CES Technology and Business Cycle Fluctuations*

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Abstract  

This paper contributes to a rapidly rising literature that brings the CES specification of the production function into the analysis of business cycle fluctuations. Using US data, we estimate by Bayesian-Maximum-Likelihood methods a medium-sized DSGE model with a CES rather than Cobb-Douglas (CD) technology. The main empirical result is to confirm decisively the importance of CES rather than CD production functions. We estimate a elasticity of substitution of elasticity well below unity at 0.16 and in a marginal likelihood race assuming equal prior model probabilities, CES beats the CD production decisively. The principle reason for this result is that the CES specification captures movements of factor shares. The marginal likelihood improvement is matched by the ability of the CES model to fit the data in terms of second moments. A comparison with a DSGE-VAR further confirms the ability of the CES model to reduce model misspecification. The consequences for optimal monetary policy of different elasticities of substitution is also examined. We find these not to be significant unless a zero lower bound constraint on the nominal interest rate is imposed. But the main message for DSGE models is that we should dismiss once and for all the use of CD for business cycle analysis.

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1 Introduction

This paper aims to extend the DSGE model developed by Christiano et al. (2005) and Smets and Wouters (2007) to allow for a richer and more data coherent specification of the production side of the economy. The idea is to enrich what has become the workhorse DSGE model by substituting the usual Cobb-Douglas production assumption with a more general CES production function which allows for cyclical variations in factor shares, estimation of the capital/labour elasticity of substitution and biased technical change.

The CES production function has been used extensively in many area of economics since the middle of the previous century (Solow (1956) and Arrow et al. (1961)). Thanks to La Grandville (1989), who introduced the concept of normalization, it has been extensively used in growth theory. Indeed La Grandville (1989) showed that it was possible to obtain a perpetual growth in income per-capita, even without any technical progress. Furthermore factor substitution and the bias in technical change feature an important role in many other areas of economics but, until recently have been largely disregarded in business cycle analysis. On the empirical side León-Ledesma et al. (2010) show that normalization improves empirical identification.\footnote{They show that using a normalized approach permits to overcome the ‘impossibility theorem’ stated by Diamond et al. (1978) and simultaneously identify the elasticity of substitution and biased technical change.}

The concepts of biased technical change and imperfect factor substitutability between factors of production has been introduced in business cycle analysis by Cantore et al. (2010b). They show that the introduction of a normalized CES production function into an otherwise standard RBC and/or NK DSGE model significantly changes the response of hours worked to a technology shock under both price-setting mechanisms and that such response might change as well within each model depending on the parameters related to the production process (developing a threshold rule for the “impact” of a technology shock on hours worked). They also show how the introduction of biased technical change and imperfect substitutability allow movement in factor shares which appear to fluctuate at business cycle frequencies in the data but are theoretically constant under the Cobb-

\footnote{The value of the substitution elasticity has been linked to differences in international factor returns and convergence (e.g., Klump and Preissler (2000), Mankiw (1995)); movements in income shares (Blanchard (1997), Caballero and Hammour (1998), Jones (2003)); the effectiveness of employment creation policies (Rowthorn (1999)), etc. The nature of technical change, on the other hand, matters for characterizing the welfare consequences of new technologies (Marqueti (2003)); labour-market inequality and skills premia (Acemoglu (2002)); the evolution of factor income shares (Kennedy (1964), Acemoglu (2003)) etc.}
Douglas specification. Indeed there is mounting evidence in the literature\(^3\) that whilst constant factor shares might be a good approximation for growth models where the time span considered is very long, at business cycle frequencies those shares are not constant. This is clearly showed in Figure 1 for the US data used to estimate our model.

![Figure 1: US Labour Share (Source: Department of Labor, U.S. Bureau of Labor Statistics)](image)

Furthermore Cantore et al. (2010a) test empirically the model(s) developed by Cantore et al. (2010b) using rolling-windows Bayesian techniques in order to check if the documented time-varying relation between hours worked, productivity and output (see Fisher (2006) and Galí and Gambetti (2009) among others) can be explained using the threshold rule. They are also the first to show that introducing a CES production function in an estimated RBC model significantly improves the fit of the model with respect to the Cobb-Douglas case.

Apart from Cantore et al. (2010b) and Cantore et al. (2010a) usually DSGE models continue using the Cobb-Douglas assumption even if the empirical evidence provided through the years has now definitely ruled out the possibility of unitary elasticity of substitution (see among others Antràs (2004), Klump et al. (2007), Chirinko (2008) and León-Ledesma et al. (2010)).

In this paper we show that by introducing a CES production function in a medium-scale DSGE model in the spirit of Christiano et al. (2005) and Smets and Wouters (2007) makes it possible to exploit the movements of factor shares we observe in the data to improve significantly the performance of the model. There are two main results, one empirical and one related to optimal monetary policy. Our empirical result is that in terms of model posterior probabilities, impulse responses, second moments and autocorrelations, the assumption of a CES technology significantly improves the model fit. Using US data, we estimate by Bayesian-Maximum-Likelihood (BML) methods a elasticity of substitution of elasticity between the capital/labour ratio and the wage rate/capital cost ratio to be 0.16, a value at the low end of the literature using other methods of estimation.\textsuperscript{4}

Our result regarding optimal policy is that the choice of elasticity of substitution does not drastically alter the outcome unless with impose a form of zero lower bound (ZLB) constraint on the nominal interest rate. The way this works in through a higher volatility introduced by capital-labour substitutability which tightens the ZLB constraint especially under discretion. It follows that a CD production function rather than CES exaggerates the ZLB problem.

The rest of the paper is organized as follows. Section 2 describes the model with particular attention paid to the normalization of the CES production function. Section 3 sets out the BML estimation of the model. Section 4 examines the ability of the model to capture the main characteristics of the actual data as described by second moments and the impulse response functions of an estimated “DSGE-VAR” hybrid. Section 5 compares the variance decomposition of the structural shocks for the CES and Cobb-Douglas formulations. Section 6 examines optimal monetary policy and Section 7 concludes the paper.

2 The Augmented SW Model

Here we present, concisely the augmented SW model with a wholesale and a retail sector, Calvo prices and wages, CES production function, adjustment costs of investment and variable capital utilization. Figure 2 illustrates the model structure. The model equilibrium conditions are presented in non-linear form. The novel feature is the introduction

\textsuperscript{4}See, for example, Table 2 in Rowthorn (1999), Chirinko (2008) and León-Ledesma et al. (2010)).
of a CES production function in the wholesale sector, instead of the usual Cobb-Douglas form. This generalization then allows for the identification of both labour-augmenting and capital-augmenting technology shocks. As in Smets and Wouters (2007) we use a household utility function compatible with a balance growth path in the steady state, but we adopt a more standard functional form used in the RBC literature. However we do not adopt Kimball aggregators for final output and composite labour.\footnote{The motivation for generalizing Dixit-Stiglitz aggregators is to bring estimates of price and wage contract lengths into line with micro-econometric evidence. In fact our estimates for US data are compatible with the simpler Dixit-Stiglitz formulation.} Again as in their paper we introduce a monopolistic trade-union that allows households to supply homogeneous labour. Then as long as preference shocks are symmetric households are identical in equilibrium and the complete market assumption is no longer required for aggregation.

The supply-side of the economy consists of competitive retail sector producing final output and a monopolistically competitive wholesale sector producing differentiated goods using the usual inputs of capital and work effort. Households consume a bundle of differentiated commodities, supply labour and capital to the production sector, save and own the monopolistically competitive firms in the goods sector. Capital producers provide the capital inputs into the wholesale sector.\footnote{There are other differences with Smets and Wouters (2007): (i) Our price and wage mark-up shocks follow an AR(1) process instead of the ARMA process chosen by SW; (ii) in SW the government spending shock is assumed to follow an autoregressive process which is also affected by the productivity shock; (iii) we have a preference shock instead of the risk-premium shock. Chari \textit{et al.} (2009) criticized the risk premium shock arguing that has little interpretation and in unlikely to be invariant to monetary policy. We prefer our somewhat simpler set-up and we expect none of the differences to affect the main focus of the paper which is on the comparison between CD and CES production functions.}

We set out the model first without specifying the form of the utility and production functions in order to obtain a flexible framework in which it will be easy to stick different functional forms.

The sequencing of decisions is as follows\footnote{Sequencing matters for the monopolistic trade-unions and intermediate firms who anticipate and exploit the downward-sloping demand for labour and goods respectively. Different set-ups with identical equilibria are common in the literature. Monopolistic prices can be transferred to the retail sector. When it comes to introducing financial frictions, for example, as in Gertler and Karadi (2009) the introduction of separate capital producers as in our set-up is convenient, but not essential in the SW model without such frictions.}

1. Each household supplies homogeneous labour at a price $W_{h,t}$ to a trade-union.

   Households choose their consumption, savings and labour supply given aggregate consumption (determining external habit). In equilibrium all household decisions...
are identical.

2. Capital producing firms that at time convert final output into new capital which is sold on to intermediate firms.

3. A monopolistic trade-union differentiates the labour and sells type \( N_t(j) \) at a price \( W_t(j) \) to a labour packer in a sequence of Calvo staggered wage contracts. In equilibrium all households make identical choices of total consumption, savings, investment and labour supply.

4. The competitive labour packer forms a composite labour service according to a constant returns CES technology \( N_t = \left( \int_0^1 N_t(j)^{(\zeta-1)/\zeta} \, dj \right)^{\zeta/(\zeta-1)} \) and sells onto the intermediate firm.

5. Each intermediate monopolistic firm \( f \) using composite labour and capital rented from purchased from capital producers to produce a differentiated intermediate good which is sold onto final goods firm at price \( P_t(f) \) in a sequence of Calvo staggered price contracts.

6. Competitive final goods firms use a continuum of intermediate goods according to another constant returns CES technology to produce aggregate output \( Y_t = \left( \int_0^1 Y_t(f)^{(\mu-1)/\mu} \, df \right)^{\mu/(\mu-1)} \).

We now solve the model by backward induction starting with the production of final goods.

### 2.1 Final Goods

Each final goods firms minimizes the cost \( \int_0^1 P_t(f)Y_t(f) \, df \) of producing the final output \( Y_t = \left( \int_0^1 Y_t(f)^{(\zeta-1)/\zeta} \, df \right)^{\zeta/(\zeta-1)} \). This leads to the standard result for the Dixit-Stiglitz aggregator

\[
Y_t(f) = \left( \frac{P_t(f)}{P_t} \right)^{-\zeta} Y_t \tag{1}
\]

\[
P_t = \left[ \int_0^1 P_t(f)^{1-\zeta} \, df \right]^{\frac{1}{1-\zeta}} \tag{2}
\]

\[
P_t Y_t = \int_0^1 P_t(f)Y_t(f) \, df \tag{3}
\]
where $P_t$ is an aggregate price index. Note that (1) and (3) imply (2).

2.2 Intermediate Firms

In the intermediate goods sector each good $f$ is produced by a single firm $f$ using composite labour and capital with a technology:

$$Y_t(f) = (1 - c)F(ZK_t, ZN_t, N_t, U_tK_t)$$ (4)

where $c$ are fixed costs of production and $U_t$ allows for variable capital utilization. The parameter $c$ is pinned down by a free-entry condition that drives profits in the steady state to zero. Given that at this stage we do not specify the form of the production function we allow for all the possible specification of technology shocks. Calling $ZK_t$ capital-augmenting and $ZN_t$ labour-augmenting we are in the case of Hicks neutrality if $ZK_t = ZN_t > 0$, Solow neutrality if $ZK_t > 0$ and $ZN_t = 0$ and Harrod neutrality in the case of $ZK_t = 0$ and $ZN_t > 0$. Then minimizing costs $P_tRR_t^KU_t(f)K_t(f) + W_tN_t(f)$ leads
\[
\frac{W_t}{P_t} \equiv MPL_t = MC_t(f)F_{N,t} \quad (5)
\]
\[
RR^K_t \equiv MPK_t = MC_t(f)F_{K,t} \quad (6)
\]

where \(MPL_t\) and \(MPK_t\) are the marginal products of labour and capital respectively, \(RR^K_t\) is the real cost of capital. As usual the firm’s cost minimizing real marginal costs \((MC_t(f))\) is given by the Lagrange multiplier related to the production function constraint.

Pricing by the firm follows the standard Calvo framework supplemented with indexation. At each period there is a probability of \(1 - \xi_p\) that the price is set optimally.\(^8\) The optimal price derives from maximizing discounted profits. For those firms and workers unable to reset, prices are indexed to last period’s aggregate inflation, with indexation parameter \(\gamma_p\). With indexation parameter \(\gamma_p \geq 0\), this implies that successive prices with no re-optimization are given by

\[
P_{0}^1(t) = P_{0}^0(f), P_{0}^0(t)\left(\frac{P_t}{P_{t-1}}\right)^{\gamma_p}, P_{0}^0(t)\left(\frac{P_{t+1}}{P_t}\right)^{\gamma_p}, \ldots.
\]

For each intermediate producer \(f\) the objective is at time \(t\) to choose \(\{P_{0}^0(t)\}\) to maximize discounted profits

\[
E_t \sum_{k=0}^{\infty} \xi_k^p \Lambda_{t,t+k} Y_{t+k}(f) \left[ P_{0}^0(f) \left(\frac{P_{t+k-1}}{P_{t-1}}\right)^{\gamma_p} - P_{t+k} MC_{t+k} \right] = 0 \quad (8)
\]

subject to \(Y_{t+k}(f) = \left(\frac{P_{0}^0(f)}{P_{t+k}} \left(\frac{P_{t+k-1}}{P_{t-1}}\right)^{\gamma_p}\right)^{-\xi} Y_{t+k}\) (from (1)), where \(\Lambda_{t,t+k} \equiv \beta\frac{\Lambda_{t,t+k}/P_{t+k}}{\Lambda_{t,t}/P_t}\), is the nominal stochastic discount factor over the interval \([t, t+k]\) and \(\zeta\) is the elasticity of substitution across intermediate goods. Since firms are atomistic, the aggregate price index and the discount factor are given in their calculations.

This leads to the following first-order condition:

\[
E_t \sum_{k=0}^{\infty} \xi_k^p \Lambda_{t,t+k} Y_{t+k}(f) \left[ P_{0}^0(f) \left(\frac{P_{t+k-1}}{P_{t-1}}\right)^{\gamma_p} - MS_{p,t} P_{t+k} MC_{t+k} \right] = 0
\]

where we introduced, as usual in the literature, a time varying mark-up of prices over marginal costs \(MS_{p,t} = \frac{\zeta}{(\zeta - 1)} e P_t\) with \(e P_t\) being the price mark-up shock. Then by the law of large numbers the evolution of the price index is given by

\[
P_{t+1}^{1-\xi_p} = \xi_p \left(\frac{P_t}{P_{t-1}}\right)^{\gamma_p\left(1-\xi_p\right)} + (1 - \xi_p)(P_{t+1}^0(f))^{1-\xi_p}
\]

\(^8\)Thus we can interpret \(\frac{1}{1-\xi_p}\) as the average duration for which prices are left unchanged.
2.3 Labour Packer

As with final goods firms, the labour packer minimizes the cost $\int_0^1 W_t(j)N_t(j) dj$ of producing the composite labour service $N_t = \left(\int_0^1 N_t(j)^{(\mu-1)/\mu} dj\right)^{\mu/(\mu-1)}$. This leads to the standard result for the Dixit-Stiglitz aggregator

$$N_t(j) = \left(\frac{W_t(j)}{W_t}\right)^{-\mu} N_t$$  \hspace{1cm} (10)

$$W_t = \left[\int_0^1 W_t(j)^{1-\mu} dj\right]^{1/\mu}$$  \hspace{1cm} (11)

$$W_t N_t = \int_0^1 W_t(j)N_t(j) dj$$  \hspace{1cm} (12)

where $W_t$ is an aggregate wage index. Note that (10) and (12) imply (11).

2.4 Trade-Unions

Wage setting by the trade-union again follows the standard Calvo framework supplemented with indexation. At each period there is a probability $1 - \xi_w$ that the wage is set optimally. The optimal wage derives from maximizing discounted profits. For those trade-unions unable to reset, wages are indexed to last period’s aggregate inflation, with wage indexation parameter $\gamma_w$. Then as for price contracts the wage rate trajectory with no re-optimization is given by $W^0_t(j)$,

$$W^0_t(j) \left(\frac{P_t}{P_{t-1}}\right)^{\gamma_w}, \ W^0_t(j) \left(\frac{P_{t+1}}{P_{t-1}}\right)^{\gamma_w}, \cdots.$$  \hspace{1cm} (13)

The trade union that buys homogeneous labour at a price $W_{h,t}$ and converts it into a differentiated labour service of type $j$. The trade union time $t$ then chooses $W^0_t(j)$ to maximize

$$E_t \sum_{k=0}^{\infty} \xi_w^k \Lambda_{t,t+k} N_{t+k}(j) \left[ W^0_t(j) \left(\frac{P_{t+k-1}}{P_{t-1}}\right)^{\gamma_w} - W_{h,t+k} \right]$$  \hspace{1cm} (13)

where $N_t(j)$ is given by (10) so that $N_{t+k}(j) = \left(\frac{W^0_t(j)}{W_{t+k}} \left(\frac{P_{t+k-1}}{P_{t-1}}\right)^{\gamma_w}\right)^{-\eta} N_{t+k}$ and $\eta$ is the elasticity of substitution across labour varieties. By analogy with (8) this leads to the following first-order condition

$$E_t \sum_{k=0}^{\infty} \xi_w^k \Lambda_{t,t+k} N_{t+k}(j) \left[ W^0_t(j) \left(\frac{P_{t+k-1}}{P_{t-1}}\right)^{\gamma_w} - MS_{w,t} W_{h,t+k} \right] = 0$$  \hspace{1cm} (14)
where \( MS_{w,t} = \frac{\eta}{(\eta - 1)} \epsilon W_t \) is the time varying wage mark-up with \( \epsilon W_t \) being the wage mark-up shock. Then by the law of large numbers the evolution of the wage index is given by

\[
W_{t+1}^{1-\eta} = \xi_w \left( W_t \left( \frac{P_t}{P_{t-1}} \right)^{\gamma_w} \right)^{1-\eta} + (1 - \xi_w)(W_t^0(j))^{1-\eta} \tag{15}
\]

### 2.5 Capital Producers

Capital producing firms convert \( I_t \) of output into \((1 - S(X_t))I_t \) of new capital sold at a real price \( Q_t \). They then maximize expected discounted profits

\[
E_t \sum_{k=0}^{\infty} D_{t,t+k} [Q_{t+k}ZI_{t+k}(1 - S(I_{t+k}/I_{t+k-1}))I_{t+k} - I_{t+k}] 
\]

where \( D_{t,t+k} = \beta \frac{\lambda_{t+k}}{\lambda_{t,t+k}} \) is the real stochastic discount factor over the interval \([t, t+k]\]. This results in the first-order condition

\[
Q_t ZI_t \left(1 - S(X_t) - X_t S'(X_t)\right) + E_t \left[ D_{t,t+1} Q_{t+1} ZI_{t+1} S'(X_{t+1}) \frac{I_{t+1}^2}{I_t^2} \right] = 1 \tag{16}
\]

Capital accumulation is given by

\[
K_{t+1} = (1 - \delta)K_t + (1 - S(X_t))I_t ZI_t; \tag{17}
\]

where \( \delta \) is the depreciation rate, \( ZI_t \) is the investment specific shock, \( X_t = \frac{I_t}{I_{t-1}} \) and \( S() \) satisfies \( S', S'' \geq 0 \); \( S(1 + g) = S'(1 + g) = 0 \).

Demand for capital by firms must satisfy

\[
E_t[R_{t+1}] = \frac{E_t [F_{K,t} + (1 - \delta)Q_{t+1}]}{Q_t} \tag{18}
\]

In (18) the rhs is the gross return to holding a unit of capital in from \( t \) to \( t + 1 \). The lhs is the gross return from holding bonds, the opportunity cost of capital. We complete this
set-up with the functional form

\[ S(X) = \phi X (X_t - (1 + g))^2 \] (19)

where \( g \) is the balanced growth rate.

Owners of physical capital can control the intensity at which capital is utilized in production. As in Christiano et al. (2005) and Smets and Wouters (2007) we assume that using the stock of capital with intensity \( U_t \) produces a cost of \( a(U_t)K_t \) units of the composite final good. The functional form is chosen consistent with the literature:

\[ a(U_t) = \gamma_1(U_t - 1) + \frac{\gamma_2}{2}(U_t - 1)^2 \] (20)

and satisfies \( a(1) = 0 \) and \( a'(1), a''(1) > 0 \). Note that \( \frac{\gamma_1}{\gamma_2} = \frac{1 - \phi}{\phi} \) in the Smets and Wouters (2007) set-up. In order to compare results we will estimate \( \phi \).

2.6 The Household Problem

The Household problem is standard and can be summarized by:

Utility: \( \Lambda_t = \Lambda(C_t, L_t) \) (21)

Euler: \( \Lambda_{C,t} = \beta E_t [R_{t+1} \Lambda_{C,t+1}] \) (22)

Labour Supply: \( \frac{\Lambda_{h,t}}{\Lambda_{C,t}} = -MRS_t \equiv -\frac{W_{h,t}}{P_t} \) (23)

Leisure: \( L_t \equiv 1 - N_t \) (24)

For later use it is useful to write the Euler consumption equation as

\[ 1 = E_t [R_{t+1} D_{t,t+1}] \] (25)

2.7 Monetary Authority, Aggregation and Equilibrium

Nominal and real interest rates are related by the Fischer equation

\[ E_t[R_{t+1}] = E_t \left[ \frac{R_{n,t}}{\Pi_{t+1}} \right] \] (26)
where the nominal gross interest rate is a policy variable, typically given by a simple Taylor-type rule:

\[
\log \left( \frac{R_{n,t}}{R_n} \right) = \alpha_R \log \left( \frac{R_{n,t-1}}{R_n} \right) + \alpha_{\pi} \log \left( \frac{\Pi_t}{\Pi} \right) + \alpha_Y \log \left( \frac{Y_t}{Y} \right) + \epsilon_{m,t}
\]  

(27)

where, we define the output gap as the deviation between the output and its steady-state value.

The resource constraint must take into account relative price dispersion across varieties and wage dispersion across firms. By writing \( Y_t(f)^W = F(Z_t, N_t(j), U_tK_{t-1}) \). At firm level supply must equal demand:

\[
(1 - c)F(Z_t, \left( \frac{W_t(j)}{W_t} \right)^{-\mu} N_t, U_tK_{t-1}) = (C_t + I_t + G_t + a(U_t)K_{t-1}) \left( \frac{P_t(f)}{P_t} \right)^{-\zeta}
\]  

(28)

Integrating across all firms, taking into account that the capital-labour ratio is common across firms and that the wholesale sector is separated from the retail sector we obtain

\[
(1 - c)F(Z_t, \int_0^1 \left( \frac{W_t(j)}{W_t} \right)^{-\mu} dj N_t, U_tK_{t-1}) = (C_t + I_t + G_t + a(U_t)K_{t-1}) \int_0^1 \left( \frac{P_t(f)}{P_t} \right)^{-\zeta} df
\]  

(29)

where the price dispersion is given by \( \Delta_{p,t} = \int_0^1 \left( \frac{P_t(f)}{P_t} \right)^{-\zeta} df \) and wage dispersion is given by \( \Delta_{w,t} = \int_0^1 \left( \frac{W_t(j)}{W_t} \right)^{-\mu} dj \). As shown in Appendix B:

\[
\Delta_{p,t} = \xi_{\Pi_t} \Delta_{p,t-1} + (1 - \xi) \left( \frac{P_0}{P_t} \right)^{-\zeta}
\]  

(30)

\[
\Delta_{w,t} = \xi_{w} \Pi_{w,t} \Delta_{w,t-1} + (1 - \xi_{w}) \left( \frac{W_0}{W_t} \right)^{-\mu}
\]  

(31)

Then (28) takes the form:

\[
Y_t = (1 - c) \frac{Y_t}{\Delta_{p,t} \Delta_{w,t}} = C_t + I_t + G_t + a(U_t)K_{t-1}
\]  

(33)

\[9\] Where by simplicity we call \( Z_t \) a vector containing each type of biased and un-biased technical change defined in (4).
2.8 Representation of Price-Wage Dynamics as Difference Equations

In order to set up the model in DYNARE we need to represent the price and wage dynamics as difference equations. Both sides of the foc for pricing (8) and wage (14), are of the form considered in Appendix A. Using the Lemma, inflation dynamics are given by:

\[
\Pi_{p,t} \equiv \frac{P_t}{P_{t-1}} = \pi_t + 1
\]  

\[
\frac{P_0}{P_t} \equiv \frac{J_{p,t}}{H_{p,t}}
\]  

\[
\tilde{\Pi}_{p,t} \equiv \frac{\Pi_{p,t}}{\Pi_{p,t-1}}
\]  

\[
MS_{p,t} \equiv \frac{\zeta}{\zeta - 1} eP_t
\]

where we introduced a price mark-up shock \( MS_{p,t} \) and by:

\[
H_{p,t} - \xi_p \beta E_{t}[\tilde{\Pi}_{p,t+1}^{\zeta-1} H_{p,t+1}] = Y_t \Lambda_{C,t}
\]  

\[
J_{p,t} - \xi_p \beta E_{t}[\tilde{\Pi}_{p,t+1}^{\zeta} J_{p,t+1}] = MS_{p,t} Y_t MC_{t} \Lambda_{C,t}
\]  

\[
1 = \xi_p \tilde{\Pi}_{p,t}^{\zeta-1} + (1 - \xi_p) \left( \frac{J_{p,t}}{H_{p,t}} \right)^{1-\zeta}
\]

For staggered wage setting, symmetrically, wage dynamics are given by defining:

\[
\Pi_{w,t} \equiv \frac{W_t}{W_{t-1}} \Pi_{p,t}
\]  

\[
\frac{W_0}{W_t} \equiv \frac{J_{w,t}}{H_{w,t}}
\]  

\[
MS_{w,t} \equiv \frac{\mu}{\mu - 1} eW_t
\]

where we introduced a wage mark-up shock \( MS_{w,t} \). Aggregate wage dynamics are then
given by

\[ H_{w,t} - \xi_w \beta E_t \left[ \Pi_{w,t+1}^{\mu} \left( \frac{\Pi_{p,t+1}^{\omega}}{\Pi_{p,t}^{\omega}} \right)^{\mu-1} \right] H_{w,t+1} = N_t \Lambda C, t \] (44)

\[ J_{w,t} - \xi_w \beta E_t \left[ \Pi_{w,t+1}^{\mu} \left( \frac{\Pi_{p,t+1}^{\omega}}{\Pi_{p,t}^{\omega}} \right)^{\mu} \right] = -MS_{w,t} N_t MU_t^N \] (45)

\[ \xi_w \left[ \Pi_{w,t} \frac{\Pi_{p,t}^{\omega}}{\Pi_{p,t-1}^{\omega}} \right]^{\mu-1} + (1 - \xi_w) \left( \frac{J_{w,t}}{H_{w,t}} \right)^{1-\mu} = 1 \] (46)

2.9 The Normalized CES Production Function

The production function is assumed to be CES as in Cantore et al. (2010b) which nests Cobb-Douglas as a special case and admits the possibility of neutral and non-neutral technical change. Here we adopt the ‘re-parametrization’ procedure described in Cantore and Levine (2011) in order to normalize the CES production function:

\[ Y_t^{W} = \left[ \alpha_k (ZK_t U_t K_t)^{\psi} + \alpha_n (ZN_t N_t)^{\psi} \right]^{\frac{1}{\psi}} ; \psi \neq 0 \& \alpha_k + \alpha_n \neq 1 \\
= (ZK_t U_t K_t)^{\alpha_k} (ZN_t N_t)^{\alpha_n} ; \psi \rightarrow 0 \& \alpha_k + \alpha_n = 1 \] (47)

where \( Y_t^{W} \), \( K_t \), \( N_t \) are wholesale output, capital and labour inputs respectively at time \( t \) and \( \psi \) is the substitution parameter and \( \alpha_k \) and \( \alpha_n \) are sometimes referred as distribution parameters. As explained earlier, the terms \( ZK_t \) and \( ZN_t \) capture capital-augmenting and labour-augmenting technical progress respectively. Calling \( \sigma \) the elasticity of substitution between capital and labour,\(^{10}\) with \( \sigma \in (0, +\infty) \) and \( \psi = \frac{\sigma-1}{\sigma} \) then \( \psi \in (-\infty, 1) \). When \( \sigma = 0 \Rightarrow \psi = -\infty \) we have the Leontief case, whilst when \( \sigma = 1 \Rightarrow \psi = 0 \) (47) collapses to the usual Cobb-Douglas case.

From the outset a comment on dimensions would be useful. Technology parameters are not measures of efficiency as they depend on the units of output and inputs (i.e., is

\(^{10}\) The elasticity of substitution for the case of perfect competition, where all the product is used to remunerate factor of productions, is defined as the elasticity of the capital/labour ratio with respect to the wage/capital rental ratio. Then calling \( W \) the wage and \( R + \delta \) the rental rate of capital we can define the elasticity as follows:

\[ \sigma = \frac{d K}{d R + \delta} \frac{R + \delta}{W} \]

See La Grandville (2009) for a more detailed discussion.
not dimensionless\(^\text{11}\) and the problem of normalization arises because unless \(\psi \to 0\), \(\alpha_n\) and \(\alpha_k\) in (47) are not shares and in fact are not dimensionless.

Marginal products of labour and capital are respectively

\[
F_{N,t} = \frac{Y_W^t}{N_t} \left[ \frac{\alpha_n(ZN_tN_t)^\psi}{\alpha_k ZK_tU_tK_t^\psi + \alpha_n(ZN_tN_t)^\psi} \right] = \alpha_nZN_t^\psi \left( \frac{Y_W^t}{N_t} \right)^{1-\psi} \tag{48}
\]
\[
F_{K,t} = \frac{Y_W^t}{K_t} \left[ \frac{\alpha_k ZK_tU_tK_t^\psi}{\alpha_k ZK_tU_tK_t^\psi + \alpha_n(ZN_tN_t)^\psi} \right] = \alpha_k(U_tZK_t)^\psi \left( \frac{Y_W^t}{K_t} \right)^{1-\psi} \tag{49}
\]

The equilibrium of real variables depends on parameters defining the RBC core of the model \(\varrho\), \(\sigma_c\), \(\delta\), \(\psi\), \(\alpha_k\) and \(\alpha_n\), and those defining the NK features. Of the former, \(\varrho\), \(\psi\) and \(\sigma_c\) are dimensionless, \(\delta\) depends on the unit of time, but unless \(\psi = 0\) and the technology is Cobb-Douglas, \(\alpha_k\) and \(\alpha_n\) depend on the units chosen for factor inputs, namely machine units per period and labour units per period. To see this rewrite the wholesale firm’s foc (5) and (6) in terms of factor shares

\[
\frac{W_tN_t}{P_t^W Y_t^W} = \alpha_nZN_t^\psi \left( \frac{Y_t^W}{N_t} \right)^{-\psi} \tag{50}
\]
\[
\frac{(R_t - 1 + \delta)K_t}{P_t^W Y_t^W} = \alpha_k(U_tZK_t)^\psi \left( \frac{Y_t^W}{K_t} \right)^{-\psi} \tag{51}
\]

where \(P_t^W \equiv MC_tP_t\) is the price of wholesale output. Then we have

\[
\frac{W_tN_t}{(R_t + \delta)} = \frac{\alpha_n}{\alpha_k} \left( \frac{ZK_tU_tK_t}{ZN_tN_t} \right)^{-\psi} \tag{52}
\]

Thus \(\alpha_n\) (\(\alpha_k\)) can be interpreted as the share of labour (capital) iff \(\psi = 0\) and the production function is Cobb-Douglas. Otherwise the dimensions of \(\alpha_k\) and \(\alpha_n\) depend on those for \(\left( \frac{ZK_tU_tK_t}{ZN_tN_t} \right)^\psi\) which could be for example, (effective machine hours per effective person hours)\(^\psi\). In our aggregate production functions we choose to avoid specifying unit of capital, labour and output.\(^\text{12}\) It is impossible to interpret and therefore to calibrate or

\(^{11}\)For example for the Cobb-Douglas production function in the steady state, \(Y = K^\alpha(AN)^{1-\alpha}\), by dimensional homogeneity, the dimensions of \(A\) are (output per period)\(^{1/\alpha}\) / ((person hours per period)\(^1\) × (machine hours per period)\(^{1-\alpha}\)). For some this poses a fundamental problem with the notion of a production function - see Barnett (2004). Units can be chosen so that when \(N = 1\) and \(K = 1\), then \(Y = 1\) implying \(A = 1\). For the equilibrium to be independent of the choice of units, it follows that it must be independent of the steady state value \(A\). This is readily demonstrated in what follows.

\(^{12}\)Unlike the physical sciences where particular units are explicitly chosen so dimension-dependent parameters pose no problems. For example the fundamental constants such as the speed of light is expressed
estimate these ‘share’ parameters.

There are two ways to resolve this problem; ‘re-parameterize’ the dimensional parameters $\alpha_k$ and $\alpha_n$ so that they are expressed in terms of dimensionless ones all parameters to be estimated or calibrated (see Cantore and Levine (2011)), or ‘normalize’ the production function in terms of deviations from a steady state. We consider these in turn.

2.9.1 Re-parametrization of $\alpha_n$ and $\alpha_k$

On the balanced-growth path (bgp) consumption, output, investment, capital stock, the real wage and government spending are growing at a common growth rate $g$ driven by exogenous labour-augmenting technical change $ZN_{t+1} = (1 + g)ZN_t$, but labour input $N$ is constant.\(^{13}\) As is well-known a bgp requires either Cobb-Douglas technology or that technical change must be driven solely by the labour-augmenting variety (see, for example, Jones (2005)). Then $ZK_t = ZK$ must also be constant along the bgp.

On the bgp let capital share and wage shares in the wholesale sector be $\alpha$ and $1 - \alpha$ respectively. Then using (50) and (51) we obtain our re-parameterization of $\alpha_n$ and $\alpha_k$:

\[
\alpha_k = \alpha \left( \frac{\bar{Y}_t}{ZKU_tK_t} \right)^\psi
\]
\[
\alpha_n = (1 - \alpha) \left( \frac{\bar{Y}_t}{ZN_tN} \right)^\psi
\]

Note that $\alpha_k = \alpha$ and $\alpha_n = 1 - \alpha$ at $\psi = 0$, the Cobb-Douglas case.\(^{14}\) This completes the stationarized equilibrium defined in terms of dimensionless RBC core parameters $\varrho$, $\sigma_c$, $\psi, \pi$ and $\delta$ which depends on the unit of time, plus NK parameters. In (53) and (54) dimensional parameters are expressed in terms of other endogenous variables $Y^W$, $N$ and $K$ which themselves are functions of $\theta \equiv [\sigma, \psi, \pi, \delta, \cdots]$. Therefore $\alpha_n = \alpha_n(\theta)$, and $\alpha_k = \alpha_k(\theta)$ which expresses why we refer to this procedure as reparameterization.

There is one more normalization issue: the choice of units at some point say $t = 0$ on the steady state bgp. We use for simplicity $Y_0 = ZN_0 = ZK = 1$\(^{15}\) but, as it

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\(^{13}\)If output, consumption etc are defined in per capita terms then $N$ can be considered as the proportion of the available time at work and is therefore both stationary and dimensionless.

\(^{14}\)And as argued before if $\pi \in (0, 1)$ $\alpha_k + \alpha_n = 1$ iff $\psi = 0$.

\(^{15}\)By assuming $Y_0 = 1$ we implicitly assume $Y^W_0 = \frac{\bar{Y}_0}{\bar{W}_0}$. 

is straightforward to show that having expressed the model in terms of dimensionless parameters through re-parameterization makes the steady state ratios of the endogenous variables of the model independent of this choice.

2.9.2 The Production Function in Deviation Form

This simply bypasses the need to retain $\alpha_k$ and $\alpha_n$ and writes the dynamic production function in deviation form about its steady state as

\[
\frac{Y_t^W}{\bar{Y}_t^W} = \left[ \frac{\alpha_k(ZK_tU_tK_t)^\psi + \alpha_n(ZN_tN_t)^\psi}{\alpha_k(ZKU_tK_t)^\psi + \alpha_n(\bar{Z}N_tN_t)^\psi} \right]^{\frac{1}{\psi}}
\]

\[
\left[ \frac{\alpha_k \left( \frac{ZK_tU_tK_t}{ZKU_tK_t} \right)^\psi + \alpha_n \left( \frac{ZN_tN_t}{ZN_tN_t} \right)^\psi}{\alpha_k + \alpha_n \left( \frac{ZK_tK_t}{ZK_tK_t} \right)^\psi} \right]^{\frac{1}{\psi}}
\]

Then from (53) and (54) we can write this simply as

\[
\frac{Y_t^W}{\bar{Y}_t^W} = \left[ (1 - \alpha) \left( \frac{ZK_tU_tK_t}{ZKU_tK_t} \right)^\psi + \alpha \left( \frac{ZN_tN_t}{ZN_tN_t} \right)^\psi \right]^{\frac{1}{\psi}}
\]

(55)

as in Cantore et al. (2010b). The steady-state normalization now consists of $\bar{Z}N_0 = \bar{Y}_0 = ZK = 1^{16}$ and is characterized entirely by fixed shares of consumption, investment and government spending and by labour supply as a proportion of available time, all dimensionless quantities apart from the unit of time.

Using either of these two approaches, as showed by Cantore and Levine (2011), the steady state ratios of the endogenous variables and the dynamics of the model are not affected by the starting values of output and the two source of shocks ($\bar{Y}_0, \bar{Z}N_0, ZK$) which only represent choice of units. Crucially, this implies also that changing $\sigma$ does not change our steady state ratios and factor shares, impulse response functions are directly comparable, and parameter values are consistent with their economic interpretation.

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16Which is almost identical to the one used in Cantore et al. (2010b) although they normalize as well hours worked to 1 using the accounting identity $\bar{Y} = (\bar{R} + \delta)\bar{K} + \bar{W}\bar{N}$. 

16
2.10 Utility Function

The household utility function is chosen to be compatible with a balanced-growth steady state and allows external habit-formation:

\[ \Lambda_t = e^{B_t}((C_t - \chi C_{t-1})^{(1-\phi)(1 - N_t)^\phi})^{1-\sigma_c - 1} \]

\[ \Lambda_{C,t} = e^{B_t}(1 - \phi)(C_t - \chi C_{t-1})^{(1-\phi)(1-\sigma_c)^{-1}}((1 - N_t)^\phi(1-\sigma_c)) \] (57)

\[ \Lambda_{N,t} = -e^{B_t}\phi(C_t - \chi C_{t-1})^{(1-\phi)(1-\sigma_c)^{-1}}(1 - N_t)^\phi(1-\sigma_c)^{-1} \] (58)

Where \( \chi \) represents the habit formation parameter and \( eB_t \) a preference shock.

2.11 Shocks

To close the model we need to specify the law of motion of the shocks:

\[ \log ZK_t - \log ZK = \rho_{ZK}(\log ZK_{t-1} - \log ZK) + \epsilon_{ZK,t} \] (59)

\[ \log ZN_t - \log ZN = \rho_{ZN}(\log ZN_{t-1} - \log ZN) + \epsilon_{ZN,t} \] (60)

\[ \log ZI_t - \log ZI = \rho_{ZI}(\log ZI_{t-1} - \log ZI) + \epsilon_{ZI,t} \] (61)

\[ \log G_t - \log G = \rho_{C}(G_{t-1} - \log G) + \epsilon_{G,t} \] (62)

\[ \log eP_t - \log eP = \rho_{eP}(eP_{t-1} - eP) + \epsilon_{eP,t} \] (63)

\[ \log eW_t - \log eW = \rho_{eW}(eW_{t-1} - eW) + \epsilon_{eW,t} \] (64)

\[ \log eB_t - \log eB = \rho_{eB}(eB_{t-1} - eB) + \epsilon_{eB,t} \] (65)

In total the model has these 7 AR(1) shocks plus the shock to the monetary policy rule.

3 Estimation

We estimate the linear version of the model around zero steady state inflation by Bayesian methods using DYNARE. We use the same data set as in Smets and Wouters (2007) in first difference at quarterly frequency. Namely, these observable variables are the log differences of real GDP, real consumption, real investment and real wage, log hours worked, the log difference of the GDP deflator and the federal funds rate. As in Smets and Wouters (2007), hours worked are derived from the index of average hours for the non-farm business sector.
and we divide hourly compensation from the same sector by the GDP price deflator to obtain the real wage. All series are seasonally adjusted. All data are taken from the FRED Database available through the Federal Reserve Bank of St. Louis and the US Bureau of Labour Statistics. The sample period is 1984:1-2004:4. A full description of the data used can be found in Smets and Wouters (2007).

The corresponding measurement equations for the 7 observables are, using lower case letters to express variables in log-deviations from the steady state:

\[
\begin{align*}
    dy &= y_t - y_{t-1} + ctrend \\
    dc &= c_t - c_{t-1} + ctrend \\
    di &= i_t - i_{t-1} + ctrend \\
    dw &= w_t - w_{t-1} + ctrend \\
    \Pi_{obs} &= \pi_t + conspire \\
    R_{obs} &= rn_t + consrn \\
    h_{obs} &= n_t + conslab
\end{align*}
\]

The four observable are taken in first difference while inflation, nominal interest rate and hours worked are used in levels. We introduce a common trend to the real variables and a specific one to inflation, nominal interest rate and hours worked.

### 3.1 Bayesian Methods

Bayesian estimation entails obtaining the posterior distribution of the model’s parameters, say \( \theta \), conditional on the data. Using the Bayes’ theorem, the posterior distribution is obtained as:

\[
p(\theta|Y^T) = \frac{L(Y^T|\theta)p(\theta)}{\int L(Y^T|\theta)p(\theta)d\theta}
\]

where \( p(\theta) \) denotes the prior density of the parameter vector \( \theta \), \( L(Y^T|\theta) \) is the likelihood of the sample \( Y^T \) with \( T \) observations (evaluated with the Kalman filter) and \( \int L(Y^T|\theta)p(\theta)d\theta \) is the marginal likelihood. Since there is no closed form analytical expression for the posterior, this must be simulated.

One of the main advantages of adopting a Bayesian approach is that it facilitates a
formal comparison of different models through their posterior marginal likelihoods, computed using the Geweke (1999) modified harmonic-mean estimator. For a given model \( m_i \in M \) and common data set, the marginal likelihood is obtained by integrating out vector \( \theta \),

\[
L(Y^T|m_i) = \int_{\Theta} L(Y^T|\theta, m_i) p(\theta|m_i) d\theta
\]

(74)

where \( p_i(\theta|m_i) \) is the prior density for model \( m_i \), and \( L(Y^T|m_i) \) is the data density for model \( m_i \) given parameter vector \( \theta \). To compare models (say, \( m_i \) and \( m_j \)) we calculate the posterior odds ratio which is the ratio of their posterior model probabilities (or Bayes Factor when the prior odds ratio, \( \frac{p(m_i)}{p(m_j)} \), is set to unity):

\[
PO_{i,j} = \frac{p(m_i|Y^T)}{p(m_j|Y^T)} = \frac{L(Y^T|m_i)p(m_i)}{L(Y^T|m_j)p(m_j)}
\]

(75)

\[
BF_{i,j} = \frac{L(Y^T|m_i)}{L(Y^T|m_j)} = \exp(LL(Y^T|m_i)) / \exp(LL(Y^T|m_j))
\]

(76)

in terms of the log-likelihood. Components (75) and (76) provide a framework for comparing alternative and potentially misspecified models based on their marginal likelihood. Such comparisons are important in the assessment of rival models, as the model which attains the highest odds outperforms its rivals and is therefore favoured.

Given Bayes factors, we can easily compute the model probabilities \( p_1, p_2, \ldots, p_n \) for \( n \) models. Since \( \sum_{i=1}^{n} p_i = 1 \) we have that \( \frac{1}{p_1} = \sum_{i=2}^{n} BF_{i,1} \), from which \( p_1 \) is obtained. Then \( p_i = p_1 BF(i,1) \) gives the remaining model probabilities.

### 3.2 Likelihood Comparison of Models

We compare 4 different model specifications in order to see if the introduction of factor substitutability and/or the biased technical change improves the fit of the estimation. In the first row of Table 1 we present the likelihood density of the model with the CD production function where only the labour-augmenting technology shock is present. In the second row we introduce the CES and calibrate the elasticity of substitution to 0.4, following the literature as in Cantore *et al.* (2010b), and introduce the capital-augmenting shock whilst in rows 3 and 4 we estimate \( \sigma \) in a model with and without the latter shock. Strictly speaking a meaningful likelihood comparison that provides information about \( \sigma \)
is only possible between row 1 and 3 (where we can compare like for like).

Table 1 reveals that Models with the CES production function clearly outperforms its CD counterpart with a posterior probability of 100%. This suggests that incorporating a CES production function offers substantial improvements in terms of the model fitness to the data in the US economy. The differences in log marginal likelihood are substantial. For example, the log marginal likelihood difference between the first two specifications is 10.60 corresponding to a posterior Bayes Factor of 33735. As suggested by Kass and Raftery (1995), the posterior Bayes Factor needs to be at least $e^3 \approx 20$ for there to be a positive evidence favouring one model over the other.

<table>
<thead>
<tr>
<th>Model</th>
<th>$\sigma$</th>
<th>Technology shocks</th>
<th>Log data density</th>
<th>Difference with CD</th>
</tr>
</thead>
<tbody>
<tr>
<td>CD</td>
<td>1</td>
<td>ZL</td>
<td>-460.58</td>
<td>0</td>
</tr>
<tr>
<td>-</td>
<td>calibrated=0.4</td>
<td>ZK &amp; ZL</td>
<td>-449.98</td>
<td>10.60</td>
</tr>
<tr>
<td>CES1</td>
<td>estimated=0.16</td>
<td>ZL</td>
<td>-447.60</td>
<td>12.98</td>
</tr>
<tr>
<td>CES2</td>
<td>estimated=0.16</td>
<td>ZK &amp; ZL</td>
<td>-447.84</td>
<td>12.74</td>
</tr>
</tbody>
</table>

Table 1: Marginal Likelihood comparison between CD and CES specifications

3.3 Estimation Results

The joint posterior distribution of the estimated parameters is obtained in two steps. First, the posterior mode and the Hessian matrix are obtained via standard numerical optimization routines. The Hessian matrix is then used in the Metropolis-Hastings (MH) algorithm to generate a sample from the posterior distribution. Two parallel chains are used in the Monte-Carlo Markov Chain Metropolis-Hastings (MCMC-MH) algorithm. Thus, 250,000 random draws (though the first 30% ‘burn-in’ observations are discarded) from the posterior density are obtained via the MCMC-MH algorithm, with the variance-covariance matrix of the perturbation term in the algorithm being adjusted in order to obtain reasonable acceptance rates (between 20%-30%).

Estimation results from posteriors maximization are presented in Appendix C. We used the same priors as Smets and Wouters (2007) for common parameters whereas we used a loose prior for the elasticity of substitution between capital and labour in order to see if the data are informative about the value of this parameter. A few structural parameters are kept fixed or calibrated based on some parameters being estimated in the estimation procedure, in accordance with the usual practice in the literature (see Table 2).
This is done so that the calibrated parameters reflect steady state values of the observed variables.

<table>
<thead>
<tr>
<th><strong>Calibrated parameter</strong></th>
<th><strong>Symbol</strong></th>
<th><strong>Value</strong></th>
</tr>
</thead>
<tbody>
<tr>
<td>Discount factor</td>
<td>$\beta$</td>
<td>0.99</td>
</tr>
<tr>
<td>Depreciation rate</td>
<td>$\delta$</td>
<td>0.025</td>
</tr>
<tr>
<td>Growth rate</td>
<td>$g$</td>
<td>$\delta$</td>
</tr>
<tr>
<td>Substitution elasticity of goods</td>
<td>$\zeta$</td>
<td>7</td>
</tr>
<tr>
<td>Substitution elasticity of labour</td>
<td>$\mu$</td>
<td>7</td>
</tr>
<tr>
<td>Variable capital utilization</td>
<td>$\gamma_1$</td>
<td>$\frac{1}{\beta} + \delta - 1$</td>
</tr>
<tr>
<td>Fixed cost</td>
<td>$c$</td>
<td>$1 - MC = 0.1429$</td>
</tr>
<tr>
<td>Preference parameter</td>
<td>$\varrho$</td>
<td>$\frac{1-h}{1-h(v^1(1-\chi)/\alpha-1)}$</td>
</tr>
</tbody>
</table>

**Implied steady state relationship**

- Government expenditure-output ratio $g_y$
- Investment-output ratio $i_y$
- Consumption-output ratio $c_y$ $1 - g_y - i_y$

Table 2: Calibrated Parameters

First we focus on the posterior estimates obtained using the most general CES model, CES2. As shown in Tables 6 and 7, the point estimates under the CES assumption are tight and plausible. In particular, focusing on the parameters characterizing the degree of price stickiness and the existence of real rigidities, we find that the price indexation parameters are estimated to be smaller than assumed in the prior distribution (in line with those reported by Smets and Wouters (2007)). The estimates of the $\gamma'$s imply that inflation is intrinsically not very persistent in the CES model specifications. The posterior mean estimates for the Calvo parameters, $\xi_p$ and $\xi_w$, imply an average price contract duration of around 2.27 quarters and an average wage contract duration of around 1.98 quarters, respectively. These results are in general consistent with the findings from empirical works on the DSGE modelling in the US economy. It is interesting to note that the risk-aversion parameter ($\sigma_c$) is estimated to be less than assumed in the prior distribution, indicating that the inter-temporal elasticity of substitution (proportional to $1/\sigma_c$) is estimated to be about 0.85 in the US, which is plausible as suggested in much of RBC literature. As expected, the policy rule estimates imply a fairly strong response ($\alpha_\pi$) to expected inflation by the US Fed Reserve and the degree of interest rate smoothing ($\alpha_r$) is fairly strong.

Figure 14 in Appendix C plots the prior and posterior distributions for the above CES model. The location and the shape of the posterior distributions are largely independent.
of the priors we have selected since priors are broadly less informative. Most of the posterior distributions are roughly symmetric implying that the mean and median coincide. According to Figure 14, there is little information in the data for some parameters where prior and posterior overlap. Perhaps the most notable finding comes from the estimation of the parameter $\sigma$ - our key parameter in the CES setting. As a result of assuming a very diffuse prior with large standard deviation, we find that the data is very informative about this parameter (as clearly shown in the figure, curves do not overlap each other and are very different) and the point estimate of $\sigma$ in Table 7 is close to the plausible values. This further provides strong evidence to support the empirical importance of the CES assumption.

We now turn on the comparisons between parameter estimates under CD and CES specification. Parameter posteriors that are quantitatively different$^{17}$ from the estimation using a Cobb-Douglas specification are underlined in Tables 6 and 7.

Starting with the parameters related to the exogenous shocks (Table 6) we notice that the estimated standard deviations of the newly introduced capital-augmenting technology shock is very small but, probably because of its introduction, the standard deviation of the investment specific shock reduces significantly (from 5.49 in the CD specification to 3.28 in the CES case). The two shocks are clearly related and it is very likely that when $Z_K$ is absent $Z_I$ is capturing “capital-biased” technological progress. We also notice that the estimated standard deviations of the mark-up shocks are lower under the CES specification and the standard deviation of the preference shock is 0.6 lower. The autoregressive parameters of the exogenous shocks are not affected significantly by the CES choice.

Posterior estimation of the investment adjustment costs parameter ($\phi^X$) reduces by 1.32 points when we estimate the model under CES showing once again how introducing factor-biased technical change affects significantly the estimation of ‘investment-related’ parameters. The parameters of the utility function also appear to be affected by the choice of the production function ($\sigma_c$ reduces by 0.87 and $\chi$ reduces by around 0.3). Regarding the parameters associated with sticky prices and wages only the probability of no price-adjustment ($\xi_p$) changes significantly, decreasing from 0.80 to 0.56. Monetary

$^{17}$Difference in posteriors up to 0.05 were not considered quantitatively relevant here.
policy weights (except the weight on inflation which increases slightly), real and nominal trends estimations are not affected by the introduction of factor substitutability and biased technical change.

4 Model Validation

After having shown the model estimates and the assessment of relative model fit to its other rivals with different restrictions, we use them to investigate a number of key macroeconomic issues in the US. The model favoured in the space of competing models may still be poor (potentially misspecified) in capturing the important dynamics in the data. To further evaluate the absolute performance of one particular model against data, it is necessary to compare the model’s implied characteristics with those of the actual data (and an identified VAR model).

In this section, we address the following questions: (i) can the models capture the underlying characteristics of the actual data? (ii) what are the impacts of the structural shocks on the main macroeconomic time series?

4.1 Standard Moment Criteria

Summary statistics such as first and second moments have been standard as means of validating models in the literature on DSGE models, especially in the RBC tradition. As the Bayes factors (or posterior model odds) are used to assess the relative fit amongst a number of competing models, the question of comparing the moments is whether the models correctly predict population moments, such as the variables’ volatility or their correlation, i.e. to assess the absolute fit of a model to macroeconomic data.

To assess the contributions of assuming different specifications of production function in our estimated models, we compute some selected second moments and present the results in this subsection. Table 3 presents the (unconditional) second moments implied by the above estimations and compares with those in the actual data. In particular, we compute these model-implied statistics by solving the models at the posterior means obtained from estimation. The results of the model’s second moments are compared with the second moments in the actual data to evaluate the models’ empirical performance.

In terms of the standard deviations, Model CD generates relatively high volatility
compared to the actual data (except for the interest rate and the CD production assumes constant labour share). Overall, the estimated models are able to reproduce broadly acceptable volatility for the main variables of the DSGE model and all model variants can successful replicate the stylized fact in the business cycle research that investment is more volatile than output whereas consumption is less volatile. In line with the Bayesian model comparison, the NK models with CES technology fit the data better in terms of implied volatility, getting closer to the data in this dimension (we highlight the ‘best’ model (performance) in bold). Note that both of our CES models clearly outperform the CD model in capturing the volatilities of all variables except for empolyment and does extremely well at matching the consumption, investment and real wage volatilities in the data. Furthermore by not imposing a constant labour share as in the CD model we are capable of capturing about half the volatility observed in the data. As suggested by the likelihood comparison, the differences in generating the moments between the CES specification with only the shock ZK and the CES with both ZK and ZL shocks are qualitatively very small.

<table>
<thead>
<tr>
<th>Model</th>
<th>Output</th>
<th>Consumption</th>
<th>Investment</th>
<th>Real wage</th>
<th>Inflation</th>
<th>Interest rate</th>
<th>Employment</th>
<th>Labour share</th>
</tr>
</thead>
<tbody>
<tr>
<td>Data</td>
<td>0.59</td>
<td>0.53</td>
<td>1.80</td>
<td>0.60</td>
<td>0.25</td>
<td>0.64</td>
<td>2.47</td>
<td>3.82</td>
</tr>
<tr>
<td>Model CD</td>
<td>0.79</td>
<td>0.68</td>
<td>1.89</td>
<td>0.76</td>
<td>0.43</td>
<td>0.47</td>
<td>2.99</td>
<td>0</td>
</tr>
<tr>
<td>Model CES1</td>
<td>0.71</td>
<td>0.64</td>
<td>1.88</td>
<td>0.65</td>
<td>0.42</td>
<td>0.48</td>
<td>4.18</td>
<td>2.57</td>
</tr>
<tr>
<td>Model CES2</td>
<td>0.71</td>
<td>0.64</td>
<td>1.88</td>
<td>0.65</td>
<td>0.43</td>
<td>0.49</td>
<td>4.30</td>
<td>2.63</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Cross-correlation with Output</th>
</tr>
</thead>
<tbody>
<tr>
<td>Data</td>
</tr>
<tr>
<td>Model CD</td>
</tr>
<tr>
<td>Model CES1</td>
</tr>
<tr>
<td>Model CES2</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Autocorrelations (Order=1)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Data</td>
</tr>
<tr>
<td>Model CD</td>
</tr>
<tr>
<td>Model CES1</td>
</tr>
<tr>
<td>Model CES2</td>
</tr>
</tbody>
</table>

Table 3: Selected Second Moments of the Model Variants

Table 3 also reports the cross-correlations of the eight observable variables vis-a-vis output. All models perform successfully in generating the positive contemporaneous correlations of consumption and investment observed in the data. The data report that the real wage is countercyclical. Both our CES models, perform successfully in generating the negative contemporaneous real wage-output correlation observed in the data, suggesting
that assuming CES production helps to capture this labour market dimension. The highlighted numbers in this category together with the evidence above show that the feature of CES in the model is particularly important in characterizing the investment and real wage dynamics. However, as evidence from the implied volatilities confirms, the main shortcoming of all the models, including the preferred ones, is the difficulty at replicating the cross-correlations of output with employment and mimicking the volatility observed in the employment data. This is not a very surprising result because there are no labour market frictions assumed in all the models under investigation. All models fail to predict the positive correlation between output and interest rate and CES models have problem in replicating the negative contemporaneous cross-relation between inflation and output. This is consistent with the work of Smets and Wouters (2003) as they find that the implied cross-correlations with the interest rate and inflation are not fully satisfactory. Nevertheless, looking at the highlighted figures, the results in general show that, in the models where the CES specification is present, cross-correlations of endogenous variables are generally closer to those in the actual data. This further strengthens the empirical relevance of the CES assumption.

All models appear to match well the autocorrelations (order=1) of the endogenous variables output, investment, inflation, interest rate and employment. Using the CES model, output is less autocorrelated at order 1, whilst investment and inflation are more autocorrelated than those in the data.

To summarize this sub-section, overall Bayesian Maximum-likelihood based methods suggest that the ability of the model’s second moments to fit those of the data generally match the outcome of the likelihood race. The two CES models deliver a better fit to the actual data for most of the second moment features in Table 3. However, as noted above, the differences in the second moments of the two competing CES variants are very small.

4.2 Autocorrelation Functions

We have so far considered autocorrelation only up to order 1. To further illustrate how the estimated models capture the data statistics and persistence in particular, we now plot the autocorrelations up to order 10 of the actual data and those of the endogenous variables generated by the model variants in Figure 3.
Of particular interest is that, when assuming CES production, the implied autocorrelograms fit very well the observed autocorrelation of inflation, interest rate and investment, whilst the CD model generates much less sluggishness and is less able to match the autocorrelation of inflation and interest rate observed in the data from the second lag onwards. Overall we find that, with nominal price stickiness in the models and highly correlated estimated price markup shocks, inflation persistence can be captured closely in DSGE models when CES production is assumed.

When it comes to output, all models perform well in matching the observed output persistence. However, the employment (hours worked) is more autocorrelated in all models than in the data, but now the model with the CD feature gets closer to the data for higher order autocorrelation. All models match reasonably well the autocorrelations of investment, consumption and real wage. To summarize, the results for higher order autocorrelations for the most part show that the DSGE models under the more general CES
production function are better at capturing the main features of the US data, strengthening the argument that the assumption of CES helps to improve the model fit to data.

4.3 Comparison with a DSGE-VAR

An alternative way of validating the model performance is to follow Del Negro and Schorfheide (2004) and Del Negro et al. (2007) and to compare it with a hybrid model that is a combination of an unrestricted VAR and the VAR implied by the estimated DSGE model. We then go on to compare the estimated DSGE model this ‘DSGE-VAR’ in terms of their impulse response functions (IRFs). We also investigate the impact on IRFs of changing the production function. Since we have demonstrated that there is little difference between the CES variants in terms of matching the data, this exercise is only performed for the best CES (one-shock) model.

The DSGE-VAR approach uses DSGE model itself to construct a prior distribution for the VAR coefficients so that DSGE-VAR estimates are tilted toward DSGE model restriction, thus identifying the shocks for the IRFs. This method constructs the DSGE prior by generating dummy observations from the DSGE model, and adding them to the actual data and leads to an estimation of the VAR based on a mixed sample of artificial and actual observations. The ratio of dummy over actual observations (called the hyper-parameter $\lambda$) controls the variance and therefore the weight of the DSGE prior relative to the sample. For extreme values of this parameter (0 or $\infty$) either an unrestricted VAR or the DSGE is estimated. If $\lambda$ is small the prior is diffuse. When $\lambda = \infty$, we obtain a VAR approximation\(^{18}\) of the log-linearized DSGE model. As $\lambda$ becomes small the cross-equation restrictions implied by the DSGE model are gradually relaxed. The empirical performance of a DSGE-VAR will depend on the tightness of the DSGE prior. Details on the algorithm used to implement this DSGE-VAR are to be found in Del Negro and Schorfheide (2004) and Del Negro et al. (2007).

We fit our VAR to the same data set used to estimate the DSGE model. We consider a VAR with 4 lags.\(^{19}\) We use a data-driven procedure to determine the tightness of prior endogenously based on the marginal data density. Our choices of the optimal $\lambda$ are 1.0 for

\(^{18}\)The accuracy of the approximation depends on the invertibility of the DSGE model’s moving average components and on the number of autoregressive lags included (Del Negro et al. (2007)).

\(^{19}\)This choice of the lag length maximizes the marginal data density associated with the DSGE-VAR($\hat{\lambda}$).
Model CD and 1.2 for Models CES, and this is found by comparing different VAR models using the estimates of the marginal data density (Figure 4). In particular, we iterate over a grid that contains the values of \( \lambda = [0.43; 0.8; 1; 1.1; 1.2; 1.4; 1.5; 1.6; 2; 5; 10; \infty] \), we find that \( \lambda = 1.0 \) and \( \lambda = 1.2 \) have the highest posterior probability for all models. Note that 0.43 is the smallest \( \lambda \) value for which we have a proper prior. Overall, the DSGE-VAR(4) with \( \lambda = 1.0, 1.2 \) have the highest posterior probability.\(^{20}\) This implies that the mixed sample that is used to estimate the VAR has a higher weight on the DSGE model (artificial observations) than on the VAR (actual observations). \( \hat{\lambda} \) represents how much the economic model (DSGE) is able to explain the real data (here \( \hat{\lambda}_{CES} > \hat{\lambda}_{CD} \)). More importantly, the results from comparing across different models show that Models CES consistently outperform Model CD and CD is strongly rejected in favour of CES when the weight on the DSGE model becomes higher (when \( \lambda \) tends to \( \infty \)).

The improved performance of the CES models over the CD applies at the optimum \( \lambda \), and the log of the marginal likelihood difference (LL) is around 8. This implies a Bayes factor of \( \exp(8) \approx 2981 \) favouring the CES models. Beyond the optimum the LL far more rapidly for the CD model reaching a difference close to that for the actual linearized DSGE model reported earlier. Overall the LL plots then confirm the fact and the degree to which the CES models are less misspecified.

Turning to the impulse responses, Figures 15–22 in Appendix D depict the mean responses corresponding to a positive one standard deviation shock. The endogenous variables of interest are the observables in the estimation and each response is for a 20 period (5 years) horizon. All DSGE impulse responses are computed simulating the vector of DSGE model parameters at the posterior mean values reported in Tables 6 and 7. The impulse responses for VAR(4) are obtained using the DSGE-VAR identification procedure using the best-fitting DSGE-VAR(4) with \( \lambda = 1.2 \). The surface between the black dashed lines in each panel covers the space between the first and ninth posterior deciles of the VAR’s IRFs.

Overall, we find that the sign and magnitude of the DSGE and VAR impulse responses are quite similar implying that the DSGE model mimics the DSGE-VAR model in, at least, some dimensions. Clearly the most important difference comes from fluctuations

\(^{20}\)Alternatively, one can simply find the ‘optimal’ \( \hat{\lambda} \) by estimating the parameter \( \lambda \) as one of the deep parameters (see, for more details, Adjemian et al. (2008)).
in factor shares under the CES specification. Fluctuations of shares translate as well in different IRFs of interest rate and wage in the two models. Indeed in Figure 15 (labour-augmenting shock) we can see how the response of the nominal interest rate is significantly different. Figure 17 (investment shock) turns out be very interesting, given the highlighted difference in the estimation of parameters related to investment in section 3.3. We note how both, wage and interest rate, present a more sluggish response to an investment specific shock under CES and, as a result, a quite different response of consumption and inflation. Although they have similar shapes, the IRFs under CES are quantitatively different. In particular, the shock amplifies the initial responses of some variables. The disagreement in the IRFs to this particular shock can be explained by the large estimate of the shock standard deviation reported in Table 6.

If we look closely at the responses to monetary policy shock (Figure 10), we find that both in the models and in the data, consumption, investment and output display a hump-shaped response to the policy shock. For this shock the IRFs from both DSGE models are in agreement with the data and when comparing the performance from CES and CD the difference is quantitatively small. Turning to the IRFs to productivity shock

Figure 4: Marginal Likelihood as a Function of $\lambda$
(labour-augmenting), using both the CES and CD settings is able to provide reasonable responses. More importantly, if we compare each response from each DSGE model with that from the VAR model, we find that overall the discrepancy between VAR and DSGE is relatively smaller under the CES production assumption. This suggests that the DSGE model misspecification is larger with the CD production than with the CES. If we study carefully the responses to the other shocks, we can generally find the similar conclusion that CES helps reduce the discrepancy although the IRFs to the investment-specific shock are the exception. In Figure 19 we can see how the reaction of wage and interest rate are once again very different after a government spending shock. Indeed it the introduction of CES specification increases the magnitude of the responses a lot. As a result of those differences in the dynamics of factor prices we notice how investment, consumption and inflation also present an increase in the amplification of the government spending shock. For price and wage mark-up shock (Figures 21 and 22) we notice non-negligible differences in the responses of interest rate, inflation and hours worked.

To sum up, there also exists some evidence from IRFs in favour of the CES assumption in DSGE models, but the evidence from the IRFs is not as strong as that obtained by comparing the moments and the marginal likelihood comparison amongst models which more clearly reject the CD specification.

5 Variance Decomposition of Business Cycle Fluctuations

This section investigates the contribution of each of the structural shocks to the forecast error variance of the observable variables in the models, i.e. the underlying sources of fluctuations, at various horizons. The results are based on the models’ posterior distribution reported in Tables 6 and 7. The results are summarized in Figures 23 and Tables 4, 5 in Appendix D.

In the short run, within a year (t=1,4), movements in real GDP are primarily driven by the exogenous government spending shock and supply-side shocks (with the dominant influence of around 70%). For instance, most of the unexpected output fluctuations are mainly explained by the government spending shock (around 30-40% depending on the model specifications) and the two mark-up shocks (around 10-20% from each shock). Within the one-year horizon, the government spending shock dominate, accounting for
the biggest part of the output forecast error variance under Model CD.

Not surprisingly, in the medium to long run the supply shocks and the exogenous spending shock together continue to dominate, but the contribution of government spending shock to output variability become smaller from medium to long run and the wage mark-up shock explain a bigger part of the long-run variations in output. This is especially the case when the model adopts the CES function form. In contrast, the monetary policy shock and preference shock have little impact on output variability, regardless of forecast horizon. Based on the estimation sample, the investment shock and labour-augmenting productivity shock are found to be moderate factors behind both short-run and longer-run movements in output (account for around 10%-11% and around 7%-8%, respectively). In terms of determining the main driving forces of output, we compare all three specifications under investigation and find similar and consistent results. Results from the CES model with two technology shocks show that, in line with its estimated standard deviation, the capital-augmenting technology shock offers a qualitatively very small impact to the output fluctuations.

Under the estimated interest rate rule we find that the monetary policy shock is by far the most determinant influence to the nominal interest rate in the short run (1 quarter), which explains around 40% of its variance under the assumption of CES technology. The second largest component is the investment shock. In fact, this shock starts to dominate from the medium to long run and the contributions of policy shock declines quite sharply toward the longer horizon. However, models with CES specification tells a slightly different story. They show that the main driving factor in the long-run development of nominal interest rate is mark-up shocks. As expected, the preference shock explains a big part of consumption variation in the short-medium run, whilst the investment shock contributes the largest fraction of investment movements in the short run (within a year). In terms of explaining the consumption and investment fluctuations, we do not find notable differences whether CD or CES is assumed.

Interestingly, the CD model suggests that the shocks that explain most of inflation variance in the short run are the two mark-up shocks but the investment shock becomes more influential from medium to long run (nearly 20%). In contrast, our estimated CES models show that inflation fluctuations are mostly affected by the policy shock in the
short and medium runs but the the main driving factor becomes the wage mark-up shock, dominates the investment shock, in the long run. There are only limited effects on inflation from the productivity shocks and various demand shocks. One possible reason for this is, according to Smets and Wouters (2007), that the estimated slope of the NK Phillips curve is small so that only large and persistent changes in the marginal cost will have an impact on inflation. Finally, the short to medium run contributions of the selected shocks to the forecast error variance of hours worked are broadly similar across the three models. In the long run, there are different results between the CES and CD assumptions. In particular, the model adopting the CD production suggests that the two mark-up shocks both contribute significantly to the variation in hours worked whereas, when we use the more general CES setting in our DSGE model, the wage mark-up shock clearly becomes the completely dominant force behind the long-run movements in hour worked from the mid-80’s onwards. This finding from the CES model seems to be more plausible.

Overall, the results in this exercise mainly show that, over the sample period, the supply-side shocks account for much of the medium to long-run variance which is in line with the business cycle literature and identified VAR studies in industrialized economies. The disturbances from government expenditures are also important at explaining the dynamics of macro-variables in the US economy.

6 Optimal Policy

What are the policy implications of capital-labour substitutability for monetary policy? In this section we address this question. We examine three monetary policy regimes: the ex ante optimal policy with commitment (the Ramsey problem), the time consistent optimal policy (discretion) and a Taylor-type interest rate rule of the form (27) without the i.i.d, shock and with welfare-optimized feedback parameters. Notice that this is a Taylor-type rules as in Taylor (1993) that responds to deviations of output from its deterministic steady state values and not from its flexi-price outcomes. Such a rules has the advantage that it can be implemented using readily available macro-data series rather than from model-based theoretical constructs (see Schmitt-Grohe and Uribe (2007)). We first consider policy ignoring zero-lower bound (ZLB) issues for the nominal interest rate.
6.1 Optimal Policy Ignoring the Interest Rate Zero Lower Bound

Table 7 sets out the welfare outcomes using the inter-temporal household utility for three specifications of the elasticity of substitutions $\sigma$: the estimated value $\sigma = 0.159$, the case of a Cobb-Douglas production function $\sigma = 1$ and a value $\sigma = 2$ indicating a degree of substitutability between capital and labour. Apart from these changes, all other parameters are set at their estimated means of the posterior distributions for the best-fitting model with $\sigma = 0.159$.\textsuperscript{21}

<table>
<thead>
<tr>
<th>$\sigma$</th>
<th>Policy</th>
<th>Rule: $[\rho_r, \theta_\pi, \theta_y]$</th>
<th>Welfare Loss $c_e(%)$</th>
<th>$\sigma^2$</th>
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</thead>
<tbody>
<tr>
<td>0.159</td>
<td>Optimal (OPT)</td>
<td>n.a.</td>
<td>4.609</td>
<td>0.542</td>
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<tr>
<td>0.159</td>
<td>Time Consistent (TC)</td>
<td>n.a.</td>
<td>6.328</td>
<td>0.037</td>
</tr>
<tr>
<td>0.159</td>
<td>Optimized Simple (SIM)</td>
<td>[0.851, 0.703, 0.012]</td>
<td>4.980</td>
<td>0.008</td>
</tr>
<tr>
<td>1</td>
<td>Optimal (OPT)</td>
<td>n.a.</td>
<td>4.630</td>
<td>0.540</td>
</tr>
<tr>
<td>1</td>
<td>Time Consistent (TC)</td>
<td>n.a.</td>
<td>6.402</td>
<td>0.038</td>
</tr>
<tr>
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<td>4.989</td>
<td>0.008</td>
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<td>2</td>
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<td>0.557</td>
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<tr>
<td>2</td>
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<td>n.a.</td>
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<td>0.037</td>
</tr>
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<td>Optimized Simple (SIM)</td>
<td>[0.740, 0.666, 0.01]</td>
<td>5.233</td>
<td>0.008</td>
</tr>
</tbody>
</table>

Table 7. Optimal Policy with and without Commitment. No ZLB Imposed

From Table 7 we see that the gains from commitment are modest, with a consumption equivalent of order $c_e = 0.04\%$ and the costs of simplicity are very small with $c_e$ less that 0.01\%. This is true for all three specifications of $\sigma$ and the welfare implications of raising $\sigma$ from 0.159 to 2 are tiny. The main effect of allowing a greater degree of capital-labour substitutability is to flatten the demand for labour relationship and allow real wage changes to bring about larger changes in employment and output. This brings about higher volatility of the real economy and a higher cost of business cycle fluctuations and reflected in the increase in the welfare loss. The latter is very small, but more crucially the variance of the nominal interest rate rises. Monetary policy is more active, in other words,

\textsuperscript{21}To derive the welfare in terms of a consumption equivalent percentage increase ($c_e \equiv \frac{\Delta C}{C} \times 10^{2}$), expanding $\Lambda(C, 1 - N_t)$ as a Taylor series, a $\Delta \Lambda = A C \Delta C = CA_C C_e \times 10^{-2}$. Losses $X$ reported in the Table are of the order of variances expressed as percentages and have been scaled by $1 - \beta$. Thus $X \times 10^{-4} = \Delta \Lambda$ and hence $c_e = \frac{\Delta C}{CA_C C_e}$. For the steady state of this model, $CA_C = 0.4689$. It follow that a welfare loss difference of $X = 1$ gives a consumption equivalent percentage difference of around $c_e = 0.02\%$. 

33
as capital-labour substitutability increases. Between policy regimes the constraint implied
by the Taylor rule lessons the immediate the response of the interest rate to shocks (see
in the impulse responses described next) resulting in a lower variance of the interest rate.
But a lack of commitment induces the opposite and the volatility of the interest rate is
very high. This has important implications for the ZLB considered in the next subsection.

Figures 5 - 10 show the impulse responses to a 1% increase in labour-augmenting
technology, government spending, price and wage mark-up, investment and preference
shocks respectively. As usual, all responses are expressed in percentage deviations about
the deterministic steady state of the particular variable under consideration.
Figure 5: IRFs with Labour Technology Shock

Figure 6: IRFs with Government Spending Shock
We see the familiar impulse responses in a NK model across all three monetary policy regimes. For a technology shock output immediately rises and, inflation falls. The optimal policy is to commit to a sharp monetary relaxation before gradually returning to the steady state. Both consumption and leisure rise (the latter a familiar result in the NK literature) and hours fall. The productivity shock results in a fall in the marginal cost and a rise in the real wage. Consumption and investment rise, the latter in response to a fall in the real interest rate. Real variables - output, hours and consumption differ little between optimal and time consistent policy for all shocks which explains the small welfare differences in Table 7 for all shocks combined.

With government spending held fixed at its steady state we can explore a fiscal stimulus in Figure 6. The increase in government spending is financed by non-distortionary tax. An increase in demand acts as a fiscal stimulus; with $G/Y = 0.2$ in the steady state the impact multiplier is below unity in our estimated model and almost identical across all policy regimes. Consumption is crowded out, inflation rises and more so for the time consistent policy which elicits an interest rate rise, again for all regimes.

For a both price and wage mark-up shocks output, consumption, investment, hours fall. Inflation and the nominal interest rise, except for the optimal policy where we see an immediate fall in the latter before it resumes its path below its steady state. The investment shock causes output and hours to rise but initially crowds out consumption. Finally the preference shock causes the marginal utility of consumption to rise an that of work effort to fall. Subsequently consumption increases, hours fall and output and investment decline.

\[ \Delta Y_t = \frac{Y_t}{G_t} \times \text{irf.} \]
Figure 7: IRFs with Price Mark-up Shock

Figure 8: IRFs with Wage Mark-up Shock
Figure 9: IRFs with Investment Shock

Figure 10: IRFs with Preference Shock
6.2 Interest Rate Zero Lower Bound Considerations

Table 7 indicates that the aggressive nature of the optimal rules leads to high interest rate variances resulting in a ZLB problem for all the rules to some extent. From the table with our zero-inflation steady state and nominal interest rate of 1% per quarter, optimal policy variances between 0.137 and 2.28 of a normally distributed variable imply a probability per quarter of hitting the ZLB in the range [0.0035, 0.255]. At the upper end of these ranges the ZLB would be hit every year on average. In this subsection we address this issue.

Our LQ set-up for a given set of observed policy instruments \( w_t \) considers a linearized model in a general state-space form:

\[
\begin{bmatrix}
    z_{t+1} \\
    E_t x_{t+1}
\end{bmatrix} = A \begin{bmatrix}
    z_t \\
    x_t
\end{bmatrix} + B w_t + \begin{bmatrix}
    u_{t+1} \\
    0
\end{bmatrix}
\]  

(77)

where \( z_t, x_t \) are vectors of backward and forward-looking variables, respectively, \( w_t \) is a vector of policy variables, and \( u_t \) is an i.i.d. zero mean shock variable with covariance matrix \( \Sigma_u \).

Let \( y_t^T \equiv [z_t, x_t, w_t] \). Then welfare-based quadratic large-distortions approximation to welfare loss function at time \( t \) by \( E_t[\Omega_t] \) where

\[
\Omega_t = \frac{1}{2} \sum_{i=0}^{\infty} \beta^i [y_{t+i}^T Q y_{t+i}]
\]  

(78)

where \( Q \) is a matrix. In the absence of a lower bound constraint on the nominal interest rate the policymaker’s optimization problem is to minimize \( \Omega_0 \) given by (78) subject to (77) and given \( z_0 \). If the variances of shocks are sufficiently large, this will lead to a large nominal interest rate variability and the possibility of the nominal interest rate becoming negative.

We can impose a lower bound effect on the nominal interest rate by modifying the discounted quadratic loss criterion as follows.\(^{23}\) Consider first the ZLB constraint on the nominal on the nominal interest rate. Rather than requiring that \( R_{n,t} \geq 0 \) for any realization of shocks, we impose the constraint that the mean rate should at least \( k \)

\(^{23}\)This follow the treatment of the ZLB in Woodford (2003) and Levine et al. (2008)
standard deviation above the ZLB. For analytical convenience we use discounted averages.

Define $\bar{R}_n \equiv E_0 \left[ (1 - \beta) \sum_{t=0}^{\infty} \beta^t R_{n,t} \right]$ to be the discounted future average of the nominal interest rate path $\{R_{n,t}\}$. Our ‘approximate form’ of the ZLB constraint is a requirement that $\bar{R}_n$ is at least $k_r$ standard deviations above the zero lower bound; i.e., using discounted averages that

$$\bar{R}_n \geq k_r \sqrt{(R_{n,t} - \bar{R}_n)^2} = k_r \sqrt{R_{n,t}^2 - (\bar{R}_n)^2} \tag{79}$$

Squaring both sides of (79) we arrive at

$$E_0 \left[ (1 - \beta) \sum_{t=0}^{\infty} \beta^t \bar{R}_n^2 \right] \leq K_r \left[ E_0 \left[ (1 - \beta) \sum_{t=0}^{\infty} \beta^t R_{n,t} \right] \right]^2 \tag{80}$$

where $K_r = 1 + k_r^{-2} > 1$

We now maximize $\sum_{t=0}^{\infty} \beta^t [U(X_{t-1}, W_t)]$ subject to the additional constraint (80) alongside the other dynamic constraints in the Ramsey problem. Using the Kuhn-Tucker theorem this results in an additional term $w_r \left( R_{n,t}^2 - K(\bar{R}_n)^2 \right)$ in the Lagrangian to incorporate this extra constraint, where $w_r > 0$ is a Lagrangian multiplier. From the first order conditions for this modified problem this is equivalent to adding terms $E_0(1 - \beta) \sum_{t=0}^{\infty} \beta^t w_r (R_{n,t}^2 - 2K\bar{R}_n R_{n,t})$ where $\bar{R}_n > 0$ is evaluated at the constrained optimum. It follows that the effect of the extra constraint is to follow the same optimization as before, except that the single period loss function terms of in log-linearized variables is replaced with

$$L_t = y_t^T Q y_t + w_r (r_{n,t} - r_n^*)^2 \tag{81}$$

where $r_n^* = (K - 1)\bar{R}_n > 0$ is a nominal interest rate target for the constrained problem.

In our LQ approximation of the non-linear optimization problem we have linearized around the Ramsey steady state which has zero inflation. With a ZLB constraint, the policymaker’s optimization problem is now to choose an unconditional distribution for $r_{n,t}$, shifted to the right by an amount $r_n^*$, about a new positive steady-state inflation rate, such that the probability of the interest rate hitting the lower bound is extremely low. This is implemented by choosing the weight $w_r$ for each of our policy rules so that
where \( z_0(p) \) is the critical value of a standard normally distributed variable \( Z \) such that \( \text{prob}(Z \leq z_0) = p \). \( R^*_n = (1 + \pi^*)R_n + \pi^* \) is the steady state nominal interest rate, \( R_n \) is the shifted steady state real interest rate, \( \sigma_r^2 = \text{var}(R_n) \) is the unconditional variance and \( \pi^* \) is the new steady state positive net inflation rate. Given \( \sigma_r \) the steady state positive inflation rate that will ensure \( R_{n,t} \geq 0 \) with probability \( 1 - p \) is given by

\[
\pi^* = \max \left[ \frac{z_0(p)\sigma_r - R_n + 1}{R_n} \times 100, 0 \right] \tag{82}
\]

In our linear-quadratic framework we can write the intertemporal expected welfare loss at time \( t = 0 \) as the sum of stochastic and deterministic components, \( \Omega_0 = \tilde{\Omega}_0 + \bar{\Omega}_0 \). By increasing \( w_r \) we can lower \( \sigma_r \) thereby decreasing \( \pi^* \) and reducing the deterministic component, but at the expense of increasing the stochastic component of the welfare loss. By exploiting this trade-off, we then arrive at the optimal policy that, in the vicinity of the steady state, imposes a ZLB constraint, \( R_{n,t} \geq 0 \) with probability \( 1 - p \). Figures 11 – 13 and Table 8 show this solution to the problem for all three policy regimes with \( p = 0.0025 \); i.e., a very stringent ZLB requirement that the probability of hitting the zero lower bound is only once every 400 quarters or 100 years.

For the commitment regimes OPT and SIM as the penalty on the interest rate variance \( w_r \) increases, the actual variance \( \sigma_r^2 \) falls and the steady-state inflation shift \( \pi^* \) necessary to enforce the ZLB falls to zero. This in fact is the optimal adjustment for this model for all three values of \( \sigma \). However this outcome depends crucially on the ability of the monetary authority to commit to a particular interest rate rule. Absent such commitment a higher penalty on interest rate adjustments is self-defeating as it induces higher volatility in the economy which in turn requires a more aggressive monetary stance. Then a penalty \( w_r = 0 \) is optimal requiring a large shift in the steady-state inflation rate of over 3% per quarter. The gains from commitment now become large and range from \( c_e = 0.49\% \) to \( c_e = 0.64\% \) as \( \sigma \) rises. For the commonly assume Cobb Douglas case these gains are exaggerated by a 0.11% compared with the empirically established lower value of the elasticity.
Figure 11: Imposition of ZLB for OPT Policy

Figure 12: Imposition of ZLB for TC Policy

Figure 13: Imposition of ZLB for Simple Rule
7 Conclusions

This paper contributes to a rapidly rising literature that brings the CES specification of the production function into the analysis of business cycle fluctuations. The main empirical result is to confirm decisively the importance of CES rather than CD production functions. Indeed in a marginal likelihood race our estimated best CES model with an elasticity well below unity at 0.16 beats the CD production function by a substantial log-likelihood of 12.98. Assuming equal prior model probabilities, this implies that posterior model probabilities are 40,000:1 in favour of the CES.

The principle reason for this result is that movements of factor shares with the CES specification help substantially to fit the data. The marginal likelihood improvement is matched by the ability of the CES model to get closer to the data in terms of second moments, especially the volatilities of output, consumption and the real wage, and the autocorrelation functions for inflation and the nominal interest rate. A comparison with a DSGE-VAR further confirms the ability of the CES model to reduce model misspecification. The main message then for DSGE models is that we should dismiss once and for all the use of CD for business cycle analysis.

But see Geweke and Amisano (2011) for an alternative to a marginal likelihood race proposing instead a ‘prediction pool’ consisting of an optimal linear combination of the marginal likelihoods for the two models.

Table 8. Optimal Policy with and without Commitment. ZLB Imposed

<table>
<thead>
<tr>
<th>$\sigma$</th>
<th>Policy</th>
<th>Rule: $[\rho_r, \theta_\pi, \theta_y]$</th>
<th>Adjusted Loss</th>
<th>$c_e$ (%)</th>
<th>$\sigma_r^2$</th>
<th>$\omega_r$</th>
<th>$\pi^*$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.159</td>
<td>OPT</td>
<td>n.a.</td>
<td>4.698</td>
<td>0</td>
<td>0.125</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>0.159</td>
<td>TC</td>
<td>n.a.</td>
<td>27.63</td>
<td>0.49</td>
<td>2.077</td>
<td>0</td>
<td>3.01</td>
</tr>
<tr>
<td>0.159</td>
<td>SIM</td>
<td>[0.871, 0.676, 0.012]</td>
<td>5.009</td>
<td>0.007</td>
<td>0.125</td>
<td>$2 \times 10^5$</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>OPT</td>
<td>n.a.</td>
<td>4.729</td>
<td>0</td>
<td>0.125</td>
<td>0.009</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>TC</td>
<td>n.a.</td>
<td>27.63</td>
<td>0.60</td>
<td>2.31</td>
<td>0</td>
<td>3.22</td>
</tr>
<tr>
<td>1</td>
<td>SIM</td>
<td>[0.913, 0.507, −0.006]</td>
<td>5.018</td>
<td>0.006</td>
<td>0.125</td>
<td>$4 \times 10^4$</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>OPT</td>
<td>n.a.</td>
<td>4.983</td>
<td>0</td>
<td>0.125</td>
<td>0.01</td>
<td>0</td>
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<td>TC</td>
<td>n.a.</td>
<td>34.86</td>
<td>0.64</td>
<td>2.38</td>
<td>0</td>
<td>3.29</td>
</tr>
<tr>
<td>2</td>
<td>Simple</td>
<td>[0.945, 0.423, 0.009]</td>
<td>5.285</td>
<td>0.006</td>
<td>0.125</td>
<td>0.002</td>
<td>0</td>
</tr>
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</table>
Despite these positive results, one area where the CES model remains a concern in terms of model misspecification is in the second moments involving wages and hours. For example, both CD and CES models fail miserably in reproducing the negative correlation between output and hours; furthermore, the CES model produces far too much persistence in hours. As pointed out by Rowthorn (1999), a low capital-labour substitutability is crucial for understanding unemployment persistence. This suggests that future research should also look more closely at the labour market and introduce search-match frictions and unemployment alongside CES production.

The paper also examines the policy implications of technology with capital-labour complementarity as favoured by our empirical results against the case of CD or complementarity. We find that in the absence of a ZLB constraint they are not significant. But as a consequence of the higher volatility introduced by capital-labour substitutability, the ZLB constraint creates a significant reduction in the gains from commitment when CES technology with a low value of \( \sigma \) replaces the usual CD technology.

Our CES specification allows us to introduce a capital-augmenting shock alongside the labour-augmenting variety. However, we find this does not bring about an improvement in the model fit and the contribution of the capital-augmenting shock in the variance decomposition is small. We have noted the well-known result that a bgp requires either Cobb-Douglas technology or that technical change must be driven solely by the labour-augmenting variety. This raises an obstacle to the prospect of unifying business cycle analysis with long-term endogenous growth based on CES technology. One possible way forward is to follow León-Ledesma and Satchi (2010); they provide a model of optimal choice of CES production technology that results in a bgp with both labour and capital-augmenting technical change. Then CES prevails in the short-run but CD in the long-run, thus allowing a capital-augmenting technical change contribution to long-run growth. These authors provide an alternative production function with these properties. A possible line for further research would be to incorporate this into the SW-type model of this paper and to assess its empirical performance and policy implications.
References


Cantore, C., León-Ledesma, M., McAdam, P., and Willman, A. (2010b). Shocking Stuff:


8 Appendix

A Expressing Summations as Difference Equations

In the first order conditions for Calvo contracts and expressions for value functions we are confronted with expected discounted sums of the general form

\[ \Omega_t = E_t \left[ \sum_{k=0}^{\infty} \beta^k \eta_{t,t+k} Y_{t+k} \right] \]  

(A.1)
where \( X_{t,t+k} \) has the property \( X_{t,t+k} = X_{t,t+1}X_{t+1,t+k} \) (for example an inflation, interest or discount rate over the interval \([t, t+k]\)).

**Lemma**

\( \Omega_t \) can be expressed as

\[
\Omega_t = X_{t,t}Y_t + \beta E_t [X_{t,t+1}\Omega_{t+1}]
\]

(A.2)

**Proof**

\[
\begin{align*}
\Omega_t &= X_{t,t}Y_t + E_t \left[ \sum_{k=1}^{\infty} \beta^k X_{t,t+k}Y_{t+k} \right] \\
&= X_{t,t}Y_t + E_t \left[ \sum_{k'=0}^{\infty} \beta^{k'+1}X_{t,t+k'+1}Y_{t+k'+1} \right] \\
&= X_{t,t}Y_t + \beta E_t \left[ \sum_{k'=0}^{\infty} \beta^k X_{t,t+1}X_{t+1,t+k'+1}Y_{t+k'+1} \right] \\
&= X_{t,t}Y_t + \beta E_t [X_{t,t+1}\Omega_{t+1}]
\end{align*}
\]

B Proof of Price and Wage Dispersion Results

For prices and without indexation, in the next period, \( \xi_p \) of these firms will keep their old prices, and \( (1 - \xi_p) \) will change their prices to \( P_{t+1}^O \). By the law of large numbers, we assume that the distribution of prices among those firms that do not change their prices is the same as the overall distribution in period \( t \). It follows that we may write

\[
\Delta_{p,t+1} = \xi_p \int_{m \text{ no change}} \left( \frac{P_t(m)}{P_{t+1}} \right)^{-\zeta} \left( \frac{P_t^0}{P_{t+1}} \right)^{-\zeta} dm + (1 - \xi_p) \left( \frac{P_t^0}{P_{t+1}} \right)^{-\zeta}
\]

\[
= \xi_p \left( \frac{P_t}{P_{t+1}} \right)^{-\zeta} \int_{m \text{ no change}} \left( \frac{P_t(m)}{P_t} \right)^{-\zeta} dm + (1 - \xi_p) \left( \frac{P_t^0}{P_{t+1}} \right)^{-\zeta}
\]

\[
= \xi_p \Pi_{t+1}^{\zeta} \Delta_{p,t} + (1 - \xi_p) \left( \frac{P_t^0}{P_{t+1}} \right)^{-\zeta}
\]

(B.1)

The generalization to indexation is straightforward.
C Posterior Distribution

Figure 14: Priors and Posteriors distributions
Figure 15: Bayesian IRFs - Labour-augmenting shock
Figure 16: Bayesian IRFs - Capital-augmenting shock

\( \text{BVAR-DSGE}(\lambda = 1.1) \): the dashed lines are the first and ninth posterior deciles of the VAR’s IRFs. The bold black curve is the posterior mean of the VAR’s IRFs.

Figure 17: Bayesian IRFs - Investment-specific shock
Figure 18: Bayesian IRFs - Monetary policy shock

Figure 19: Bayesian IRFs - Government spending shock
### Table 4: Variance Decomposition - Model CES (in Percent)

<table>
<thead>
<tr>
<th></th>
<th>Forecast horizon</th>
<th>Observable variables</th>
<th>Productivity (K)</th>
<th>Productivity (L)</th>
<th>Government spending (price)</th>
<th>Investment (wage)</th>
<th>Monetary preference policy</th>
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<tbody>
<tr>
<td><strong>t=1</strong> Output</td>
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<td>22.68</td>
<td>25.02</td>
<td>7.97</td>
<td>26.64</td>
<td>5.95</td>
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<td>17.08</td>
<td>3.14</td>
<td>21.44</td>
<td>8.46</td>
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<td>4.77</td>
<td>0.00</td>
<td>12.74</td>
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<td>13.50</td>
<td>0.38</td>
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<td>8.06</td>
<td>22.66</td>
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<tr>
<td>Real wage</td>
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<td>36.17</td>
<td>0.34</td>
<td>49.75</td>
<td>1.57</td>
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<td>3.16</td>
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<td>11.53</td>
<td>4.13</td>
<td>43.91</td>
</tr>
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<td>15.85</td>
<td>21.60</td>
<td>21.75</td>
<td>7.48</td>
<td>26.56</td>
<td>5.38</td>
</tr>
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<td>6.61</td>
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<td>6.22</td>
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<td>8.85</td>
<td>22.17</td>
<td>16.63</td>
<td>3.98</td>
<td>22.40</td>
<td>10.11</td>
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<td>15.59</td>
<td>46.06</td>
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<td>16.45</td>
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<td>11.29</td>
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<td>10.02</td>
<td>28.31</td>
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<tr>
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<td>3.99</td>
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<td>6.17</td>
<td>46.90</td>
<td>1.34</td>
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<tr>
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<td>19.16</td>
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<td>7.02</td>
<td>28.85</td>
<td>6.18</td>
</tr>
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<td>8.39</td>
<td>21.16</td>
<td>15.92</td>
<td>8.02</td>
<td>21.14</td>
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</tr>
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<td>16.76</td>
<td>0.28</td>
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<td>1.53</td>
<td>16.64</td>
<td>14.83</td>
<td>14.47</td>
<td>21.00</td>
</tr>
<tr>
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<td>0.05</td>
<td>2.76</td>
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<td>35.22</td>
<td>2.88</td>
<td>47.77</td>
<td>1.62</td>
</tr>
<tr>
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<td>2.75</td>
<td>7.67</td>
<td>28.95</td>
<td>7.39</td>
<td>9.55</td>
</tr>
<tr>
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<td>1.68</td>
<td>6.31</td>
<td>32.87</td>
<td>3.42</td>
<td>32.94</td>
<td>0.51</td>
</tr>
<tr>
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<td>16.14</td>
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<td>21.28</td>
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<td>5.11</td>
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<td>46.83</td>
<td>0.20</td>
</tr>
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</table>

♢ All the variance decomposition is computed from the model solutions (order of approximation = 1). The results are based on the models’ posterior distribution.
### Table 5: Variance Decomposition - Model CD (in Percent)

♢ All the variance decomposition is computed from the model solutions (order of approximation = 1). The results are based on the models’ posterior distribution.

<table>
<thead>
<tr>
<th>Shocks of the Estimated DSGE Models</th>
<th>Forecast Observable horizon variables</th>
<th>Productivity (K)</th>
<th>Productivity (L)</th>
<th>Government spending</th>
<th>Mark-up Investment (price)</th>
<th>Mark-up Investment (wage)</th>
<th>Monetary Preference</th>
<th>Policy</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>t=1</strong></td>
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<td>3.46</td>
<td>43.84</td>
<td>10.14</td>
<td>11.01</td>
<td>10.43</td>
<td>4.63</td>
</tr>
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<td>Consumption</td>
<td>0.00</td>
<td>1.62</td>
<td>0.12</td>
<td>5.00</td>
<td>0.42</td>
<td>12.27</td>
<td>8.55</td>
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<td>3.69</td>
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<td>57.16</td>
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<td>3.63</td>
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<td>11.31</td>
<td>24.25</td>
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<td>1.98</td>
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<td>12.07</td>
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<td>6.96</td>
<td>8.57</td>
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<td>17.98</td>
<td>27.14</td>
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<td>13.33</td>
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<td>6.29</td>
<td>2.14</td>
<td>43.05</td>
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<td>14.77</td>
<td>5.71</td>
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<td>5.03</td>
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<td>2.03</td>
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### Table 6: Posterior results for the exogenous shocks

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Prior mean</th>
<th>Post. mean CD (SW07)</th>
<th>Post. mean CES</th>
<th>5% CES</th>
<th>95% CES</th>
<th>Prior</th>
<th>pstdev CES</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\rho_{ZL}$</td>
<td>0.5</td>
<td>0.9409 (0.95)</td>
<td>0.9379</td>
<td>0.8930</td>
<td>0.9844</td>
<td>beta</td>
<td>0.2</td>
</tr>
<tr>
<td>$\rho_{ZK}$</td>
<td>0.5</td>
<td>N/A (N/A)</td>
<td>0.4576</td>
<td>0.1273</td>
<td>0.7710</td>
<td>beta</td>
<td>0.2</td>
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<tr>
<td>$\rho_{ZT}$</td>
<td>0.5</td>
<td>0.9516 (0.97)</td>
<td>0.9533</td>
<td>0.9363</td>
<td>0.9778</td>
<td>beta</td>
<td>0.2</td>
</tr>
<tr>
<td>$\rho_{PT}$</td>
<td>0.5</td>
<td>0.6449 (0.71)</td>
<td>0.6745</td>
<td>0.5437</td>
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<tr>
<td>$\rho_{PW}$</td>
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</tr>
<tr>
<td>$\rho_{PB}$</td>
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<td>0.3438 (N/A)</td>
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<td>0.6045</td>
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<td>0.2797</td>
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<td>$\sigma_{M}$</td>
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<td>0.1503 (0.24)</td>
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</table>

### Table 7: Posteriors results for model parameters

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Prior mean</th>
<th>Post. mean CD (SW07)</th>
<th>Post. mean CES</th>
<th>5% CES</th>
<th>95% CES</th>
<th>Prior</th>
<th>pstdev CES</th>
</tr>
</thead>
<tbody>
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<td>$\sigma$</td>
<td>1</td>
<td>1 (1)</td>
<td>0.1594</td>
<td>0.0218</td>
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<td>gamma</td>
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<td>0.5062 (N/A)</td>
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<td>0.5159</td>
<td>beta</td>
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<td>0.8129 (0.54)</td>
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<td>0.7143</td>
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<tr>
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Figure 20: Bayesian IRFs - Preference shock

Figure 21: Bayesian IRFs - Price mark-up shock
Figure 22: Bayesian IRFs - Wage mark-up shock
Figure 23: Variance Decomposition