set of future inflation rates chosen by the foreign country. This is

ideally suited to obtaining an LQ approximation using the Hamiltonian approach, but it is not the only Nash solution. The alternative is *closed-loop Nash*, for which the sequence of foreign inflation rates is known to be dependent on the other system variables, such as output, and this is taken into account by the home policymaker when setting its inflation rate. However the latter concept has a solution that can only be obtained numerically even in the LQ case and, as far as we are aware, has not been characterized for nonlinear problems. For reasons of tractability therefore, we therefore utilize the open-loop Nash concept. The model is as before but for reasons that will become we also include a mark-up shock M_t , with expected value equal to 1.

The Ramsey problem for the home country (with a corresponding foreign problem) then can be described as one of finding the stationary points of the Lagrangian

$$\begin{aligned}
L = \sum_{t=0}^{\infty} \beta^t & \left[\frac{(kY_t^{1-\gamma}Y_t^{*\gamma})^{1-\sigma}}{1-\sigma} - \kappa \frac{N_t^{1+\phi}D_t^{1+\phi}}{(1+\phi)} + \lambda_{1t}(1 - \xi\Pi_t^{\zeta-1} - (1-\xi)Q_t^{1-\zeta}) \right. \\
& + \lambda_{2t}(Q_tH_t - \Lambda_t) + \lambda_{3t}(H_t - \beta\xi\Pi_{t+1}^{\zeta-1}H_{t+1} - k^{-\sigma}(Y_t^{1-\gamma}Y_t^{*\gamma})^{1-\sigma}) \\
& + \lambda_{4t}(\Lambda_t - \beta\xi\Pi_{t+1}^{\zeta}\Lambda_{t+1} - \frac{N_t^{1+\phi}D_t^{\phi}}{k\alpha}M_t) + \lambda_{5t}(D_t - \xi\Pi_t^{\zeta}D_{t-1} - (1-\xi)Q_t^{-\zeta}) \\
& + \lambda_{1t}^*(1 - \xi\Pi_t^{*\zeta-1} - (1-\xi)Q_t^{*1-\zeta}) \\
& + \lambda_{2t}^*(Q_t^*H_t^* - \Lambda_t^*) + \lambda_{3t}^*(H_t^* - \beta\xi\Pi_{t+1}^{*\zeta-1}H_{t+1}^* - k^{-\sigma}(Y_t^{1-\gamma}Y_t^{*\gamma})^{1-\sigma}) \\
& \left. + \lambda_{4t}^*(\Lambda_t^* - \beta\xi\Pi_{t+1}^{*\zeta}\Lambda_{t+1}^* - \kappa \frac{N_t^{*1+\phi}D_t^{*\phi}}{k\alpha}M_t^*) + \lambda_{5t}^*(D_t^* - \xi\Pi_t^{*\zeta}D_{t-1}^* - (1-\xi)Q_t^{*-\zeta}) \right]
\end{aligned} \tag{44}$$

with respect to all variables other than Π_t^* . From the first-order conditions set out in Appendix F we obtain:

Result 5

The steady state of the Ramsey problem for the home country is given by

$$\begin{aligned}
\Pi = Q = D = 1 = \Pi^* = Q^* = D^* \quad \Lambda = H = \kappa \frac{N^{1+\phi}}{k\alpha(1-\beta\xi)} = \Lambda^* = H^* \\
\kappa Y^{\sigma+\phi} = \alpha k^{1-\sigma} A^{1+\phi} = \kappa Y^{*\sigma+\phi}
\end{aligned}$$

$$\begin{aligned}
(1 - \gamma - \alpha)k &= (\lambda_3 + \lambda_3^*)(1 - \sigma)(1 - \gamma) - \lambda_3(1 + \phi) & \gamma k &= (\lambda_3 + \lambda_3^*)(1 - \sigma)\gamma - \lambda_3^*(1 + \phi) \\
\kappa N^{1+\phi} \left(1 - \lambda_3 \frac{\phi}{k\alpha}\right) &= (1 - \beta\xi)\lambda_5 & -\kappa N^{1+\phi} \lambda_3^* \frac{\phi}{k\alpha} &= (1 - \beta\xi)\lambda_5^* \\
-\lambda_1(1 - \xi)(1 - \zeta) + \lambda_2 H + \lambda_5 \zeta(1 - \xi) &= 0 & -\lambda_1^*(1 - \xi)(1 - \zeta) + \lambda_2^* H^* + \lambda_5^* \zeta(1 - \xi) &= 0 \\
\lambda_4 &= -\lambda_3 & \lambda_4^* &= -\lambda_3^* & \lambda_2 &= -\lambda_3(1 - \xi) & \lambda_2^* &= -\lambda_3^*(1 - \xi)
\end{aligned}$$

We can now expand (44) about its steady state in order to evaluate the second-order approximation in the welfare, using the steady state values of the Lagrange multipliers $\lambda_1, \dots, \lambda_5^*$. Defining home deviations $y_t = (Y_t - Y)/Y$, $a_t = (A_t - A)/A$, $\pi_t = \Pi_t - 1$, $m_t = M_t - 1$ and foreign deviations analogously, we first note that it is easy to show that $\partial L / \partial \Pi^* = 0$, so that the coefficient of π_t^* is zero. After further manipulation we can show that, up to a scalar transformation, the period- t welfare is given by:

$$\begin{aligned}
u_t^{NC} &= -\kappa \frac{Y^{1+\phi}}{2\alpha} \left(\frac{\phi + 1 - \alpha + \alpha\sigma}{\sigma + \phi} \right) \left[(\sigma + \phi)((1 - \gamma)y_t + \gamma y_t^*)^2 + \gamma(1 - \gamma)(1 + \phi)(y_t - y_t^*)^2 \right] \\
&\quad - 2(1 + \phi)((1 - \gamma)y_t a_t + \gamma y_t^* a_t^*) + \left(1 - \gamma - \frac{\alpha(\sigma + \phi)}{\phi + 1 - \alpha + \alpha\sigma} \right) y_t m_t + \gamma y_t^* m_t^* \\
&\quad + \frac{\xi\zeta}{(1 - \xi)(1 - \beta\xi)} ((1 - \gamma)\pi_t^2 + \gamma\pi_t^{*2}) \tag{45}
\end{aligned}$$

For further details see Appendix F. The first term in this expression represents consumption deviations and the second, terms of trade deviations. The presence of foreign country inflation deviations arises from the fact that they are associated with foreign country price dispersion. The latter enters the foreign price-setting decision, which in turn has an impact on the home country's consumption choice.

What is immediately noticeable in (45) is the symmetry with which the terms in output and inflation deviations from each bloc enter the quadratic approximation to the utility of the home bloc. It implies of course that it is identical to the approximation to the utility of the foreign bloc provided there are no mark-up shocks. Furthermore, a similar calculation for the cooperative utility function produces an identical approximation when mark-up shocks are zero. This replicates in a more straightforward fashion²⁰ the Benigno and Benigno (2006) result for technology shocks, that for identical economies (of this simple form) there are no gains from cooperation. However the terms involving mark-up shocks are different for the cooperative and non-cooperative approximations so there are potential gains from

²⁰The advantage of the Hamiltonian approach is clearly seen by comparing our workings in Appendix F with those in the far longer technical appendix provided with Benigno and Benigno (2006)

cooperation in the presence of markup shocks.

7 Concluding Remarks

Despite recent advances in numerical methods for nonlinear optimization problems, the ‘curse of dimensionality’ (see Judd (1998), chapter 7) will ensure the usefulness of LQ approximations to these problem, even for the case of a single policy-maker. For games involving many policy-makers a LQ approximation to calculations of players is for some equilibrium concepts, for example the closed-loop Nash equilibrium, probably essential. This paper has attempted a ‘users’ guide’ to accurate LQ approximation for researchers studying such problems. We have highlighted pitfalls already exposed in the literature and added another one in emphasizing that second-order conditions, usually ignored in the optimal policy literature, do matter.

The Hamiltonian method of Magill (1977a), which we have shown is equivalent to the Benigno-Woodford ‘large distortions’ procedure, provides an accurate LQ approximation of the household’s utility function given a linearized model economy *in the vicinity of the Ramsey commitment problem for the policymaker*. For the case of non-cooperative games, the latter is the Ramsey problem for each policy-maker given the open-loop trajectory of instruments of the other players. The question then is, given the choice of welfare which differs for cooperative and non-cooperative games, is this LQ approximation appropriate for other types of policy (for example for optimized simple rules, time-consistent policy or for other non-cooperative equilibrium concepts)? Because the Ramsey commitment problem is, *ex ante*, the best the policymaker can achieve, as Woodford (2003), chapter 6, has pointed out, this is indeed the case.²¹ Thus LQ approximation provides a tractable framework for comparing both cooperative and non-cooperative rules with and without commitment, and different forms of non-cooperative equilibria (closed-loop versus open-loop, different choices of instruments) using the *same* LQ approximation of the problem facing each policymaker. Future research involving the authors will pursue precisely this agenda.

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²¹He writes that “... this calculation (of the quadratic approximation) need only be done once, and not separately for each type of policy that one may wish to study”.

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A Proof of Theorem 3

The basic idea is that the the optimal policy depends on the initial condition and the instruments and, in the case of an RE system, the jumps in the non-predetermined variables. Given the latter, one can take a dynamic programming approach to the problem to prove (a): taking variations about the optimal path, one may write the value function V_t at time t as a constant plus $\frac{1}{2}x_t^T P_t x_t$. Using (2.1), one can write the value function V_{t-1} (ignoring constants) as

$$V_{t-1} = \frac{1}{2} \max \left\{ \beta(f_X x_{t-1} + f_W w_t)^T P_t (f_X x_{t-1} + f_W w_t) + \begin{bmatrix} x_{t-1} & w_t \end{bmatrix} \begin{bmatrix} H_{XX} & H_{XW} \\ H_{WX} & H_{WW} \end{bmatrix} \begin{bmatrix} x_{t-1} \\ w_t \end{bmatrix} \right\} \quad (\text{A.1})$$

with respect to w_t . The stated conditions for a maximum, and the update of P_t are straightforward to derive from this.

To prove (b), recall that from Currie and Levine (1993), we have the result under RE that V_0 is given by $\frac{1}{2}(x_0^{pT}(P_{11} - P_{12}P_{22}^{-1}P_{21})x_0^p + p_0^T P_{22}^{-1}p_0)$ where x_t^p are the deviations in the predetermined variables, p_0 is the initial value of the Lagrange multipliers associated with the non-predetermined variables (and is the source of the time inconsistency problem), and $P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}$ is written conformably with predetermined and non-predetermined variables respectively. Clearly if P_{22}^{-1} is not non-negative definite, then the value of V_0 can be set arbitrarily large by appropriate choice of p_0 ; in such a case, a solution to the problem which tends to a steady state optimum does not exist.

B Proof of Theorem 4

We first deal with the utility function, using the notation for deviations from steady state introduced earlier:

$$\begin{aligned}
& \sum_{t=0}^{\infty} \beta^t V(X_{t-1}, Z_{t-1}, W_t) = \sum_{t=0}^{\infty} \beta^t [U(X_{t-1}, W_t) + \tilde{V}(X_{t-1}, Z_{t-1}, W_t)] \\
&= \sum_{t=0}^{\infty} \beta^t [U(X_{t-1}, W_t) - \lambda^{*T}(X_t - f(X_{t-1}, W_t)) + \tilde{V}(X_{t-1}, Z_{t-1}, W_t)] \\
&\cong \sum_{t=0}^{\infty} \beta^t [U(X^*, W^*) + U_X x_{t-1} + U_W w_t - \lambda^{*T}(x_t - f_X x_{t-1} - f_W w_t) \\
&\quad + \frac{1}{2}(x_{t-1}^T H_{XX} x_{t-1} + 2x_{t-1}^T H_{XW} w_t + w_t^T H_{WW} w_t + \tilde{V}(X_{t-1}, Z_{t-1}, W_t))] \\
&= \sum_{t=0}^{\infty} \beta^t [U(X^*, W^*) + (U_X - \frac{1}{\beta} \lambda^{*T} + \lambda^{*T} f_X) x_{t-1} + (U_W + \lambda^{*T} f_W) w_t \\
&\quad + \frac{1}{2}(x_{t-1}^T H_{XX} x_{t-1} + 2x_{t-1}^T H_{XW} w_t + w_t^T H_{WW} w_t + \tilde{V}(X_{t-1}, Z_{t-1}, W_t))] \quad (\text{B.1})
\end{aligned}$$

where $H = U(X, W) + \lambda^{*T} f(X, W)$, and its second derivatives are evaluated at (X^*, W^*) . Hence, using (11), the linear terms in x_t, w_t vanish. Since by assumption there are no first-order terms in the expansion of \tilde{V} at the efficient level, it follows that this expansion has no first-order terms present.

Now consider the constraints. Firstly the resource constraint is in steady state at the efficient level, so that an expansion about the latter will contain no constant term. Secondly, the constraint involving Z , by appropriate choice of $\tau = \tau^*$ is also in a zero-inflation steady state at the efficient level, so that any approximation of its dynamics about the efficient level will omit a constant term. This completes the proof.

C Proof of Result 1

First-order conditions are given by:

$$\begin{aligned}
& (Y_t - Z_t)^{-\sigma} - \kappa \frac{Y_t^\phi}{A_t^{1+\phi}} D_t^{1+\phi} - \lambda_{1t} h_C \\
& - \lambda_{5t} \frac{\kappa(1+\phi)}{\alpha} \frac{Y_t^\phi}{A_t^{1+\phi}} D_t^\phi - \lambda_{4t} ((Y_t - Z_t)^{-\sigma} - \sigma Y_t (Y_t - Z_t)^{-\sigma-1}) = 0 \\
& -(Y_t - Z_t)^{-\sigma} + \frac{1}{\beta} \lambda_{1t-1} - \lambda_{4t} \sigma Y_t (Y_t - Z_t)^{-\sigma-1} = 0
\end{aligned}$$

$$\begin{aligned}
& \beta(1-\zeta)\xi\lambda_{2,t+1}\Pi_{t+1}^{\zeta-2} - \lambda_{4t}\xi\beta(\zeta-1)\Pi_{t+1}^{\zeta-2}H_{t+1} - \lambda_{5t}\xi\beta\zeta\Pi_{t+1}^{\zeta-1}\Lambda_{t+1} - \xi\beta\zeta\lambda_{6,t+1}\Pi_{t+1}^{\zeta-1}D_t = 0 \\
& -\lambda_{2t}(1-\xi)(1-\zeta)Q_t^{-\zeta} + \lambda_{3t}H_t + \zeta(1-\xi)\lambda_{6t}Q_t^{-\zeta-1} = 0 \\
& \lambda_{3t}Q_t + \lambda_{4t} - \xi\Pi_t^{\zeta-1}\lambda_{4,t-1} = 0 \\
& -\lambda_{3t} + \lambda_{5t} - \xi\Pi_t^{\zeta}\lambda_{5,t-1} = 0 \\
& -\kappa N_t^{1+\phi}D_t^\phi + \lambda_{6t} - \xi\beta\Pi_{t+1}^{\zeta}\lambda_{6,t+1} - \frac{\kappa\phi}{\alpha}N_t^{1+\phi}D_t^{\phi-1}\lambda_{5t} = 0
\end{aligned}$$

There are also boundary conditions: initial values of the backward-looking variables, and $\lambda_{40} = \lambda_{50} = 0$, since H_t and Λ_t are forward-looking, plus a terminal condition. A sufficient terminal condition is that the dynamic system described by the first-order conditions and original constraints is saddle-path stable. In fact all that concerns us for LQ approximation is the steady state and the requirement of saddle-path stability. By standard control theory the latter is ensured if the discount factor β is sufficiently close to unity

D Proof of Proposition

(i) Firstly, we require the coefficient of π_t^2 inside the brackets of (25), $\alpha + (1 + \phi)\lambda_5$, to be positive. A little calculation shows that (with $\alpha > 1 - h_C$) this term is greater than $1 - h_C$ provided that $\sigma > 1$. Ignoring the shock term a_t , if we now consider the remaining terms as a quadratic function of y_t and $y_t - z_t$, then this quadratic will always be positive provided that (a) $\alpha + (1 + \phi)\lambda_5 > 0$, (b) $\frac{\sigma}{1-h_C}(1 - \frac{\lambda_5(1+\sigma)}{1-h_C}) > 0$ and (c) $\phi(\alpha + (1 + \phi)\lambda_5)\frac{\sigma}{1-h_C}(1 - \frac{\lambda_5(1+\sigma)}{1-h_C}) - \frac{\lambda_5^2\sigma^2}{(1-h_C)^2} > 0$. (a) has already been shown, and since by assumption $\alpha > 1 - h_C$ it follows that $\lambda_5 < 0$, so the left hand side of (b) is greater 0. After some manipulation we can show that after multiplying (c) through by $(1 - h_C)^2$ the left hand side becomes

$$\begin{aligned}
& (\phi^3 - \phi - \sigma)(1 - h_C)^2 + 2\alpha(1 - h_C)(\sigma\phi^2 + \phi + \sigma) + (\phi\sigma^2 - \phi - \sigma)\alpha^2 \\
& > (\phi^3 + \sigma\phi^2)(1 - h_C)^2 + \alpha(1 - h_C)(\sigma\phi^2 + \phi + \sigma) + (\phi\sigma^2 - \phi - \sigma)\alpha^2 \quad (\text{D.1})
\end{aligned}$$

where the inequality holds when $\alpha > 1 - h_C$. Thus the sufficient condition $\phi\sigma^2 - \phi - \sigma > 0$ is likely to be considerably more stringent a condition than is required.

(ii) With $\alpha < 1 - h_C$, it is clear that $\lambda_5 > 0$, so that (a) above is satisfied. After a little manipulation, it is easy to show that $\frac{\sigma}{1-h_C}(1 - \frac{\lambda_5(1+\sigma)}{1-h_C}) = \frac{\sigma}{(1-h_C)^2(\sigma+\phi)}(\alpha(1 + \sigma) - (1 - h_C)(1 - \phi))$, so that (b) is satisfied if $\alpha(1 + \sigma) > (1 - h_C)(1 - \phi)$. Finally,

using the condition $\alpha < 1 - h_C$,

$$\begin{aligned} & (\phi^3 - \phi - \sigma)(1 - h_C)^2 + 2\alpha(1 - h_C)(\sigma\phi^2 + \phi + \sigma) + (\phi\sigma^2 - \phi - \sigma)\alpha^2 \\ & > (\phi^3 - \sigma - \phi)(1 - h_C)^2 + \alpha(1 - h_C)(\sigma\phi^2 + \phi + \sigma) + (\phi\sigma^2 + \sigma\phi^2)\alpha^2 \quad (\text{D.2}) \end{aligned}$$

where the inequality holds when $\alpha < 1 - h_C$. Once again the sufficient condition $\phi^3 - \phi - \sigma > 0$ is likely to be considerably more stringent a condition than is required.

E Proof of Result 5 and Equation (45)

The result follows from the following first-order conditions

$$\begin{aligned} \frac{\partial L}{\partial Y_t} &= \frac{(1 - \gamma)k^{-\sigma}}{Y_t} (Y_t^{1-\gamma} Y_t^{*\gamma})^{1-\sigma} [k - (\lambda_{3t} + \lambda_{3t}^*)(1 - \sigma)] - Y_t^\phi D_t^\phi \left[D_t + \lambda_{4t} \frac{1 + \phi}{k\alpha} \right] = 0 \\ \frac{\partial L}{\partial Y_t^*} &= \frac{\gamma k^{-\sigma}}{Y_t^*} (Y_t^{1-\gamma} Y_t^{*\gamma})^{1-\sigma} [k - (\lambda_{3t} + \lambda_{3t}^*)(1 - \sigma)] - Y_t^\phi D_t^{*\phi} \left[D_t^* + \lambda_{4t}^* \frac{1 + \phi}{k\alpha} \right] = 0 \\ \frac{\partial L}{\partial D_t} &= -Y_t^{1+\phi} D_t^\phi - \lambda_{4t} \frac{\phi}{k\alpha} Y_t^{1+\phi} D_t^{\phi-1} + \lambda_{5t} - \beta \xi \lambda_{5,t+1} \Pi_{t+1}^\zeta = 0 \\ \frac{\partial L}{\partial D_t^*} &= -\lambda_{4t}^* \frac{\phi}{k\alpha} Y_t^{*1+\phi} D_t^{*\phi-1} + \lambda_{5t}^* - \beta \xi \lambda_{5,t+1}^* \Pi_{t+1}^{*\zeta} \\ \frac{\partial L}{\partial Q_t} &= -\lambda_{1t}(1 - \xi)(1 - \zeta)Q_t^{-\zeta} + \lambda_{2t}H_t + \zeta(1 - \xi)\lambda_{5t}Q_t^{-\zeta-1} = 0 \\ \frac{\partial L}{\partial Q_t^*} &= -\lambda_{1t}^*(1 - \xi)(1 - \zeta)Q_t^{*-\zeta} + \lambda_{2t}^*H_t^* + \zeta(1 - \xi)\lambda_{5t}^*Q_t^{*-\zeta-1} = 0 \\ \\ \frac{\partial L}{\partial H_t} &= \lambda_{2t}Q_t + \lambda_{3t} - \xi\lambda_{3,t-1}\Pi_t^{\zeta-1} = 0 \\ \frac{\partial L}{\partial H_t^*} &= \lambda_{2t}^*Q_t^* + \lambda_{3t}^* - \xi\lambda_{3,t-1}^*\Pi_t^{*\zeta-1} = 0 \\ \frac{\partial L}{\partial \Lambda_t} &= -\lambda_{2t} + \lambda_{4t} - \xi\lambda_{4,t-1}\Pi_t^\zeta = 0 \\ \frac{\partial L}{\partial \Lambda_t^*} &= -\lambda_{2t}^* + \lambda_{4t}^* - \xi\lambda_{4,t-1}^*\Pi_t^{*\zeta} = 0 \\ \frac{\partial L}{\partial \Pi_t} &= (1 - \zeta)\xi\lambda_{1t}\Pi_t^{\zeta-2} - \lambda_{3t}\xi(\zeta - 1)\Pi_t^{\zeta-2}H_t - \lambda_{4t}\xi\zeta\Pi_t^{\zeta-1}\Lambda_t - \xi\zeta\lambda_{5t}\Pi_t^{\zeta-1}D_{t-1} = 0 \end{aligned}$$

To prove result (45), first we drop all t -subscripts for purposes of conciseness.

Firstly, we note that

$$\frac{\partial L}{\partial \Pi^*} = -\xi(\zeta - 1)\Pi^{*\zeta-1}\lambda_1^* - \xi(\zeta - 1)\Pi^{*\zeta-2}H^*\lambda_3^* - \xi\zeta\Pi^{*\zeta-1}\Lambda^*\lambda_4^* - \xi\zeta\Pi^{*\zeta-1}D\lambda_5^* \quad (\text{E.1})$$

which is easily shown to equal 0. Next, the coefficient of $\frac{1}{2}y_t^2$ is given by

$$\begin{aligned} Y^2 \frac{\partial^2 L}{\partial Y^2} &= \frac{Y^{1+\phi}}{\alpha} \left[\left(1 - \frac{\lambda_3 + \lambda_3^*}{k} (1 - \sigma)\right) (1 - \gamma) ((1 - \sigma)(1 - \gamma) - 1) - \phi\alpha - \frac{\lambda_4}{k} \phi(1 + \phi) \right] \\ &= -\frac{Y^{1+\phi}}{\alpha} \frac{\phi + 1 - \alpha + \alpha\sigma}{\phi + \sigma} (1 - \gamma)(\phi + \sigma - \sigma\gamma + \gamma) \end{aligned} \quad (\text{E.2})$$

that of $y_t y_t^*$ is given by

$$\begin{aligned} Y Y^* \frac{\partial^2 L}{\partial Y \partial Y^*} &= \frac{Y^{1+\phi}}{\alpha} (1 - \gamma) \gamma (1 - \sigma) \left(1 - \frac{\lambda_3 + \lambda_3^*}{k} (1 - \sigma)\right) \\ &= -\frac{Y^{1+\phi}}{\alpha} \frac{\phi + 1 - \alpha + \alpha\sigma}{\phi + \sigma} (1 - \gamma) \gamma (1 - \sigma) \end{aligned} \quad (\text{E.3})$$

and that of $\frac{1}{2}y_t^{*2}$ is given by

$$\begin{aligned} Y^{*2} \frac{\partial^2 L}{\partial Y^{*2}} &= \frac{Y^{1+\phi}}{\alpha} \left[\left(1 - \frac{\lambda_3 + \lambda_3^*}{k} (1 - \sigma)\right) \gamma ((1 - \sigma)\gamma - 1) + \frac{\lambda_3^*}{k} \phi(1 + \phi) \right] \\ &= -\frac{Y^{1+\phi}}{\alpha} \frac{\phi + 1 - \alpha + \alpha\sigma}{\phi + \sigma} \gamma (\phi + \gamma\sigma + 1 - \gamma) \end{aligned} \quad (\text{E.4})$$

Summing these yields the first two terms in (45), expressed as deviations in total output and the terms of trade. The second-order terms in π_t^2 are derived from the sum of terms in $\frac{\partial^2 L}{\partial \Pi^2}$, $2\frac{\partial^2 L}{\partial \Pi \partial H}$, $2\frac{\partial^2 L}{\partial \Pi \partial \Lambda}$, $2\frac{\partial^2 L}{\partial \Pi \partial H}$, $\frac{\partial^2 L}{\partial Q^2}$:

$$\begin{aligned} &-\frac{1}{2}\xi(\zeta - 1)\pi_t^2 [(\zeta - 2)\lambda_1 + (\zeta - 2)\lambda_3 H + \zeta\lambda_4 H + \zeta\lambda_5] - \pi_t h_t \xi(\zeta - 1)\lambda_3 \\ &\quad - \pi_t \lambda_t \zeta \lambda_4 + q_t h_t \lambda_2 + \frac{1}{2}q_t^2 (1 - \xi)\zeta [(1 - \zeta)\lambda_1 - (\zeta + 1)\lambda_5] \end{aligned} \quad (\text{E.5})$$

where $h_t = H_t - H$, $\lambda_t = \Lambda_t - \Lambda$, $q = Q_t - 1$. Substituting $h_t + Hq_t = \lambda_t$, $\xi\pi_t = (1 - \xi)q_t$ yields the required term of (45). The coefficient of $\frac{1}{2}\pi_t^{*2}$ is given by

$$\frac{\partial^2 L}{\partial \Pi^{*2}} = -\xi(\zeta - 1)[(\zeta - 2)\lambda_1^* + (\zeta - 2)\lambda_3^* H + \zeta\lambda_4^* H + \zeta\lambda_5^*] \quad (\text{E.6})$$

Finally, it is easy to show that all partial second derivatives $\partial^2 L / \partial \Pi^i \partial Y^i = 0$ (where $\Pi^i = \Pi, \Pi^*$, $Y^i = Y, Y^*$), so that there are no cross-terms of the form $\pi_t y_t$ in the approximation.

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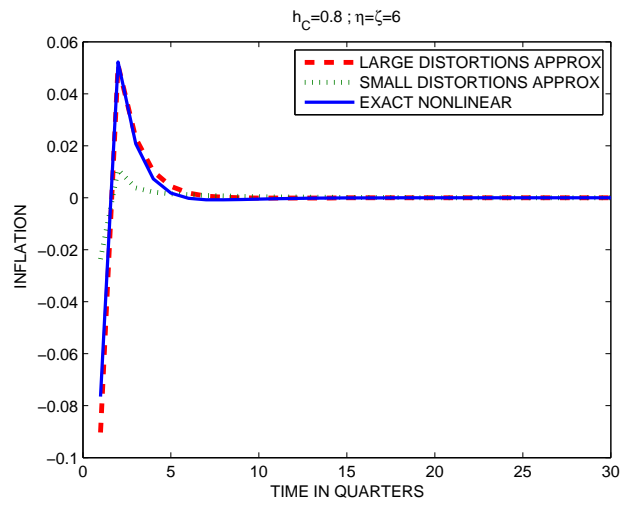
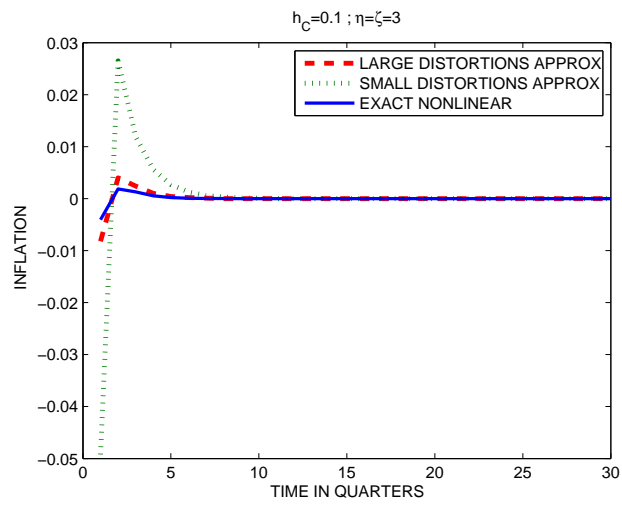
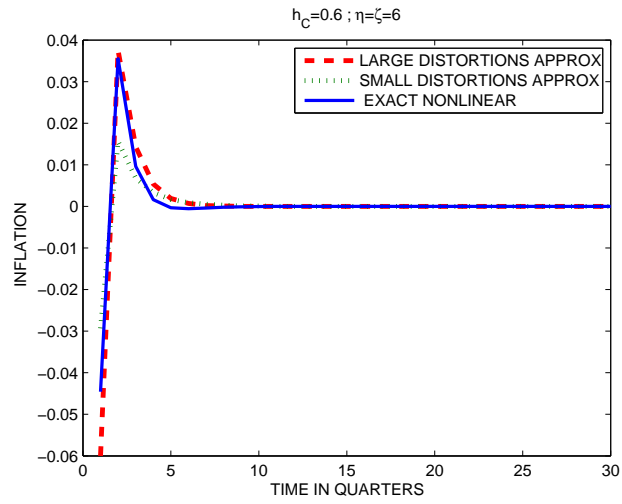


Figure 1: Comparisons of Approximations for Inflation

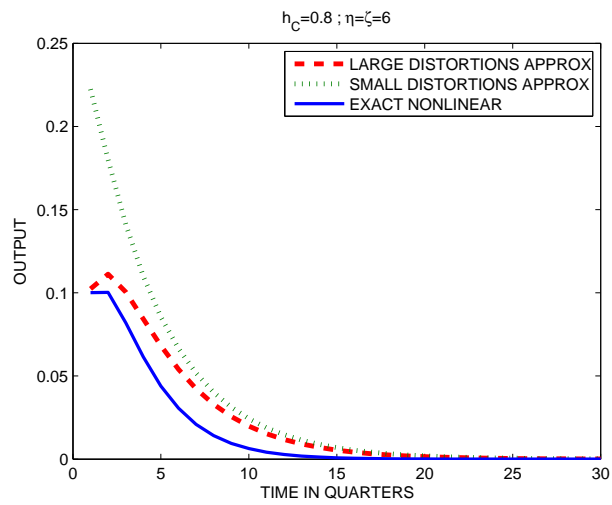
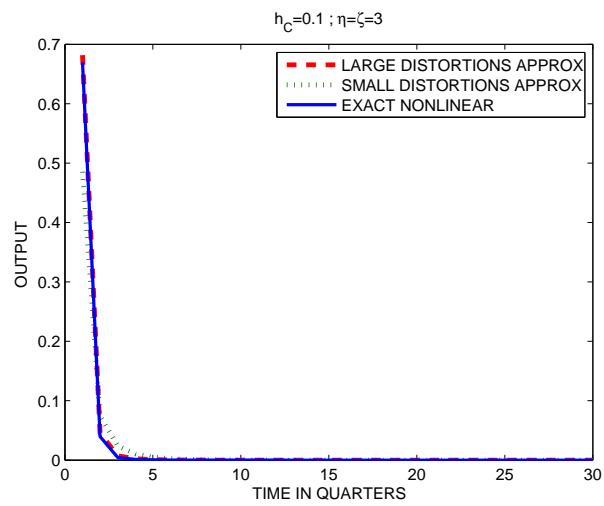
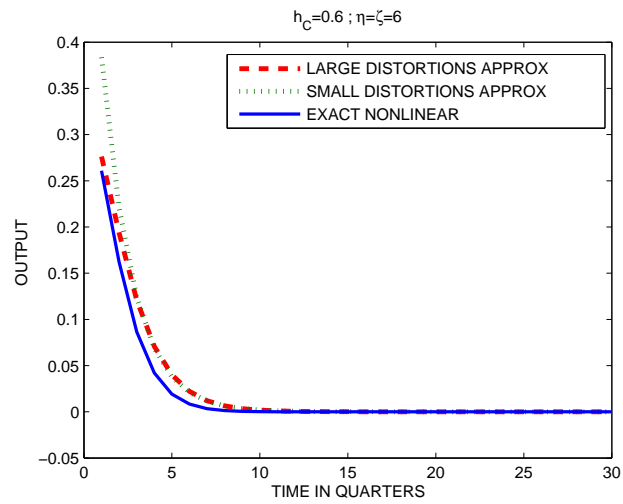


Figure 2: Comparisons of Approximations for Output