Monotonicity of entropy for real multimodal maps

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Abstract

In [16], Milnor posed the Monotonicity Conjecture that the set of parameters within a family of real multimodal polynomial interval maps, for which the topological entropy is constant, is connected. This conjecture was proved for quadratic by Milnor & Thurston [17] and for cubic maps by Milnor & Tresser, see [18] and also [5]. In this paper we will prove the general case.

1 Introduction and Statement of Results.

Let \( P^d \) be the space of \( d \)-modal real polynomials \( f \) of degree \( d + 1 \) which map \([-1, 1]\) into itself, such that all its \( d \) critical points are distinct and real (from this it follows \( f \) has negative Schwarzian derivative), inside \((-1, 1)\) and so that \( f \) is anchored: \( f((-1, 1)) \subset (-1, 1) \). To each such map one can assign a shape \( \epsilon \in \{+, -\} \) which describes whether \( f \) is increasing or decreasing on its first lap. The set of maps \( P^d \subset P^d \) with shape \( \epsilon \) can be parametrized by the coefficients of the polynomial, or more suitably by the critical values of \( f \), see [14, Section II.4] or [17]. In this paper we will prove the following

**Theorem 1.** For each \( \epsilon \in \{+, -\}, \, d \in \mathbb{N} \) and \( s \geq 0 \), the set

\[
\{f \in P^d_\epsilon \mid h_{\text{top}}(f) = s\}
\]

is connected.

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In the unimodal case $d = 1$, this is the monotonicity of topological entropy for real quadratic maps (which was proved by Milnor & Thurston and others). For the bimodal case $d = 2$ (i.e., the cubic case), using the fact that then the parameter space is two-dimensional, the above theorem was proved previously by Milnor & Tresser, see [18]. In fact, prior to that paper it was shown in [5] that, again in the special case that $d = 2$, the above statement follows from another conjecture (density of Axiom maps for cubic maps). Using the Jordan theorem, and the fact that on some curves in the parameter space, the bimodal family behaves essentially like a one-parameter family of unimodal maps, Milnor and Tresser were able to use density of Axiom A within the space of ‘quadratic-like’ maps. In this paper, we shall use a more general approach. The crucial new ingredient for our proof is the recent result that Axiom A maps are dense in $P^d$, for any $d$, see Theorem 3 below.

More precisely, we relate the class $P^d$ to the class of stunted sawtooth maps $S^d$ (see Section 3.1) as follows:

**Theorem 2.** There exists a map $\Psi : P^d \rightarrow S^d$ such that

- $\Psi$ is ‘almost continuous’, ‘almost surjective’ and ‘almost injective’ (this statement is made precise in a series of lemmas in subsection 3.5);
- There exists a connected set $[\Psi(f)] \ni \Psi(f)$ such that the topological entropy of any map $T \in [\Psi(f)]$ is equal to the topological entropy of $f$;
- If $K$ is closed and connected then $\Psi^{-1}(K) = \{f; [\Psi(f)] \cap K \neq \emptyset\}$ is connected.

Theorem 1 then follows from the statement that the subsets in $S^d$ of constant entropy - and which correspond to polynomials - are connected, see Theorem 7. We shall show later on, see Example 5, that $\Psi(P^d)$ is not closed. Moreover, the inverse of the map $\Psi$ is not continuous, see Example 6.

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### 2 Ingredients for the proof.

Let $B(f)$ be the union of the basins of periodic attractors of $f$, i.e., $B(f)$ consists of all points $x$ so that $f^n(x)$ tends to a (possibly one-sided) periodic attractor. Note that if $f$ has non-hyperbolic periodic attractors, $B(f)$ need not be open. We say that two $d$-modal maps $f, g : [-1, 1] \rightarrow [-1, 1]$ are partially conjugate if there is a homeomorphism $h : [-1, 1] \rightarrow [-1, 1]$ such that
Figure 1: The maps \( f \) and \( g \in \overline{\mathcal{PH}(f)} \setminus \mathcal{PH}(f) \). The map \( g \) maps \( \tilde{c} \) to the fixed point in the boundary of the interval.

- \( h \) maps \( B(f) \) onto \( B(g) \);
- \( h \) maps the \( i \)-th critical point of \( f \) to the \( i \)-th critical point of \( g \);
- \( h \circ f(x) = g \circ h(x) \) for all \( x \notin B(f) \).

The partially hyperbolic deformation space \( \mathcal{PH}(f) \) is the set of maps \( g \in P_{d}^d \) which are partially conjugate to \( f \). Note that \( \mathcal{PH}(f) \) makes sense also when \( f \notin P_{d}^d \); in this case \( f \notin \mathcal{PH}(f) \). Let \( \mathcal{PH}^o(f) \) be the set of \( g \in \mathcal{PH}(f) \) such that \( g \) has only hyperbolic periodic points. Obviously \( \mathcal{PH}^o(f) \subset \mathcal{PH}(f) \subset \overline{\mathcal{PH}(f)} \) and the last inclusion can be strict. For example, assume that \( f, g \in P_{2}^2 \) both have an attracting fixed point \( p \) with immediate basin \( B \), and attracting one critical point \( c \). If \( g \) has a second critical point \( \tilde{c} \) such that \( g(\tilde{c}) \in \partial B \), while \( f \), being a small perturbation of \( g \), has \( f(\tilde{c}) \in B \), then \( g \in \overline{\mathcal{PH}(f)} \setminus \mathcal{PH}(f) \), see Figure 1.

The main ingredients for the main result of this paper are the following:

**Theorem 3** (Rigidity Theorem, see [10]). Let \( f, g \in P_{d}^d \). Assume that \( f \) and \( g \) are partially conjugate and that there exists a conformal conjugacy between \( f \) and \( g \) restricted to the immediate basins of periodic attractors of \( f \) and \( g \). Then \( f = g \).

The quadratic case of Theorem 3 was proved independently by Lyubich and Graczyk & Świątek, and was used by Milnor & Tresser in their proof of Theorem 1 for the cubic case, see [18]. Since the method of proof in [18] uses planar topology (cubic families are parametrized by two parameters), we need some additional tools in order to deal with the ‘partially hyperbolic’ case and extend the above rigidity theorem to:

**Theorem 4** (Description of the partially hyperbolic deformation space). Let \( f \in P_{d}^d \). Then \( \mathcal{PH}^o(f) \) is homeomorphic to an open ball of dimension equal to the number of critical points in \( B(f) \).
We should remark here that for the proof of this theorem it is important that the maps we consider are real. In general, it is not obvious how to deform a map with an attracting and a repelling orbit to one with a parabolic orbit, or vice versa to deform a map with a parabolic point to a ‘subhyperbolic’ map in such a way that the Julia set remains topologically the same, see [7, 8]. The proof of Theorem 4 is given in Section 4.

3 Proof of the Main Theorem.

3.1 The space of stunted sawtooth maps $\mathcal{S}^d$.

Fix the number of turning points $d$ and the shape $\epsilon$ of the polynomials in the space $P^d_\epsilon$ we will consider. From now on we will drop the symbol $\epsilon$. Following [18], it will be useful to introduce a class of piecewise linear maps with plateaus. Fix the slope $\lambda = d + 2$ and let $e = d\lambda/(\lambda - 1)$. Define $S: [-e, e] \to \mathbb{R}$ as the piecewise linear map with $d$ turning points

- with $d + 1$ laps $I_0, \ldots, I_d$ of monotonicity and with turning points $c_1, \ldots, c_d$ in $-d + 1, -d + 3, \ldots, d - 3, d - 1$;
- with slope $\pm \lambda$;
- which has shape $\epsilon$;
- and such that $f(\{-e, e\}) \subset \{-e, e\}$.

It is not hard to check that such a choice is possible and that each turning point of $S$ is mapped outside $[-e, e]$. The space of $\mathcal{S}^d$ of stunted seesaw maps consists of

\begin{figure}[h]
\centering
\includegraphics[width=0.3\textwidth]{sawtooth.png}
\caption{A sawtooth and two stunted sawtooth maps (with different third plateaus).}
\end{figure}

continuous maps $T$ with plateaus $Z_{i,T}$, $i = 1, \ldots, d$, satisfying,
• $Z_{i,T}$ is a closed symmetric interval around $c_i$;
• $T$ and $S$ agree outside $\cup_i Z_{i,T}$;
• $T|Z_{i,T}$ is constant and $T(Z_{i,T}) \in [-e,e]$.
• $Z_{i,T}$ have pairwise disjoint interiors.

So maps in $S^d$ are allowed to have touching plateaus (i.e., sharing one boundary point).

It is convenient to use the $d$ ‘signed’ extremal values $\zeta \in [-e,e]^d$ to parametrize $S^d$:

$$\zeta_i = \begin{cases} T(Z_{i,T}) & \text{if } S(c_i) \text{ assumes a maximum;} \\ -T(Z_{i,T}) & \text{if } S(c_i) \text{ assumes a minimum.} \end{cases}$$

Note that $\zeta_i + \zeta_{i+1}$ is equal the length of the interval $T([Z_i, Z_{i+1}])$ and so

$$\zeta_i \geq -\zeta_{i+1} \quad \text{for } i = 1, \ldots, d - 1,$$

with equality when the plateaus $Z_i$ and $Z_{i+1}$ touch. Thus we can identify $S^d$ with

$$\{ \zeta = (\zeta_1, \ldots, \zeta_d) : \zeta_i \in [-e,e], \; \zeta_i \geq -\zeta_{i+1} \},$$

and we call the parts of $\partial S^d$ with $\zeta_i = -\zeta_{i+1}$ for some $i$ the oblique boundary. Using

the parameters $\zeta = (\zeta_1, \ldots, \zeta_d)$, $\zeta \to h_{top}(T_{\zeta})$ is non-decreasing in each component.

A consequence of this is that $\{T \in S^d : h_{top}(T) = s\}$ is contractible, see Theorem 6.1 in [18].

However, not every $T \in S^d$ can be used to parametrize polynomials in $f \in P_{d+1}^d$.

This restriction is caused by the non-existence of wandering intervals, corresponds to wandering pairs for $T \in S^d$, cf. page 14.

Definition 1. We say that a pair of plateaus $(Z_i, Z_j)$ is a wandering pair if there exists $n \geq 0$ such that $T^n(J)$ is a point, where $J := [Z_i, Z_j]$ is the convex hull of $Z_i$ and $Z_j$. We say that $T$ is non-degenerate if for every wandering pair $(Z_i, Z_j)$, the corresponding interval $J$ is eventually mapped into a periodic plateau. Let $S^d_\ast$ denote the set of non-degenerate maps $T \in S^d$, and let $R^d_\ast = S^d \setminus S^d_\ast$.

Theorem 5. The space $\{T \in S^d_\ast : h_{top}(T) = s\}$ is contractible.

Proof. The proof is deferred to Section 5.

Example 1. It is possible to parametrize the family $P^d$ by critical values, see Theorem II.4.1 of [14]. The following example shows that it is not true that topological entropy depends monotonically on each of these parameters separately. Let
Let $f = f_{a,b} : [0, 1] \to [0, 1]$ be a bimodal map such that $f(0) = 0$ and $f(1) = 1$, such that $a$ and $b$ are the images of the critical points $c_1 < c_2$. So $f_{a,b} \in P^2$. In this example we show that there are values of $b$ such that the map $a \mapsto h_{\text{top}}(f_{a,b})$ is not monotone.

Start with $b = 0$ and $a_*$ such that $c_1 = f^2(c_1) < c_2 < f(c_1)$. Next take $b_* > 0$ small such that $\nu_2 = \frac{f_0^{-1}}{0^2}c_2$ for $n$ large. The point $c_2$ is now super-attracting, of period $n$. Note that $f^n$ has a local minimum at $c_2$. Because $b_* \approx 0$, there still is a 2-periodic attracting point $p$ close to $c_1$.

Claim: $a \mapsto h_{\text{top}}(f_{a,b_*})$ is not monotone.

It is clear that $h_{\text{top}}(f_{1,b_*})$ is close to $\log 3$, which is larger than $h_{\text{top}}(f_{b_*}) = 0$. For $a$ is close to $a_*$, there is an attracting $n$-periodic point $q(a)$ close to $c_2$, and $q(a_*) = c_2$. If $a$ increases, so do all points $f^k_{a,b_*}(c_2)$ for $1 \leq k \leq n$. Hence, if $B$ is a small neighborhood of $c_2$, $f^n_{a,b_*}|B$ has two branches for $a \geq a_*$ and four branches for $a < a_*$. If $a > a_*$ is so large that $q(a)$ disappears in a saddle-node bifurcation, then $h_{\text{top}}(f_{a,b_*})$ decreases compared to $h_{\text{top}}(f_{a_*})$. On the other hand, if $a < a_*$ is so small that $q(a)$ has lost its stability in a period doubling bifurcation and the full cascade of period doubling bifurcations has been gone through, then $h_{\text{top}}(f_{a,b_*})$ increases compared to $h_{\text{top}}(f_{a_*})$.

This process is illustrated in Figure 3 for the family $f(x) = 2ax^3 - 3ax^2 + b$ for $a = b - 0.515$. This cubic map has critical points 0 and 1 and $f(0) = b$, and $f(1) = b - a = 0.515$. To some extent, it resembles the strategy of [9]. The entire graph of the the entropy function of the cubic family of polynomials can be found in [2].

### 3.2 Plateaus and preplateaus.

Note that if $T$ has touching plateaus, then it is constant on some lap(s) of $S$, and $T$ is $d$-modal only in a degenerate sense. We define the preplateau $W(T)$ of a map $T \in S^d$ to be the set of points $x$ which eventually map into the interior of the union of the plateaus of $T$, i.e.,

$$W(T) = \cup_{k \geq 0} T^{-k}(\text{int}(\cup_{i=1}^d Z_{i,T})).$$

Because of the possibility of plateaus touching each other, we take the interior of the union rather than the union of the interiors. By definition, if $W'$ and $W''$ are components of $W(T)$ such that $T^n(W') \cap W'' \neq \emptyset$, then $T^n(W') \subset W''$. (Indeed, if $T^n(W')$ intersects but is not contained in $W''$, then there exists $x \in W' \setminus \text{int}(\cup_{i=1}^d Z_{i,T})$ such that $T^n(x) \in \partial W''$. Since $T^k(x) \notin \text{int}(\cup_{i=1}^d Z_{i,T})$ for all $k = 0, 1, \ldots, n$, $T^n(x)$ is eventually mapped into $\text{int}(\cup_{i=1}^d Z_{i,T})$ which is impossible since $T^n(x) \in \partial W''$.) If $Z_{i,T}$ contains a periodic point for some $i$, then the preplateau serves as an analog of the basin of periodic attractor of a map in $P^d$. However, it is possible that $W(T)$
Figure 3: Non-monotonicity of entropy for the map \( f_b(x) = 2ax^3 - 3ax^2 + b \) with \( a = b + 0.515 \).

contains a periodic orbit in its boundary, and not in its interior, see e.g. the two stunted sawtooth maps in Figure 2.

Define

\[
\langle T \rangle = \{ \tilde{T} \in S^d; W(\tilde{T}) = W(T) \} \quad \text{and} \quad [T] = \text{closure}(\langle T \rangle).
\] (2)

To clarify this definition, let us consider an example: take for \( T_1 \) (resp. \( T_2 \)) the map with the lower (resp. higher) third plateau drawn in Figure 2, then \( \langle T_1 \rangle \) consists of all maps \( \tilde{T} \) for which the third plateau \([a(\tilde{T}), b(\tilde{T})]\) (with \( a(\tilde{T}) < b(\tilde{T}) \) and containing \( c_3 \)) has a fixed point in the half-open interval \([a(\tilde{T}), b(\tilde{T})]\). For all these maps \( \tilde{T} \), each point in \((a(T_1), b(T_1))\) is eventually mapped into \((a(\tilde{T}), b(\tilde{T}))\). However, the map \( T_2 \) for which \( b(T_2) \) is a fixed point is not contained in \( \langle T_1 \rangle \). So \( \langle T_1 \rangle \) is neither an open nor a closed subset of \( S^d \). Note that the third plateau has period 2 for maps in \( \langle T_2 \rangle \setminus \{T_2\} \).

A natural representative for an equivalence class \( \langle T \rangle \) is the map \( \tilde{T} \) so that if \( Z_i \) is mapped into a component \( W \) of \( W(T) \) then \( \tilde{T}(Z_i) \) is equal to the midpoint of \( W \). If a component of \( W(T) \) contains several plateaus, then these plateaus will touch for \( \tilde{T} \).
3.3 Properties of $\langle T \rangle$ and $[T]$.

Let $W_i$ denote the component of $W(T)$ containing $c_i$. It can happen that $W_i$ contains more than one critical point, so that $W_i = W_{i'}$ is possible also for $i \neq i'$. Let us say that $W_i$ of $T$ is critically related to $W_j$ if the following two conditions are satisfied:

1. there is $n > 0$ such that either $T^n(W_i) \subset W_j$ or $T^n(W_i) \subset \partial W_j$ but then for all small neighborhoods $U$ of $W_i$, $T^n(U) \cap W_j = \emptyset$. (Recall here that components $W_i$ are open. If $W_i$ contains an even number of plateaus, then the latter possibility cannot occur.)

2. for each $0 < m < n$ and $1 \leq j \leq d$, both $T^m(W_i) \cap W_j = \emptyset$ and $T^m(W_i) \not\subset \partial W_j$ whenever there exists a small neighborhood $U$ of $W_i$ with $T^m(U) \cap W_j \neq \emptyset$.

In this case we will write $W_i \rightarrow W_j$. If the first alternative holds in part 1 of this definition, then we call this an interior critical relation.

Thus we have a directed graph $G_W$ on preplateaus. Note that unless $T|W_i$ is constant, $W_i$ must have an outgoing arrow. The definition of the arrows $W_i \rightarrow W_j$ is chosen precisely this way to fulfill the following assertion:

**Lemma 1.** The graph $G_W$ is constant on $\langle T \rangle$.

*Proof.* To prove this assertion, consider $\tilde{T} \in \langle T \rangle$ and two components $W$ and $W'$ of $W(T) = W(\tilde{T})$ for which $T(W) \subset W'$. Then $W$ is a component of $T^{-1}(W')$. If $\tilde{T}(W) \cap \partial W' \neq \emptyset$ then $\tilde{T}|W$ is constant, because boundary points of $\tilde{T}^{-1}(\partial W')$ are not contained in $W(T)$. On the other hand, $\tilde{T}(W) \cap W' = \emptyset$ is only possible if $\tilde{T}$ is constant on a neighbourhood of $W$, which would contradict $W(T) = W(\tilde{T})$. It follows that $T(W) \subset W'$ implies that either $\tilde{T}(W) \subset W'$ or $\tilde{T}(W) = \partial W'$ and then for any neighbourhood $U$ of $W$, $\tilde{T}(U) \cap W' = \emptyset$. Similarly, $T(W) \subset \partial W'$ also implies these two possibilities. The lemma follows. \hfill $\square$

The main result of this subsection describes the shape of $\langle T \rangle$.

**Lemma 2.** The map $T \mapsto \langle T \rangle$ is such that $\langle T \rangle$ is connected and is the product of finitely many polygons and its dimension is equal to the number of $i$’s, $1 \leq i \leq d$, such that $W_i$ has an outgoing arrow. In particular, if $T$ has no critical relations, then $\langle T \rangle = \{ T \}$.

We should emphasize that some parts of the boundary of the polygons are contained in $\langle T \rangle$ whereas others are not. This lemma implies that if $[T_1] = [T_2]$ then $\langle T_1 \rangle = \langle T_2 \rangle$. 

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Proof. Let us identify \( \langle T \rangle \) with its parameter region. When \( \hat{T} \in \partial \langle T \rangle \) then some components of \( W(T') \) either merge or disappear for some sequence of maps \( T' \to \hat{T} \).

This can happen in various ways: by a plateau mapping to the boundary of \( W(T) \) (corresponding to the (hc) and (sn) situations described below), by having a periodic point in the boundary of a plateau (the (pd) situation) or by having two touching plateaus (the (tp) situation).

Let us be more formal and prove the lemma, by showing that \( \langle T \rangle \) is the Cartesian product of sets of the following type:

1. Assume that \( W_{i_1} \to \cdots \to W_{i_m} \to W_{i_1} \) is a loop in the graph \( G_W \) of the critical relations of \( T \) and let \( J \) be the collection of indices \( j \) so that \( Z_j \subset \cup_{k=1}^m W_{i_k} \). If each \( W_{i_1}, \ldots, W_{i_m} \) contains only one plateau, then the projection of \( \langle T \rangle \) onto the components \( \zeta_j, j \in J \), is an \( \tilde{m} \)-dimensional polytope, where \( \tilde{m} = \# J \geq m \) with its \( (\tilde{m} - 1) \)-dimensional boundary faces parallel to the coordinate axes, see Figure 5. If a component \( W_{i_k} \) contains more than one plateau, then the projection of \( \langle T \rangle \) onto the components \( \zeta_j, j \in J \) is a finite union of \( \tilde{m} \)-dimensional polytopes, with each \( (\tilde{m} - 1) \)-dimensional boundary face either parallel to the coordinate axes or a subset of the oblique boundary of \( S_d \); the closure of these polytopes is connected, see Figure 4.

2. For each critical relation \( W_i \to W_j \) not contained in any loop, the projection of \( \langle T \rangle \) onto the \( \zeta_{i'} \)-component is a (half)open interval for each \( i' \) such that \( c_{i'} \in W_i \). (Thus these parameters form a ‘cartesian’ rectangle, except for the oblique boundaries caused by touching plateaus.)

3. If \( W_i \) has no outgoing arrow in the graph \( G_W \) of critical relations, then the projection of \( \langle T \rangle \) onto the \( \zeta_{i'} \)-component is a single point for each \( i' \) such that \( c_{i'} \in W_i \).

To prove Statement 1, let us rename \( i_t \to t \) for simplicity, and let

\[
W_t =: (x_t, \hat{x}_t)
\]

be the component of the preplateau \( W \) containing \( c_t \). By maximality of the component, \( x_t \) and \( \hat{x}_t \) are (pre)periodic, where \( x_t \) is always chosen to be periodic. If \( W_t \) contains only one plateau (or an odd number of plateaus), then \( T(\hat{x}_t) = T(x_t) \), so \( \hat{x}_t \) is preperiodic. If \( W_t \) contains an even number of critical points, then \( \hat{x}_t \) could be periodic, see the left of Figure 4, and \( x_t \) and \( \hat{x}_t \) could even be on the same orbit.

Let \( s_t \) be such that \( T^{s_t}(x_t) = x_{t+1} \) (throughout this proof, we compute the indices mod \( m \) in \( \{1, 2, \ldots, m\} \), so \( m + 1 \equiv 1 \), or if \( x_t \) and \( \hat{x}_t \) are on the same orbit, then \( x_{m+1} = \hat{x}_t \) and \( \hat{x}_{m+1} = x_1 \). Let \( g_t : W_t \to W_{t+1} \) be the restriction of \( T^{s_t} \) to \( W_t \). Clearly \( g_t \), \( Z_t \) and \( W_t \) depend on \( T \), but in order not to overload notation we will not
always indicate this dependence, but denote the objects corresponding to $\tilde{T} \in \langle T \rangle$ by $\tilde{g}_t$, $\tilde{Z}_t$, and $\tilde{W}_t$.

We claim that $\langle T \rangle$ projects into a subregion satisfying:

(A) $\tilde{T}$ is such that $\tilde{g}_t(\tilde{Z}_j) \in [x_{t+1}, \hat{x}_{t+1}]$ for each $t$ and each plateau $\tilde{Z}_j \subset \overline{W}_t$; moreover, if $\tilde{g}_t(\tilde{Z}_j)$ is equal to $x_{t+1}$ for some plateau $\tilde{Z}_j \subset \overline{W}_t$ then $\tilde{g}_t(\overline{W}_t)$ is constant (and so $x_{t+1}$ and $\hat{x}_{t+1}$ cannot both be periodic, see Figure 4.)

(B) If $\bigcup_{t=1}^m W_t$ contains a periodic orbit, say $u_1 \xrightarrow{\tilde{g}_1} u_2 \xrightarrow{\tilde{g}_2} \cdots \xrightarrow{\tilde{g}_{m-1}} u_m \xrightarrow{\tilde{g}_m} u_1$, then $\bigcup_{t=1}^m W_t$ are components of the basin attraction of this orbit and $u_t \in \text{int}(\bigcup_{c_j \in W_t} Z_j)$ for at least one $t$.

Obviously $\tilde{g}_t(\tilde{Z}_t) \in [x_{t+1}, \hat{x}_{t+1}]$. If $\hat{x}_{t+1} = \tilde{g}_t(Z_t)$, then there exists a point $y \in \partial Z_t$ such that $\tilde{g}_t(y) \in \partial W_{t+1}$, so $y$ is never mapped into a plateau of $\tilde{T}$. By the same argument, $\tilde{g}_t(Z_t) = x_{t+1}$ is possible only when $W_t = \text{int}(\bigcup_{c_j \in W_t} Z_j)$ (so if $W_t$ contains an odd number of critical points, then all plateaus must touch, and if $W_t$ contains an even number of critical points, this is impossible). This proves Statement (A).

Note that periodic orbits $\text{orb}(p)$ that avoid $\text{int}(\bigcup_i Z_i)$ are the same as periodic orbits of the non-stunted map $S$. Hence the location of such orbits is independent of the parameters, as long as none of its points is “swallowed” by the interior of a plateau. Obviously, $\bigcup_{t=1}^m W_t$ cannot contain periodic points that avoid plateaus. This proves Statement (B).

Note that (A) and (B) together imply that there are two possibilities:
(i) $\bigcup_{t=1}^m W_t$ contains a periodic point so that each $W_t$, $t = 1, \ldots, m$ is a component of its basin of attraction and this periodic point is in the interior of some plateau or
(ii) the first return map to each of the intervals $W_t$ is constant.

It follows that $\langle T \rangle$ projects even onto a region satisfying (A) and (B).

For each map $\tilde{T}$ in the boundary of the $\tilde{m}$-dimensional projection of $[T]$ the set $W_{t,T}$ is no longer constant for $T'$ near $\tilde{T}$ because of one of the following bifurcations:

**Saddle node:** $\tilde{T}$ is such that $g_t(W_t) = x_{t+1}$ for some $t$. (As remarked under (A) above, this can only happen if $W_t$ contains an odd number of plateaus and $\text{int}(\bigcup_{Z_j \in W_t} Z_j) = W_t$, so all the plateaus in $W_t$ touch.) In other words, $x_t$ is a one-sided periodic attractor (and this is why we call this case a saddle-node). If $\tilde{T}$ projects to a boundary point of this type, then $\tilde{T} \in \langle T \rangle$, provided the projection to other coordinates render no obstructions.

**Homoclinic:** $\tilde{T}$ is such that $g_t(\partial Z_j) = \hat{x}_{t+1}$ for at least one $t$ and $Z_j \subset W_t$. This case only occurs if there are several plateaus in the orbit of $W_t$ (because if there is only one plateau, then the (pd) case below will happen before (hc)), and corresponds to the situation that there exists a sequence of maps $\langle T \rangle \owns T_n \rightarrow \tilde{T}$ so that $g_{t,T_n}$ maps some interior point of $W_{t,T} = W_{t+1,T}$ closer and closer to the boundary of $W_{t+1,T} = W_{t+1,T_n}$ (this is why we call this case homoclinic). Maps $\tilde{T}$ that project to such boundary points belong to $[T] \setminus \langle T \rangle$.

**Period doubling:** $\tilde{T}$ is such that there is at least one periodic orbit $u_1 \xrightarrow{g_1} u_2 \xrightarrow{g_2} \cdots \xrightarrow{g_{m-1}} u_m \xrightarrow{g_m} u_1$ as in (B) above except that it avoids $\text{int}(\bigcup_i Z_i)$. In this case, as $\tilde{T}$ can be approximated by maps as in (B), $W_1, \ldots, W_m$ remain components of the basin of attraction of this orbit. In other words, $u_i$ is a two-sided attractor, and this is why this case is referred to as period doubling. Since $u_i \notin W(\tilde{T})$, $W_i$ are no longer components of $W(\tilde{T})$ and so maps $\tilde{T}$ that project to such boundary points belong to $[T] \setminus \langle T \rangle$.

**Touching plateaus:** $\tilde{T}$ is such that $W_t$ contains at least two more neighboring critical points $c_i$ and $c_{i+1}$, and $\zeta_i = -\zeta_{i+1}$, see (1). This means that the plateaus $Z_i$ and $Z_{i+1}$ are touching. This happens only at the oblique boundary of the entire parameter space, and hence at the boundary of $\langle T \rangle$. (Figure 4 gives an impression for if $W_t$ contains two plateaus.) If $\tilde{T}$ projects to a boundary point of this type, then $\tilde{T} \in \langle T \rangle$, provided the projection to other coordinates render no obstructions. Touching plateaus are the reason why the interior of $\langle T \rangle$ need not be connected, see Figure 4. However, this part of the boundary belongs to $\langle T \rangle$, so $\langle T \rangle$ is connected.

Figure 5 gives an impression what the projection of $\langle T \rangle$ looks like for a loop $W_1 \rightarrow W_2 \rightarrow W_1$, where each $W_t$ contains a single critical point.

Note that for each of these boundary components, at least one parameter $\zeta_j$ is fixed.
Figure 5: The projection of $\langle T \rangle$ onto a parameter subspace corresponding to a 2-loop in the graph $G_W$ of the critical relations. The codes $\text{sn}$ (saddle node), $\text{hc}$ (homoclinic) and $\text{pd}$ (period doubling) indicate parts of the boundary discussed in the proof Lemma 2. On the right side, the graphs of $T^{s_1}$ and return map $T^{s_2} \circ T^{s_1}$ to $W_1$ (respectively $T^{s_2}$ and return map $T^{s_1} \circ T^{s_2}$ to $W_2$) are shown for the indicated parameters on the left. Note that the diagonal has only meaning for the maps $T^{s_2} \circ T^{s_1}$ and $T^{s_1} \circ T^{s_2}$.

unless $T \in \partial S^d$, so the boundary components are parallel to the coordinate axes or subsets of $S^d$. This completes the proof of Statement 1.

In order to prove Statement 2, assume that $W_i \rightarrow W_j$, say $T^s(Z_i) \in W_j$, where $W_i$ is not part of a loop. Then there is an interval of $\zeta_i$-values (leaving all other parameters fixed) for which $\tilde{T}(Z_i) = \zeta_i$ and $\tilde{T}(Z_i) \in W_j$. Since $W_j$ is open, this parameter interval is open too, except when one of the following occurs.

(i) If $W_i = W_i'$ for some $i \neq i'$, then varying $\zeta_i$ may cause $Z_i$ to touch $Z_i'$, and in this case the interval of $\zeta_i$-values can be half-open.

(ii) $W_i = \text{int}(Z_i)$ and $T^n(W_i) \in \partial W_j$ and for all small neighborhoods $U$ of $W_i$, $T^n(U) \cap W_i = \emptyset$. (Remember that in this case we also write $W_i \rightarrow W_j$.) This situation causes the interval of $\zeta_i$-values to be half-open.

Finally, to prove Statement 3, observe that preimages of plateaus are dense in $[-e, e]$. If $W_i$ has no outgoing arrow in the graph $G_W$, then $T^n(W_i)$ is disjoint from all plateaus for all $n \geq 1$ and so $T(W_i)$ is disjoint from all preimages of plateaus. It follows that $T(W_i)$ is a single point ($W_i$ can consist of only one block of touching plateaus), and since $T(W_i)$ is the accumulation point of components of $W(T)$, any change in parameter $\zeta_i$ will produce a map which is no longer in $\langle T \rangle$. Hence the
projection of \( \langle T \rangle \) onto the \( i \)-th coordinate is a single point. In particular, if there are no critical relations at all, then \( \langle T \rangle \) is a singleton.

\[ \]

3.4 The map \( \Psi : P^d \to S_*^d \).

Let us review some basic kneading theory, see [17], and also [14] and [18]. Given a piecewise monotone \( d \)-modal map \( f : [-1, 1] \to [-1, 1] \) with turning points \( c_1, \ldots, c_d \) (in particular, \( f \) is strictly monotone on each component of \( [-1, 1] \setminus \cup_i c_i \)), one can associate to each point \( x \in [-1, 1] \) an itinerary \( \hat{I}_f(x) \) consisting of a sequence \( (i_0, i_1, \ldots) \) of symbols from the alphabet \( \{I_0, c_1, I_1, c_2, \ldots, c_d, I_d\} \). Recall that the critical points are distinct and inside \( (-1, 1) \), so all the \( I_i \) are non-degenerate intervals. It is well-known that \( x \mapsto \hat{I}_f(x) \) is monotone w.r.t. the signed lexicographic ordering and that therefore the \( i \)-th kneading sequence

\[ \nu_i := \lim_{x \to c_i} \hat{I}_f(x) \]  

is well-defined. Let \( \sigma \) be the shift map on the space of symbol sequences. Note that the sequence \( \nu_i \) does not contain any of the symbols \( c_1, \ldots, c_d \). The kneading invariant \( \nu(f) \) of \( f \) is defined as

\[ \nu(f) := (\nu_1, \ldots, \nu_d). \]

Any kneading invariant which is realized by some piecewise monotone \( d \)-modal map is called \textit{admissible}.

To each map \( f \in P^d \) one can associate \textit{uniquely} a stunted sawtooth map as follows. Let \( \nu(f) = (\nu_1, \ldots, \nu_d) \) be the kneading invariant of \( f \), and let \( s_i \) be the unique point in the \( (i + 1) \)-th lap \( I_i \) of \( S \) such that

\[ \lim_{y \to s_i} \hat{S}(y) = \nu_i := \lim_{x \to c_i} \hat{I}_f(x). \]

Such a point \( s_i \) exists because all itineraries are realized by \( S \), and it is unique because \( S \) is expanding and so distinct points have different itineraries. Let \( Z_i \) be the symmetric interval around the \( i \)-th turning points of \( S \) with right endpoint \( s_i \).

Let us define \( f \mapsto \Psi(f) \in S_*^d \), by associating to \( f \) the unique stunted sawtooth map \( \Psi(f) \) which agrees with \( S \) outside \( \cup_i Z_i \) and which is constant on \( Z_i \) with value \( S(s_i) \). Note that \( f \) and \( \Psi(f) \) are conjugate restricted to the complements of the basins of their periodic attractors and the complement of the preimages of plateaus of \( \Psi(f) \). We again say that \( f \) and \( \Psi(f) \) are \textit{partially conjugate}. This holds, because each periodic attractor of \( f \) has a critical point in its immediate basin (because \( f \)
has negative Schwarzian derivative) and so $\Psi(f)$ will have a corresponding periodic plateau. Moreover (recall Definition 1),

$$f \mapsto \Psi(f) \in S^d_+$$

because $T \in R^d_+$ would have to correspond to a map $f$ for which there exists a critical point $c_j$ which is not in a basin of a periodic attractor and for which there also exists an interval connecting two adjacent critical points is eventually mapped into $c_j$; this is not possible because polynomials have no wandering intervals.

![Diagram](image)

**Figure 6:** Unimodal sawtooth and stunted sawtooth maps

**Example 2.** Consider the family $f_\lambda(x) = \lambda x(1 - x)$ and the map $\lambda \mapsto \Psi(f_\lambda) \in S^2$. Let $p_1$ be the orientation reversing fixed point of $S$ and $p_2$ the fixed point of $S^2$ closest to 0, see Figure 6. For $\lambda \in (0, 2]$, $f_\lambda$ has an attracting orientation preserving fixed point in $(0, 1/2]$ (or a critical fixed point if $\lambda = 2$), $\nu(f_\lambda) = I_1 T_0$ and $\Psi(f_\lambda) \equiv -e$ (the plateau is maximal). For $\lambda > 2$ close to 2, the attracting fixed point of $f$ is to the right of the critical point, $\nu(f_\lambda)$ is equal to $I_1 T_1$ and the map $\Psi(f_\lambda)$ has plateau $Z = [-p_1, p_1]$. So the map $f \mapsto \Psi(f)$ is not continuous when the critical point of $f$ becomes a fixed point. In other words, within a hyperbolic component of the parameter set, $f \mapsto \Psi(f)$ has discontinuities. Moreover, $f \mapsto \Psi(f)$ is not surjective. Increasing $\lambda$ from 2, $\nu(f_\lambda)$ and $\Psi(f_\lambda)$ will remain constant, until the critical point has period 2. At this parameter $\nu(f_\lambda)$ changes to $I_1 T_1 T_0$ and $\Psi(f_\lambda)$ has a plateau $[p_2, -p_2]$. Note that

$$[\Psi(f_\lambda)] = \{T \in S^2; [-e, e] \supseteq Z_T \supseteq [-p_1, p_1]\} \text{ for } \lambda \in (0, 2]$$

and

$$[\Psi(f_\lambda)] = \{T \in S^2; [-p_1, p_1] \supseteq Z_T \supseteq [p_2, -p_2]\} \text{ for } \lambda \in (2, \lambda_2],$$

where $\lambda_2 > 2$ is minimal so that the critical point has period 2 for $f_{\lambda_2}$. The next jump in $\lambda \mapsto \Psi(f_\lambda)$ occurs at the parameter $\lambda_4$, where the critical point has period 4, etc. Note that if $2 < \mu_0 < \mu_1 < \mu_2$ are the parameters where the Feigenbaum periodic doubling occurs, then $PH(f_\lambda)$ is constant for $\lambda \in (\mu_i, \mu_{i+1}]$. So $\lambda \mapsto \Psi(f_\lambda)$ and $\lambda \mapsto PH(f_\lambda)$ have discontinuities at different parameters.
3.5 The definition of $\mathcal{A}^*$, $\mathcal{A}^*$ and almost surjectivity of $\Psi$.

In analogy to the graph $\mathcal{G}_W$ on components of preplateaus, we define, for $f \in P^d$, the graph $\mathcal{G}_B$ on components of the basin of attractors analogously. Namely, let $B$ be the basin of the periodic attractors of $f$, and $B_i$ is the component of $B$ containing $c_i$. (If $f$ has a one-sided attractor $B_i$ need not be open.) There is a critical relation, denoted $B_i \to B_j$ if some iterate, say $f^s$, maps $B_i$ into $B_j$, and $f^s(B_i) \cap B_k = \emptyset$ for $0 < t < s$ and $k = 1, \ldots, d$.

**Definition 2.** We say that $f \in \mathcal{A}^*$ if for each critical relation $f^s$: $B_i \to B_j$ with $B_i$ and $B_j$ each containing an attracting periodic point, the image $f^s(c)$ of each critical point $c \in B_i$ lies in the same component of $B_j \setminus \text{Crit}(f)$ as the attracting periodic point $p_j$ in $B_j$. Here $\text{Crit}(f)$ is the set of critical points of $f$.

We say that $f \in \mathcal{A}^*$ if $f \in \mathcal{A}^*$ and if for each $B_i$ which is periodic, i.e., $f^s(B_i) \subset B_i$ for some $s$, the following holds. If $f^s(\partial B_i)$ is a single point, then $B_i$ does not contain a periodic attractor in its interior and so $\partial B_i$ is a parabolic periodic point.

**Remark 1.** It is allowed for a map $f \in \mathcal{A}^*$ to have a critical point in the boundary of some $B_i$. For example, $f(x) = (1 + \varepsilon)(x^3 + x^2) - 1$ (for $\varepsilon > 0$ small) has a parabolic fixed points $-1$ and repelling fixed point $-1$ and $\sqrt{1/(1+\varepsilon)}$, attracting fixed point $-\sqrt{1/(1+\varepsilon)}$ and two critical point $-\frac{2}{3}$ and $0$. The latter maps to $-1$, and the immediate basin of $\sqrt{1/(1+\varepsilon)}$ is $(0,0)$. So $\partial B_1$ contains a critical point.

If $f \in \mathcal{A}^*$ and $T = \Psi(f)$ then the interior of each component of the basin of a periodic attractor of $T$ coincides with a component of $W(T)$. Indeed, let $B_{i_1} \to \cdots \to B_{i_m} \to B_{i_p}$ is a loop in the critical graph $\mathcal{G}_B$. Since $f \in \mathcal{A}^*$, for each critical point $c_j \in B_{i_j}$, the itinerary $\sigma(\nu_j(f))$ is equal to the itinerary of the periodic attractor in $B_{i_{j+1}}$, where $\sigma$ denotes the left-shift. It follows that $T = \Psi(f)$ has an attracting periodic orbit $p$ so that the component of its basin containing $p$ contains each plateau $Z_j$ (for all $j$’s corresponding to the $j$’s above). Moreover, either $T$ is constant on $W$ and $p \in \partial W$ or $p$ is in the interior of the union of these plateaus. (The former holds if $W$ contains an odd number of plateaus, the latter otherwise.) It follows that $W$ is a component of $W(T)$ and the graph $\mathcal{G}_B(f)$ is isomorphic to $\mathcal{G}_W(T)$.

Let us motivate the definition of $\mathcal{A}^*$ by considering three examples.

**Example 3.** (i) If $f$ is a quadratic map with an attracting fixed point, then $\Psi(f)$ is a stunted sawtooth map which is either equal to the constant map $T_0$ corresponding to the parameter $\zeta = -e$ or to the map $T_1$ which has a plateau $[p_2, p_1]$ corresponding to the parameter $\zeta = p_1$, see Figure 6. Now $T_1 \in [T_0]$ while $T_0 \notin [T_1]$. Moreover $\Psi(\mathcal{P}H(f)) \subset \{T_0, T_1\} \subset [T_0]$. If $f \in \mathcal{A}^*$ then $\Psi(f) = T_0$ and so $\Psi(\mathcal{P}H(f)) \subset [\Psi(f)]$. If $f \notin \mathcal{A}^*$ then this inclusion does not hold.
(ii) The situation is similar if \( f \) is a cubic map with an attracting fixed point which
attracts both critical points (say with the left critical point a maximum). In this
case \( \Psi(f) \) is equal to one of the following five maps \( T_0, T_1, T_2, T_3, T_4 \) determined by
\( (\zeta_1, \zeta_2) \) equal to \((e, -e), (e, 0), (0, 0), (0, e)\) or \((-e, e)\). Now \( T_0, T_1, \ldots, T_4 \in [T_i] \) when
\( i = 0, 2, 4 \) but not when \( i = 1, 3 \). Also note that \( \Psi(\mathcal{PH}(f)) \subset \{T_0, \ldots, T_4\} \subset [T_0] = [T_2] = [T_4] \). If \( f \in \mathcal{A}^* \) then \( \Psi(f) \) is equal to \( T_0, T_2 \) or \( T_4 \), i.e., in one of the lower
corners of the corresponding region, see the right part of Figure 4.

(iii) Finally consider the example of a cubic map such that \( B_1 \rightarrow B_2 \rightarrow B_1 \) and so
that \( B_1 \) and \( B_2 \) both contain exactly one critical point. Then \( \Psi(f) \) is contained in
the polygon drawn in Figure 5. The parameters of maps in \( \{\Psi(f) : f \in \mathcal{PH}(f)\} \)
are 1, 2, 4, 6 and their symmetric counterpart under reflection in the diagonal of the
\( (\zeta_1, \zeta_2) \)-space. There are six such maps, all belonging to \( \partial[\Psi(f)] \).

The fact that \( f \in \mathcal{A}^* \) ensures that \( \Psi(f) \) corresponds to the lower corner of the region,
i.e., the map \( T \) denoted by 1 in the figure. This map has the property that \([T]\) is
equal to this polygon (this is false for maps denoted by (hc) and (pd)).

**Lemma 3.** \( \Psi \) is ‘almost surjective’: for each \( T \in S^d \) there exists a polynomial
\( f \in P^d \cap \mathcal{A}^* \) such that \( T \in [\Psi(f)] \).

In order to prove Lemma 3, we need a result from [14] concerning full families, but
let us first discuss an example from [12].

**Example 4.** Consider a bimodal piecewise monotone map \( g: [0, 1] \rightarrow [0, 1] \) with
turning points \( 0 < c_1 < c_2 < 1 \) such that \( g(0) = 0, g(1) = 1, g(c_1) \in (c_1, c_2),
g(c_2) = 0, g^2(c_1) = p \) with \( p \) a fixed point in \((0, c_1)\), see Figure 7. There is no
cubic polynomial which is conjugate to \( g \) (because the attractor 0 would have to
have a critical point in its immediate basin (in \( \mathbb{R} \), not just in \([0, 1]\) ). Note that
\( \nu_1(g) = I_1I_1 \bar{T}_0 \) and \( \nu_2(g) = I_2 \bar{T}_0 \) and it is not hard to see that there is a cubic
polynomial with \( f: [0, 1] \rightarrow [0, 1] \) with \( f(0) = 0, f(1) = 1, f(c_1) = c_2, f(c_2) = 0 \)
and that \( f \) has the same kneading invariant as \( g \). However, it is impossible to find
a cubic polynomial so that \( \tilde{I}_1(g(c_1)) = I_1I_0 \) and \( \tilde{I}_2(g(c_2)) = \bar{T}_0 \). So we could never
expect all admissible kneading invariants to be realized by polynomials.

Let \( I \) be an interval and let us say that a periodic attractor of a piecewise monotone
map \( g: I \rightarrow I \) is essential if it contains a turning point in its immediate basin. We
say that \( g \) has no wandering intervals, if each interval \( J \) for which \( J, g(J), g^2(J), \ldots \)
are all pairwise disjoint necessarily intersects the basin of some periodic attractor. If
\( g \) has no wandering interval, then each interval \( J \) for which \( g^n|J \) is a homeomorphism
for all \( n \) is necessarily contained in the basin of periodic attractor, see [14].

It is well-known, see [14], that maps in \( P^d \) do not have wandering intervals and that
all their attractors are essential. Moreover,
Theorem 6. (Fullness of families in $P^d$, see [14]) For each piecewise monotone map $g$ with $d$ turning points and shape $\epsilon$ such that

1. $g$ has no wandering intervals and no inessential attractors;
2. each periodic turning point is an attractor (this is automatically satisfied if $g$ is $C^1$),

there exists a polynomial $f \in P^d$ which is conjugate to $g$.

Proof of Lemma 3. Take $T \in S^d$. Since $T$ is not piecewise monotone (because of its plateaus), we cannot apply Theorem 6 directly. We start therefore with surgeries making $T$ piecewise monotone, by replacing $T|W_i$ (where $W_i \ni c_i$) by an affinely scaled copy of a map of the correct type (i)-(iv) in Figure 8.

Assume $T^{s_i}(W_i) \subset W_j$ for some minimal $s_i > 0$, and $W_i$ contains $q_i$ plateaus. Write $W_i = [a_i, b_i]$ and $W_j = [a_j, b_j]$. There are four shapes that $T^{s_i}|W_i$ can take:

(i) $T^{s_i}(a_i) = T^{s_i}(b_i) = a_j$. Then we replace $T|W_i$ by the unique map $\tilde{T}$ such that $T^{s_i-1} \circ \tilde{T}|W_i$ becomes an affine copy of $L_{q_i}$ for type (i).

(ii) $T^{s_i}(a_i) = T^{s_i}(b_i) = b_j$. Then we replace $T|W_i$ by the unique map $\tilde{T}$ such that $T^{s_i-1} \circ \tilde{T}|W_i$ becomes an affine copy of $L_{q_i}$ for type (ii).

(iii) $T^{s_i}(a_i) = a_j, T^{s_i}(b_i) = b_j$. Then we replace $T|W_i$ by the unique map $\tilde{T}$ such that $T^{s_i-1} \circ \tilde{T}|W_i$ becomes an affine copy of $L_{q_i}$ for type (iii).

(iv) $T^{s_i}(a_i) = b_j, T^{s_i}(b_i) = a_j$. Then we replace $T|W_i$ by the unique map $\tilde{T}$ such that $T^{s_i-1} \circ \tilde{T}|W_i$ becomes an affine copy of $L_{q_i}$ for type (iv).

After all these surgeries have been carried out we call the new map $\tilde{T}$. 

Figure 7: The maps $g$ and $f$. 

Figure 8: **Types (i) and (ii):** $L_q : [0, 1] \to [0, 1]$ is $q$-modal with slope $\pm 5^{-d}$ (where as before $d$ is the total number of plateaus of $T$), and the $q$ critical points are put symmetrically in $\left[\frac{2}{5}, \frac{3}{5}\right]$. Therefore $L_q$ maps $(0, 1)$ strictly into itself.

**Types (iii) and (iv):** $L_q : [0, 1] \to [0, 1]$ is $q$-modal and $L_q|[\frac{1}{10}, \frac{9}{10}]$ has slope $\pm 5^{-d}$, with $q$ equally spaced critical points in $\left[\frac{1}{10}, \frac{9}{10}\right]$. For type (iii) the slope of $L_q|[0, \frac{1}{10}] \cup [\frac{9}{10}, 1]$ is $5(1 + \frac{2}{5}5^{-d})$ and for type (iv) the slope of $L_q|[0, \frac{1}{10}] \cup [\frac{9}{10}, 1]$ is $-5(1 + \frac{2}{5}5^{-d})$.

If a component $W_i$ is periodic for $\tilde{T}$, say $\tilde{T}^k(W_i) \subset W_i$, then it has a unique periodic attractor $p_i$ and either $p_i \in \partial W_i$, namely if at least one of the maps $\tilde{T}^s_i$ of which the iterate $\tilde{T}^t_i|W_i$ is composed is of type (i) or (ii), or $p_i$ is an interior point of $W_i$ otherwise. Moreover, $p_i$ attracts every point in $W_i$ under $\tilde{T}^t_i$. Indeed, if $x \in W_i$ with $\tilde{T}^n(x) \in W_j$ is such that the map $L_{q_j}$ of the correct type at the corresponding point has derivative $\pm 5^{-d}$, then $|DT^t_i(x)| \leq 5^{-d} \cdot [5(1 + \frac{2}{5}5^{-d})]^{d-1} \leq 1/4$. Alternatively, if for each $j$, $\tilde{T}^n(x) \in W_j$ is such that the map $L_{q_j}$ of the correct type at the corresponding point has derivative $\pm 5$, then $x$ itself belongs to an interval adjacent to a $j$ or $b_j$ at which $DT^t_i = \pm 5^k$, where $k \geq 1$ is the number of components $W_j$ in the cycle. Under iteration of $\tilde{T}^t_i$ these points are pushed away further into the interior of $W_i$ until they reach the part where the derivative $|DT^t_i| \leq 1/4$. Hence they are attracted to $p_i$, the midpoint of $W_i$.

Next collapse components of $W$ of $\tilde{T}$ without outgoing arrows, as well as all their preimages, to points. Denote the resulting piecewise continuous map by $T'$. Since $T \in \mathcal{S}_d^t$, the modality of $T'$ is still the same and therefore Theorem 6 guarantees the existence of a polynomial $f \in \mathcal{P}^d$ which is conjugate to $T'$. For a periodic component $W_i$ of $T'$ with interior periodic point $p_i$, let $U_i$ be the component of $W_i \setminus \text{Crit}(T)$ containing $p_i$. Then $\tilde{T}^s_i|U_i$ is affine with slope $5^{-d}|W_j|/|W_i|$, so all critical values of $\tilde{T}^s_i|W_i$ belong to $U_j$. Since $T'$ and $f$ are conjugate, $f \in \mathcal{A}^*$. Finally, since the attracting periodic point $p_i \in \partial W_i$ whenever $W_i$ has exactly one periodic and one pre-periodic boundary point, we also have $f \in \mathcal{A}^*$.

\[\square\]
3.6 Almost injectivity and almost continuity of $\Psi$.

Example 5. The space $\Psi(P^d)$ is not closed. Indeed, let $(f_n)$ be a sequence of cubic maps on $[0, 1]$ with critical points $0 < c_1 < c_2 < 1$ such that $f_n(0) \equiv 0$, $f_n(c_1) \equiv f_n(1) \equiv 1$ and $\nu_2(f_n) = I_2I_0I_2I_1 \ldots$. The limit map $f = \lim_n f_n$ satisfies $f(c_2) = c_1$ and $\nu_2(f) = I_2I_1I_2$.

If $T_n = \Psi(f_n)$, then $T_n$ is a stunted sawtooth map with $T_n(−e) = −e$, $T_n(Z_1) \equiv T_n(e) = e$ and $T_n(Z_2)$ approaches $Z_1$ from the left as $n \to \infty$. However, the map $T = \Psi(f)$ is such that $T(Z_2)$ is the right boundary point of $Z_1$. Hence $\lim_n T_n =: \tilde{T} \neq T$ and in fact $\tilde{T} \notin \Psi(P^2)$.

Lemma 4. If $f \in A^*$ then $\Psi(\mathcal{PH}(f)) \subset [\Psi(f)]$.

Proof. If $f$ has no periodic attractors, then $\mathcal{PH}(f) = \{f\}$ by Theorem 3, and so the lemma follows in this case.

Assume now that $f$ has a periodic attractor. In this case, it is important that $f \in A^*$, see Example 3. The maps $T = \Psi(f)$ and $f$ are partially conjugate and if $f \in A^*$ then (as remarked below Definition 2) the interior of a component of the basin of a periodic attractor of $T$ consists of a component of $W(T)$. Let us denote this property of $W(T)$ by $(\ast)$. While $\tilde{f}$ varies in $\mathcal{PH}(f)$, $\tilde{T} = \Psi(\tilde{f})$ and $T$ remain partially conjugate, so each component $B$ of the basin of a periodic attractor of $\tilde{T}$ is still a component of the basin of a periodic attractor of $T$ (and vice versa). However, since the boundary of a plateau of $T$ is not in $W(T)$ (unless it is adjacent to another plateau) the sets $W(\tilde{T})$ and $W(T)$ may differ. In any case, the set $W(\tilde{T})$ is a subset of $W(T)$ because $\tilde{T}$ and $T$ are partially conjugate and $(\ast)$. So if $W(\tilde{T}) \neq W(T)$, then $\tilde{T}$ has a periodic attractor $u$ which is in the boundary of some plateau $Z_{i,\tilde{T}}$ and which is in the interior of its immediate basin of attraction $B$. However, there exists $T_n \to \tilde{T}$ so that $Z_{i,\tilde{T}}$ contains $u$ in its interior (and which agrees with $\tilde{T}$ outside a neighbourhood of $Z_{i,\tilde{T}}$). By $(\ast)$, the interior of $B$ is a component of $W(T)$ and therefore $W(T_n) \cap B = W(T)$. Since we can do this for each periodic attractor of $\tilde{T}$, it follows that $T \in [T]$. \hfill $\Box$

Lemma 5. The map $\Psi : P^d \to S^d$ is ‘almost injective’ in the sense that if $f_1, f_2 \in A^*$ and $[\Psi(f_1)] \cap [\Psi(f_2)] \neq \emptyset$, then $S^d(\mathcal{PH}(f_1)) \cap S^d(\mathcal{PH}(f_2)) \neq \emptyset$.

Proof. Let us begin by introducing some terminology. Let $G_W$ be the graph of critical relations of $T$. We say that $i$ is periodic if $W_i$ belongs to a loop in this graph, and $i$ is preperiodic if $W_i$ has an outgoing arrow, but $i$ is not periodic. (Note that this use of preperiodic allows that the path from $W_i$ does not enter a loop in $G_W$.)

Given $\zeta = (\zeta_1, \ldots, \zeta_d)$, let $\zeta \pm \epsilon_i$ be shorthand for $(\zeta_1, \ldots, \zeta_i \pm \epsilon_i, \ldots, \zeta_d)$. We say that $\zeta_i$ is a boundary coordinate of $[T]$ if for every $\epsilon_i > 0$, at least one of $T_{\zeta_i+\epsilon_i}$ or $T_{\zeta_i-\epsilon_i}$
Let us now start with the proof. Take \( T_k = \Psi(f_k) \) for \( k = 1, 2 \). If \([T_1] = [T_2]\) then, as noted below the statement of Lemma 2, \( \langle T_1 \rangle = \langle T_2 \rangle \). Moreover, as remarked below Definition 2, since \( f_i \in A^* \) the interior of each component of the basin of a periodic attractor of \( T_i \) coincides with a component of \( W(T_i) \). Since \( \langle T_1 \rangle = \langle T_2 \rangle \) and since \( f_i \) and \( T_i = \Psi(f_i) \) are partially conjugate, it follows that \( f_1 \) and \( f_2 \) are partially conjugate. So in this case there is nothing to prove. So let \( \emptyset \neq M := [T_1] \cap [T_2] \) and assume that \( [T_1] \neq [T_2] \).

Let \( T_i \) be the map with smallest parameter \( \zeta_i^* \), i.e., \( \zeta_i^* = \min \{ \zeta_i : T_\zeta \in M \} \) for each \( i \). The description of \([T_k]\) in Lemma 2 shows that \( T_* \in M \). Note that \( \zeta_i^* = \max \{ \zeta_i(T_1^k), \zeta_i(T_2^k) \} \) where \( T_i^k \), \( k = 1, 2 \), is in one of the lowest corners of \([T_k]\), and hence \( Z_i,T_i = Z_i,T_1^k \cap Z_i,T_2^k \) for each \( i \). (If there is only one critical point in each \( B_i \) then \( T_k = T_k \), and otherwise \( T_k \) is equal to \( \Psi(f_k) \) for some map \( f_k \in \mathcal{PH}(f_k) \).

Since \( T_1, T_2 \in S^d \) and since \( \zeta_i^* = \max \{ \zeta_i(T_1^k), \zeta_i(T_2^k) \} \) it is easy to see that \( T_* \in S^d \) (lowering plateaus only destroys wandering pairs and does not create them). Apply Lemma 3 to obtain a map \( f_* \in A^* \) such that \( \Psi(f_*) = T_* \in [T_1] \cap [T_2] \). By Lemma 11 it follows that there exists two families of polynomials \( f_{\mu,1} \) and \( f_{\mu,2} \) with \( f_{0,k} = f_* \), with \( f_{\mu,k} \) partially conjugate to \( f_k \) (and therefore to \( T_k \)) for \( \mu > 0 \) and such that \( f_{\mu,k} \) has only hyperbolic periodic orbits for \( \mu > 0 \). It follows that \( f_* \in \overline{\mathcal{PH}(f_k)} \) for \( i = 1, 2 \).

**Lemma 6.** \( \Psi : P^d \ni f \mapsto \Psi(f) \in S^d_* \) is ‘almost continuous’ in the following sense. Assume that \( f_n \to f \) where \( f_n \in A^* \). Then any limit \( T_n \in [\Psi(f_n)] \) is contained in \([\Psi(f)]\).

**Proof.** Take \( T_n = \Psi(f_n) \). Since \( f_n \in A^* \), \( f \) can only have a periodic critical point if it has the following property: \( f^s(\partial B) = \partial B \) where \( s \) is the period of this orbit and \( B \) is its immediate basin (so \( f^s(B) \) contains an even number of plateaus). Therefore, if the \( i \)-th turning point \( \zeta_i(f) \) is periodic, then the basin of attraction corresponding to \( Z_i \) for \( \Psi(f_n) \) and \( \Psi(f) \) are equal, and all plateaus of \( \Psi(f_n) \) and \( \Psi(f) \) in the immediate basin containing \( Z_i \) touch (but the attracting periodic point in the corresponding immediate basin for \( \Psi(f_n) \) and for \( \Psi(f) \) might be in the boundary of different plateaus). So the component of \( W(T_n) \) containing \( Z_i,T_n \) is equal to the component of \( W(T) \) containing \( Z_i,T \).
If the $i$-th turning point $c_i(f)$ is not eventually mapped onto another critical point, then $\nu_i(f_n) \to \nu_i(f)$ as $n \to \infty$ (in the usual topology on sequence spaces) and so $Z_{i,\Psi(f_n)} \to Z_{i,T}$ as $n \to \infty$. If the $j$-th turning point $c_j(f)$ has a critical relation, say $f^k(c_j) = c_i$ (with $k$ the minimal such integer) where $c_i(f)$ is not eventually mapped onto another critical point, then the limit of $Z_{i,T}$ need not be $Z_{j,T}$. However, then $\sigma^k+1(\nu_i(f_n))$ tends to $\sigma^k+1(\nu_i(f))$ and so the limit of $T^k_{\nu}(Z_{j,T})$ is one of the endpoints of $Z_{i,T}$ whereas it is possible that $T^k(Z_{j,T})$ is equal to the other end point of $Z_{i,T}$. Hence the component of $W(T_n)$ containing $Z_{j,T_n}$ converges to the component of $W(T)$ containing $Z_{j,T}$.

Combining these arguments, it follows that the component of $W(T_n)$ containing $Z_{i,T_n}$ always converges to the component of $W(T)$ containing $Z_{i,T}$. It follows that any limit of $\hat{T}_n \in [T_n]$ converges to $[T]$. \hfill \Box

### 3.7 Proof of Theorem 1.

First let us prove the following

**Lemma 7.** ‘Fibers of $\Psi$ are connected’ in the following sense. Assume that $K$ is a closed and connected subset of $S^d_*$ with the property that if $T \in K$ then $[T] \subset K$. Then $\Psi^{-1}(K)$ is connected. (Note that $S^d_*$ is not a closed subset of $S^d$; so we merely assume that $K$ is a closed subset in the relative topology of $S^d_*$ meaning that if $T_n \in K$ converges to $T \in S^d_*$ then $T \in K$.)

**Proof.** Take a closed connected set $K \subset S^d_*$ and assume by contradiction that $C := \Psi^{-1}(K)$ is not connected. This means that there are disjoint open sets $U_1, U_2 \subset P^d$ so that $U_1 \cup U_2 \supset \Psi^{-1}(K)$ and $C_i := U_i \cap \Psi^{-1}(K) \neq \emptyset$, $i = 1, 2$. Write $[\Psi(C_i)] := \cup \{[\Psi(f)] \mid f \in C_i\}$.

**Claim 1:** $[\Psi(C_1)] \cup [\Psi(C_2)] \supset K$: it immediately follows from Lemma 3 that for every $T \in K$, there are indeed $f \in P^d$ such that $T \in [\Psi(f)]$.

**Claim 2:** $[\Psi(C_1)] \cap K$ is closed (again in the relative topology of $S^d_*$). To see this, take $T_n \in [\Psi(f_n)] \cap K$ for $f_n \in C_i$. By Lemma 3, we may assume that $f_n \in \mathcal{A}^*$.

By considering subsequences we may assume that $T_n \to T$ for some $T \in K$ and $f_n \to f$ for some $f \in U_i$. By Lemma 6, $T$ is contained in $[\Psi(f)]$. Since $T \in K$, also $[\Psi(f)] \subset K$. Hence $f \in C_i$, completing the proof of Claim 2.

**Claim 3:** $[\Psi(C_1)] \cap [\Psi(C_2)] \neq \emptyset$: this follows from the connectedness of $K$ and Claim 1 and 2.

Hence there exists $f \in C_i \cap \mathcal{A}^*$ such that $[\Psi(f_1)] \cap [\Psi(f_2)] \neq \emptyset$. By Lemma 5, this implies that

$$\overline{\mathcal{PH}(f_1)} \cap \overline{\mathcal{PH}(f_2)} \neq \emptyset.$$
Moreover, by Lemma 4, $\Psi(\mathcal{PH}(f_i)) \subset [\Psi(f_i)] \subset K$. Hence $\mathcal{PH}(f_i) \subset \Psi^{-1}([\Psi(f_i)]) \subset C_i$. Since $\mathcal{PH}(f_i)$ and $\mathcal{PH}(f_2)$ are both connected, this contradicts $C_i \subset U_i$ with $U_1, U_2$ disjoint.

**Proof of Theorem 1.** By Theorem 5 it follows that level sets of $h_{\text{top}}: \mathcal{S}^d_\ast \to \mathbb{R}$ are connected. Moreover, $h_{\text{top}}: P^d \to \mathbb{R}$ agrees with $h_{\text{top}} \circ \Psi$. Because the topological entropy of each map in $[T]$ is the same, Lemma 7 shows that the level sets of constant entropy lift to connected sets in $P^d$. 

### 3.8 Continuity of the ‘inverse’ of $\Psi$.

**Example 6.** The map $\Psi$ from Theorem 2 has a discontinuous inverse. In this example we present a sequence $T_n = \Psi(f_n)$ converging to $T \in \Psi(P^d)$ in the $C^0$-topology, but $f_n$ does not converge.

Let $T_n \in \mathcal{S}^3$ be a stunted sawtooth map such that $T_n(-e) = T_n(e) = e$ and $T_n(Z_1) = -e$ for all $n$. Then each $T_n$ has a fixed point $q \in (-e,c_1)$. Assume that

$$\nu_2(T_n) = I_2I_1^nT_0,$$

so $T_n^{n+1}(Z_2) = q$,

and

$$\nu_3(T_n) = \begin{cases} I_3I_1^{n-1}T_0, & \text{so } T_n(Z_3) = T_n^2(Z_2) \\ I_3I_1^{n-2}T_0, & \text{so } T_n(Z_3) = T_n^3(Z_2) \end{cases}$$

if $n$ is even,

if $n$ is odd.

Clearly $\nu_2(T_n) \to I_2I_1$ and $\nu_2(T_n) \to I_3I_1$ as $n \to \infty$, so the limit map $T = \lim_n T_n$ has touching plateaus $Z_2$ and $Z_3$, both mapping to the left boundary point of $Z_2$ which is fixed.

As none of the $T_n$ has critical relations, $T_n = \Psi(f_n)$ for a single quartic polynomial $f_n: [0,1] \to [0,1]$ satisfying $f_n(0) = f_n(1) = 1$, $f_n(c_1) = 0$ and $f_n^{n+1}(c_2) = p$ is a fixed point in $(0,c_1)$. As $n$ gets large, $c_2$ and $c_3$ will spend long time in a region $J \subset (c_1,c_2)$ where the graph of $f_n$ is very closely below the diagonal. Therefore any accumulation point of $(f_n)$ has a parabolic fixed point $r \in (c_1,c_2)$ attracting $c_2$ and $c_3$. But because

$$f_n(c_3) = \begin{cases} f_n^2(c_2) & \text{if } n \text{ is even,} \\ f_n^3(c_2) & \text{if } n \text{ is odd.} \end{cases}$$

there are two different limit maps: $f_{\text{even}} = \lim_{k \to \infty} f_{2k}$ and $f_{\text{odd}} = \lim_{k \to \infty} f_{2k+1}$, satisfying $f_{\text{even}}(c_3) = f_{\text{even}}^2(c_2)$ and $f_{\text{odd}}(c_3) = f_{\text{odd}}^2(c_2)$. Both $f_{\text{even}}$ and $f_{\text{odd}}$ have kneading sequences $\nu_2 = I_3I_1$ and $\nu_2 = I_3I_1$, so $T = \Psi(f_{\text{even}}) = \Psi(f_{\text{odd}}) \in \Psi(P^3)$. 

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4 The Proof of Theorem 4.

For any \( n \geq 1 \) and \( \epsilon \in \{+, -\} \), let \( M_{n, \epsilon} \) denote the set of all proper (i.e., the inverse of a compact set is compact) maps \( A : \mathbb{D} \to \mathbb{D} \) of degree \( n \), preserving the real axis, such that \( A \) has \( n-1 \) real critical points in \((-1, 1)\), and such that the sign of \( A'(-1) \) is \( \epsilon \). Note that \( A \in M_{n, \epsilon} \) can be written as \( z \mapsto \prod_{i=1}^{n} \frac{z-a_i}{1-a_z} \), where \( \{a_i, 1 \leq i \leq n\} \) is a subset of \( \mathbb{D} \) which is symmetric with respect to the real axis. So \( M_{n, \epsilon} \) can be considered as a subset of \( \mathbb{D}^n \) and is thus supplied with the induced topology. Using the same argument as the proof of Lemma 3.1 in [17], one sees that maps in \( M_{n, \epsilon} \) can be reparametrized by the critical values, so the space \( M_{n, \epsilon} \) is homeomorphic to a Euclidean ball of dimension \( n-1 \). In particular \( M_{n, \epsilon} \) is connected.

If \( g \) is a \( d \)-modal map, let as in Section 2 \( \mathcal{PH}(g) \) be the set of polynomials which are partially conjugate to \( g \) and denote by \( \mathcal{PH}^\epsilon(g) \) denote the subset of \( \mathcal{PH}(g) \) consisting of polynomials without parabolic cycles. (Note that if \( g \) is not a polynomial, then \( g \notin \mathcal{PH}(g) \).) To show that \( \mathcal{PH}(f) \) is connected, our strategy is to prove first that \( \mathcal{PH}^\epsilon(f) \) is connected. To this end we shall identify \( \mathcal{PH}^\epsilon(f) \) with another topological space by means of quasi-conformal surgery.

**Lemma 8.** \( \mathcal{PH}^\epsilon(f) \) is non-empty.

**Proof.** If \( f \) has no parabolic cycles then there is nothing to prove. If \( f \) has parabolic cycles, then these parabolic cycles have critical points in their basins (all critical points of \( f \) are real). So take a \( C^1 \) map \( \hat{f} \) on the real line which agrees with \( f \) outside \( B(f) \), with the same number of critical points as \( f \), and such that when \( B \) is a component of \( B(f) \) containing a (possibly one-sided) periodic attractor of \( f \) with a critical point in its immediate basin, then the critical point (or one of the critical points) of \( \hat{f} \) in \( B \) is periodic. By the Fullness Theorem 6 there exists a polynomial \( f_0 \in P^d \) which is conjugate to \( \hat{f} \). It follows that each periodic orbit of \( f_0 \) is hyperbolic (because each critical point of \( \hat{f} \) which belongs to the basin of a periodic attractor of \( \hat{f} \) in fact is in the basin of a periodic critical point). Since \( \hat{f} \) is partially conjugate to \( f \) we get \( f_0 \in \mathcal{PH}^\epsilon(f) \). \( \square \)

Let us associate spaces \( \mathcal{M}(f) \) and \( \mathcal{M}^\epsilon(f) \) to \( \mathcal{PH}(f) \). For this we will consider \( f \in P^d \) as a map acting on the complex plane, and define \( B(f) \) as the set of points in the complex plane whose iterates converge to periodic attractors (or parabolic points) of \( f \). Let \( U_1, U_2, \ldots, U_m \) be the components of \( B(f) \subset \mathbb{C} \) which contain critical points, and let \( n_i \) be the number of critical points in \( U_i \). For each \( i \) let \( s_i \) be the minimal positive integer such that \( f^{s_i}(U_i) = U_i \) for some \( 1 \leq i' \leq m \). Let us consider the space \( \mathcal{M}(f) = \prod_{i=1}^{m} M_{n_i, \epsilon_i} \), where \( \epsilon_i \) denotes the sign of \( (f^{s_i})' \) at the left endpoint of \( B_i := U_i \cap \mathbb{R} \). (If this endpoint is a critical point, then \( \epsilon_i \) is the sign of the second derivative at this point.) An element \( A = (A_1, \ldots, A_m) \in \mathcal{M}(f) \) will
be viewed as a dynamical system on the disjoint union of $m$ copies of the unit disk,

$$A : \bigcup_{i=1}^{m} \mathbb{D} \times \{i\} \to \bigcup_{i=1}^{m} \mathbb{D} \times \{i\}$$

such that $A(z, i) = (A_i(z), i')$, where $i'$ is as above. Let us say that $A \sim \tilde{A}$ if they are conjugate to each other via a component-preserving conformal map $\varphi : \bigcup \mathbb{D}_i \to \bigcup \mathbb{D}_i$ such that for each $1 \leq i \leq m$, $\varphi|\mathbb{D}_i$ is a real symmetric (i.e., $\varphi(z) = \varphi(\overline{z})$) conformal map whose restriction to the real line preserves the orientation. Here $\mathbb{D}_i = \mathbb{D} \times \{i\}$.

Let $\mathcal{M}^o(f)$ denote the subset of $\mathcal{M}(f)$ consisting of maps

$$A = (A_1, A_2, \ldots, A_m) \in \bigcup_{i=1}^{m} \mathbb{D} \times \{i\}$$

with the following property: if $A_k$ maps $\mathbb{D}_i$ onto itself, then $A_k$ has a fixed point in $\mathbb{D}_i$. In other words, if $U_{i_1}, U_{i_2}, \ldots, U_{i_k}$ is a cycle of attracting basins of $f$, then we require that $A_{i_k} \circ \cdots \circ A_{i_1}$ has a fixed point. (Note that the space $\mathcal{M}^o(f)$ associated to any map in $\mathcal{P}\mathcal{H}(f)$ is the same.) This means that $\mathcal{M}^o(f)/\sim = \prod_{i=1}^{m} M_{n_i, \epsilon_i}^o$ where $M_{n_i, \epsilon_i}$ consists of maps $A \in M_{n, \epsilon}$ which can be written as $z \mapsto z \prod_{i=1}^{n-1} \frac{z-a_i}{1-\overline{a}_i z}$. So the quotient space $\mathcal{M}^o(f)/\sim$ is connected.

Let us define a map

$$\Theta : \mathcal{P}\mathcal{H}^o(f) \to \mathcal{M}^o(f)/\sim$$

as follows. For $g \in \mathcal{P}\mathcal{H}^o(f)$, let $U_i(g) \subset \mathbb{C}$, $i = 1, 2, \ldots, m$, be the components of $B(g)$ containing critical points (so the sets $U_i$ from above correspond to the real traces of these sets). For each $i$, let $\varphi_i : U_i(g) \to \mathbb{D}$ be a real-symmetric conformal map whose restriction to the real axis is orientation-preserving, and let $A_i(g) = \varphi_i \circ g_i \circ \varphi_i^{-1}$. Then define

$$\Theta(g) = [(A_1(g), A_2(g), \ldots, A_m(g))].$$

Lemma 9. The map $\Theta$ defines a homeomorphism between $\mathcal{P}\mathcal{H}^o(f)$ and $\mathcal{M}^o(f)/\sim$. In particular, $\mathcal{P}\mathcal{H}^o(f)$ is connected.

Corollary 1. Assume that all attracting periodic orbits of $f \in \mathcal{P}^d$ are hyperbolic and denote these attracting orbits by $\text{orb}(p_i)$, $i = 1, \ldots, m$. Then the following hold.

1. For each $\lambda \in [-1, 1]^m$, there exists $\tilde{f} \in \mathcal{P}^d$ having the same number of non-repelling periodic orbits $\text{orb}(\tilde{p}_i)$ such that the multipliers of these non-repelling periodic orbits are equal $\lambda_i$, and such that $\tilde{f} \in \mathcal{P}\mathcal{H}^o(f)$.

2. There exists $\tilde{f} \in \mathcal{P}^d \cap \mathcal{A}^*$ such that $\tilde{f} \in \mathcal{P}\mathcal{H}^o(f)$. 24
Proof of Corollary 1. Since all periodic orbits of $f$ are hyperbolic, one can define the above map $\Theta$, and so is associated to $m$ maps $A_i = z \prod_{i=1}^{n-1} \frac{z-a_i}{1-\overline{a_i}z}$. Therefore the multiplier at the attracting orbit corresponds to a product of such numbers $a_i$. By deforming these factors appropriately one can obtain any $\lambda \in (-1,1)^m$. Taking limits gives (1). Taking all $\lambda_i = 1$ gives (2). \hfill \Box

Proof of Lemma 9. As $U_i(g)$ moves continuously with respect to $g$, the map $\Theta$ is continuous. By the Rigidity Theorem 3, $\Theta$ is injective. In fact, if $\Theta(g) = \Theta(\tilde{g})$ then $g$ and $\tilde{g}$ are topologically conjugate on $\mathbb{R}$, and moreover they are conformally conjugate near the corresponding periodic attractors. Therefore $g$ and $\tilde{g}$ are affinely conjugate.

Because a continuous bijective map between open subsets of Euclidean spaces is a homeomorphism (by the Invariance of Domain Theorem), it remains to prove that $\Theta$ is surjective. Let $A = (A_1, A_2, \ldots, A_m)$ be an element in $\mathcal{M}^\circ(f)$, and our aim is to construct a map $g \in \mathcal{PH}^\circ(f)$ so that $\Theta(g) = [A]$. To do this one applies quasi-conformal surgery techniques in a standard fashion. Let us therefore be brief, and refer to the exposition given in Theorem VIII.2.1 of [4] for details. Choose $f_0 \in \mathcal{PH}^\circ(f)$ and let $U_i, s_i, n_i, \varepsilon_i$ be the objects associated to $f_0$ as above. Let $\varphi_i : U_i \to \mathbb{D}$ be a real-symmetric conformal map sending the periodic attractor in $U_i$ to 0. Then $\varphi_i \circ f_0^s \circ \varphi_i^{-1} : \mathbb{D} \to \mathbb{D}$ is a map $A_i^\circ$ in $M_{n_i, \varepsilon_i}$. Define a new map $\tilde{A}_i : \mathbb{D} \to \mathbb{D}$ as follows. Take discs $\Delta(r_i) \subset \mathbb{D}$ with $r_i < 1$ close to 1 so that $\cup_i \Delta(r_i) \times \{i\}$ is mapped into itself by $A$ and by $A^\circ$. Let $\text{Ann}_i = ((A_i^\circ)^{-1} \Delta(r_i)) \setminus \Delta(r_i)$ so that $\text{Ann}_i$ is a fundamental annulus of $A_i^\circ$. Choose a smooth map $\tilde{A}_i : \mathbb{D} \to \mathbb{D}$ which agrees with $A_i$ on $\Delta(r_i)$ and with $A_i^\circ$ on $\Delta(r_i) \cup \text{Ann}_i$ and which is a smooth (and therefore quasi-conformal) covering map on the fundamental annulus $\text{Ann}_i$. Doing this for each $i$, and going back via the maps $\varphi_i$ we obtain a smooth map $\tilde{g}$ which agrees with $g$ outside $U_i$ and is conformal except in fundamental annuli which are in $U_i$. Each orbit only hits at most once a fundamental domain, and hence the standard ellipse field is distorted at most once in each orbit of $g$. Therefore, by the Measurable Riemann Mapping Theorem, one obtains a polynomial $g \in \mathcal{PH}^\circ(f)$ with $\Theta(g) = [A]$ (because by construction $\tilde{g}$ has 'the desired dynamics' in $U_i$). \hfill \Box

To complete the proof that $\mathcal{PH}(f)$ is connected, it suffices to show the following

Lemma 10. $\mathcal{PH}(f) \subset \overline{\mathcal{PH}^\circ(f)}$.

In fact, we will need a stronger statement.

Lemma 11. For each polynomial $f$ and for each $T \in S^d$ such that $\Psi(f) \in \lceil T \rceil$ there exists a family of polynomials $f_\mu$, $\mu \in [0,1]$, with coefficients depending continuously on $\mu$, with $f_0 = f$, of fixed degree and with the same modality as $T$, such that
1. $f_\mu$ is partially conjugate to $T$ for $\mu > 0$ (note that all maps in $\langle T \rangle$ are partially conjugate to each other);

2. $f_\mu$ has only hyperbolic periodic orbits for $\mu > 0$.

**Proof of Lemma 10.** Taking $T = \Psi(f)$, the result follows from the previous lemma.

**Proof of Lemma 11.** Take a polynomial $f$ and take $T \in S_d^*$ so that $\Psi(f) \in [T]$. We need to deal with the various ways that basins of periodic attractors of $f$ can bifurcate (by saddle-node, flip or homoclinic bifurcations. To do this, define $B_\mathbb{R}(f)$ to be the set of real points that are attracted to a periodic attractor of $f$, consider any real polynomial map $Q$ which is zero on the boundary points of $B_\mathbb{R}(f)$, and let

$$F_\mu = f + \mu Q.$$  

Next define a piecewise smooth interval map

$$\hat{F}_\mu = \begin{cases} f \text{ outside } B_\mathbb{R}(f), & F_\mu \text{ inside } B_\mathbb{R}(f). \end{cases}$$

One can choose $Q$ so that (simultaneously) the following bifurcations occur for $\hat{F}_\mu$ as $\mu$ becomes positive.

(i) if $\hat{F}_0$ has a two-sided attracting periodic orbit with multiplier $-1$, a generic periodic doubling/halving bifurcation occurs as $\mu$ becomes positive;

(ii) if $\hat{F}_0$ has a two-sided attracting orbit of period $n$ with multiplier 1, a generic pitch-fork bifurcation occurs as $\mu$ becomes positive (so locally the bifurcation is that of $x \mapsto (1 \pm \mu)x - x^3$).

(iii) if a periodic orbit of $\hat{F}_0$ has multiplier 1 then one can create a saddle-node pair for $\hat{F}_\mu$ as $\mu$ becomes positive.

(iv) if a critical turning point is mapped into the boundary of the basin, then one can unfold this homoclinic tangency generically.

To see how to define $Q$, see Theorem VI.1.2 in [4]. Note that the polynomial $F_\mu$ can have much higher degree than $f$. However, $\mathcal{PH}(F_\mu)$ is constant for $\mu > 0$ small.

We should emphasize that the bifurcations (i), (ii) and (iv) can be done in either direction, but that one cannot destroy the saddle-node orbit in the above way, except if it lies in the boundary of the basin of another periodic attractor as in Figure 9. In that map, there are three degeneracies: two parabolic fixed points and
a homoclinic orbit (a critical point which is mapped to the boundary of $B_R(f)$).
For any combination of bifurcations as in (i)-(iv), there exists a polynomial $Q$ which
realizes it.

![Figure 9: A critical value is mapped to the boundary of the basin, while at the same
time there is a parabolic fixed point.](image)

Take $\mu_0 > 0$ so small that the real part of the basin of $\hat{F}_\mu$ is the same for each
$\mu \in (0, \mu_0]$. By the Fullness Theorem 6 there exists a polynomial $\hat{f} \in P^d$ which is
conjugate to $\hat{F}_{\mu_0}$. As before, we can associate to $\hat{f}$ a space $M^o(\hat{f})/\sim$ and a map

$$\Theta: \mathcal{PH}^o(\hat{f}) \to M^o(\hat{f})/\sim.$$  

Let $U_1(F_\mu), \ldots, U_k(F_\mu) \subset \mathbb{C}$ be the components of $B(F_\mu)$ (in the complex plane)
which intersect $B_R(f)$ and contain critical points of $F_\mu$ Using only these domains $U_i(F_\mu)$ and
ingoring other attracting basins that $F_\mu$ might have, we get exactly as in the proof of Lemma 9 an associated element $\Theta(F_\mu) \in M^o(f)/\sim$ which depends continuously on $\mu$. (We should emphasize that $\Theta(F_\mu)$ is somewhat an abuse of notation because $F_\mu$ is not contained in $P^d$ (because $F_\mu$ has much higher degree than $f$) and only the components of the basin of $F_\mu$ whose real trace agree with $V_1, \ldots, V_k$ are taken into account for determining $\Theta(F_\mu)$.) Next define $f_\mu = \Theta^{-1}\Theta(F_\mu)$. This
means that by definition $f_\mu$ is contained in $\mathcal{PH}^o(\hat{f})$ and that $f_\mu$ is conjugate to $\hat{F}_\mu$.
By construction all periodic orbits of $f_\mu$ are hyperbolic.

**Claim 1:** The multipliers of attracting orbits of $f_\mu$ tend to those of $f$ as $\mu \to 0$.

Indeed, the multiplier of an attractor of $f_\mu$ is the same as the multiplier of the
corresponding attractor of $F_\mu$ in $U_i(F_\mu)$. Since the latter tends to the multiplier of
the corresponding attractor of $f$ as $\mu \to 0$, the claim follows.

We now need to show that $f_\mu$ to tends to $f$. In order to do this, we need to show

**Claim 2:** If a critical point of $f$ is mapped onto a periodic orbit or onto another
critical point of $f$, then the same holds for any limit of $f_\mu$.

(The fact that $F_\mu \to f$ and $f_\mu, F_\mu$ are conjugate on $V_i$ does not imply this claim.)
Since $f_\mu \in \mathcal{PH}^a(\hat{f})$, we only need to prove this claim for critical points which are contained in one of the intervals $V_i$.

Let us first consider the case that there exists an interval $V_i$, say of period $k$, containing a critical point $c$ such that $f^k(c)$ is equal to a boundary point $p$ of $V_i$. Then there is at least one additional critical point $c' \in V_i$ which is in the basin of the attracting (possibly parabolic) orbit in $V_i$. Then $F^k_\mu(c_\mu)$ tends to $p$ as $\mu \downarrow 0$, and so the Poincaré distance $d_\mu$ (on $U_i(F_\mu)$) between $c_\mu$ and $c'_\mu$ tends to infinity. (Indeed, if $p$ is parabolic for $f$, then the domains $U_i(F_\mu)$ pinch at $p$ as $\mu \downarrow 0$, and if $p$ is a repelling point then $p$ is in the boundary of $U_i(F_\mu)$. Since the Poincaré metric is invariant under conformal maps, the corresponding statements also hold for the Blaschke products determined by $\Theta(F_\mu)$. But since $\Theta(f_\mu) = \Theta(F_\mu)$, by Claim 1, this implies that the corresponding statements also hold for $f_\mu$. This proves Claim 2 in this situation.

If $f^k(c)$ is equal to a hyperbolic periodic point or a critical $p$ in the interior of $V_i$, then $F^k_\mu(c_\mu)$ tends to the corresponding point $p_\mu$, and the Poincaré distances between $c_\mu$ and $p_\mu$ tend to zero. If $f^k(c)$ is equal to a parabolic periodic point $p$ of $f$, then the domain $U_i(F_\mu)$ pinches as $p_\mu$ as $\mu \to 0$ and there is another critical point $c'_\mu$ in $U_i(F_\mu)$ so that the Poincaré distances between $c_\mu$ and $c'_\mu$ tend to infinity. Combining this again with $\Theta(f_\mu) = \Theta(F_\mu)$ and Claim 1, the corresponding statements also hold for $f_\mu$, proving Claim 2 in every situation.

By the Rigidity Theorem we conclude from Claims 1-3 that $f_\mu$ converges to $f$ as $\mu$ tends to 0.

Note that we do not state that the Julia set of $F_\mu$ is related to that of $f$. This enables us to avoid using the techniques employed in [7] and [8].

5 Level sets in $S^d_*$ are contractible.

In this section we will give a proof of Theorem 5 (which is stated as Theorem 7 below). Recall that $S^d_*$ is the collection of non-degenerate stunted sawtooth maps $T \in S^d$ and $R_*^d = S^d \setminus S^d_*$. Hence if $J := [Z_i, Z_j]$ is the convex hull of $Z_i$ and $Z_j$, and there is $n \geq 0$ such that $T^n(J)$ is a point, then $T \sin S^d_*$ means that $T^n(J)$ is eventually mapped into a periodic plateau.

Of course, if $(Z_i, Z_j)$ is a wandering pair, then all plateaus between $Z_i$ and $Z_j$ form wandering pairs. The subset $S^d_* \subset S^d$ is chosen because $\Psi: P^d \to S^d$ fails to be surjective in a serious way (whereas $\Psi: P^d \to S^d_*$ is almost surjective in the sense of Lemma 3). Indeed, if $T \in R^d_*$ has a non-preperiodic wandering pair $(Z_i, Z_j)$ and
$\Psi(f) \in [T]$ then $f$ has a wandering interval $[c_i, c_j]$. It is well-known (see e.g. [14]) that polynomials, and in fact $C^2$ interval maps with non-flat critical points, have no wandering intervals.

**Theorem 7.** Let $L(h) = \{ T \in S^d : h_{\text{top}}(T) = h \}$ and $L_*(h) = L(h) \cap S^d_*$. Then for every $h_0 \in [0, \log(d+1)]$, the level set $L_*(h_0)$ is a contractible subset of $L(h_0)$.

It is well-known that $L(h_0)$ is contractible, see Theorem 6.1 in [18]. Contractibility of $L_*(h_0)$ is much more difficult, and we have to adjust the proof of [18] in a delicate way.

The proof involves the construction of a retract $R$ composed of an entropy decreasing deformation (to contract $L(h)$ to a single point) and an entropy increasing deformations (to keep $L(h)$ within itself). The problem is to keep $L_*(h)$ within itself under continuous action of the retract. To this end we are forced to compose $R$ of altogether six deformations. We use the letters $\Gamma, \gamma, \hat{\gamma}$ to indicate entropy increasing deformations, and $\delta, \hat{\delta}, \Delta, \hat{\Delta}$ for entropy decreasing deformations. The deformation denoted by $\beta$ will not change entropy. The letters $R$ and $r$ stand for retract.

Before we are able to give the proof of this theorem we will develop the necessary ingredients.

### 5.1 The piecewise affine case.

Take a piecewise monotone map (possibly with plateaus) $T : [a, b] \to [a, b]$ with $T([a, b]) \subset [a, b]$. We say that $X$ is a $k$-periodic cycle of intervals for $T$ if the components of $X$ are closed intervals $K_0, \ldots, K_{k-1}$ with disjoint interiors such that $T(K_i) \subset K_{i+1}$ (where we take $K_k = K_0$) for $i = 0, \ldots, k - 1$. Similarly, we say that the cycle of intervals is minimal if there exists no cycle of intervals $K'_0, \ldots, K'_{k'-1}$ with $K'_0 \subsetneq K_0$. This implies that the extremal values of $T^k|K_0$ are iterates of plateaus of $T$.

**Lemma 12.** Let $F$ be a piecewise affine map with slopes $\pm \theta$, $\theta > 1$. There exists a finite number of minimal cycles of intervals $X_i$, $i = 1, \ldots, m$ of periods $k_i \geq 1$. They have disjoint interiors and

- $\theta^{k_i} \leq 2^d$ where $d$ is the number of turning points of $F$;
- the non-wandering set $\Omega(F)$ of $F$ is the union of $\bigcup X_i$, (possibly) an invariant Cantor set $C$ and a finite number of periodic points;
- $h_{\text{top}}(F|X_i) = h_{\text{top}}(F)$ for each $i$ and $h_{\text{top}}(F|C) < h_{\text{top}}(F)$;
• $F: X_i \to X_i$ is transitive (i.e., $X_i$ contains a dense orbit);

• for any nondegenerate interval $J$ there exists an iterate $n$ so that $F^n(J)$ contains a component of $X_i$ for some $i$.

Proof. If an interval $J$ contains precisely $i$ turning points, then $|F(J)| > (\theta/2^i)|J|$. Therefore if $J$ is a minimal cycle of intervals of period $k$, then because $J, \ldots, F^{k-1}(J)$ have disjoint interiors, $|F^k(J)| > (\theta^k/2^d)|J|$ which contradicts $F^k(J) = J$ unless $\theta^k/2^d \leq 1$. The above argument shows that $F$ can only be finitely renormalizable (with a bound on the period which only depends on $\theta$). If $F$ is non-renormalizable (i.e., there is no cycle of period $\geq 2$), then the non-wandering set consists of some invariant intervals $X_i$ (with $F(X_i) = X_i$), (possibly) a Cantor set $C$ and a finite number periodic points. If it is renormalizable, then consider the first return map to a renormalization interval, and repeat the argument. Note that $h_{\text{top}}(F|X_i) = h_{\text{top}}(F) = \log \theta$ because $F$ has constant slopes $\pm \theta$, see [15] or the corollary after Theorem II.7.2 in [14]. The remaining part of the lemma follows as in Proposition III.4.4 in [14]. The dynamical volume lemma [11] states that the Hausdorff dimension of an invariant measure $HD(\mu) = h_\mu/\chi_\mu$. Since $F$ has constant slope $\pm \theta$, the Lyapunov exponent $\chi_\mu := \int \log |F'| \, d\mu = \log \theta$ and (as is easy to show) $HD(C) < 1$, for any invariant measure $\mu$ on $C$ one has $h_\mu < HD(C) \log \theta < \log \theta$. Since

\[ h_{\text{top}}(F|C) = \limsup \{h_\mu; \mu \text{ is invariant measure on } C\} \]

it follows that $h_{\text{top}}(F|C) < \log(\theta)$. \hfill \qed

Proposition 1. Assume $\theta > 1$ and that $F$ is a continuous, piecewise affine map $F: [0, 1] \to [0, 1]$ with slopes $\pm \theta$. Consider a minimal cycle of intervals $X_i$. If $c$ is a turning point in $X_i$ and $J$ a neighborhood of $c$ so that $F(\partial J)$ is a single point, then the function $\tilde{F}$ defined by

\[ \tilde{F} = \begin{cases} F & \text{outside } J, \\ F(\partial J) & \text{on } J, \end{cases} \]

satisfies $h_{\text{top}}(\tilde{F}|X_i) < h_{\text{top}}(F|X_i)$.

Proof. The inequality $h_{\text{top}}(\tilde{F}) \leq h_{\text{top}}(F)$ follows directly from the definition of $\tilde{F}$; in fact $|J| \mapsto h_{\text{top}}(\tilde{F})$ is a decreasing function. However, we need to prove strict inequality. The proof uses two technical results, Lemmas 13 and 14, which we delay until the end of this subsection.

For simplicity assume that the cycle $X_i$ consists of a single interval $H$, and so there is $N = N(J)$ such that $F^N(J) \supset H$. Also take $d = \#(\text{Crit} \cap H)$.

To compute the entropy, we will use finite Markov partitions, and transition matrices. If $G$ is a transition graph associated to some Markov partition, and $A$ is the
corresponding transition matrix, then a subgraph $\mathcal{R}$ of $\mathcal{G}$ is called arome, see Block et al. [1] and also the exposition in [3], if there are no loops in $\mathcal{G} \setminus \mathcal{R}$. The rume matrix $A_{\mathcal{R}}(x) = (a_{i,j}(x))$ is given by

$$a_{i,j} = \sum_{p} x^{-l(p)},$$

where the sum runs over all simple paths $p$ in $\mathcal{G}$ from vertex $i \in \mathcal{R}$ to vertex $j \in \mathcal{R}$ such the intermediate vertices belong to $\mathcal{G} \setminus \mathcal{R}$ and $l(p)$ is the length of the path. Then the characteristic polynomial of $A$ is equal to

$$\det(A - xI_{A}) = (-x)^{\#\mathcal{G} - \#\mathcal{R}} \det(A_{\mathcal{R}}(x) - xI_{\mathcal{R}}), \tag{4}$$

where $I_{A}$ and $I_{\mathcal{R}}$ are the identity matrices of the right dimensions.

Let $\mathcal{P}_{0}$ be the partition of $H$ given by the critical set Crit, then $\mathcal{P}_{n} = \bigvee_{i=0}^{n-1} F^{-i} \mathcal{P}_{0}$ is the partition into $n$-cylinders. This partition is almost Markov w.r.t. $F$, except that the collection $\mathcal{E}$ of the $2d$ $n$-cylinders adjacent to the critical points in $H$ need not map onto a union of $n$-cylinders. Adjust $F$ locally to $\tilde{F}$ such that $\tilde{F}(E) = \bigcup\{C' \in \mathcal{P}_{n} : F(E) \cap C' \neq \emptyset\}$ for each $E \in \mathcal{E}$.

Let $\mathcal{G}$ be the corresponding transition graph, and $\tilde{A}$ its transition matrix with leading eigenvalue $\tilde{\rho}$. (This matrix has size $\#\mathcal{P}_{n} \times \#\mathcal{P}_{n}$, i.e., it depends on $n$, but we will suppress this dependence in the notation.)

Similarly, we can flatten $F$ within the $n$-cylinders $E \in \mathcal{E}$, creating a map $F_{0}$ such that $F_{0}(E) = \bigcup\{C' \in \mathcal{P}_{n} : F(E) \supset C'\}$. Call the resulting transition graph $\mathcal{G}_{0}$ with transition matrix $A_{0}$ and leading eigenvalue $\rho_{0}$. Then we have

$$\log \rho_{0} = h_{top}(F_{0}) \leq h_{top}(F) \leq h_{top}(\tilde{F}) = \log \tilde{\rho}$$

Add to $\mathcal{G}_{0}$ an $n$-path from each $E \in \mathcal{E}$ to each $C \in \mathcal{P}_{n}$ such that $\tilde{F}^{n}(E) \supset C$, and call this extended graph $\mathcal{G}'_{0}$. (Since $E \in \mathcal{P}_{n}$, $\tilde{F}^{n}|E$ is one-to-one, and therefore one such path from $E$ to $C$ suffices.) The effect of the extra paths is that there is a one-to-one correspondence between the $m$-paths (with $m \gg n$) in $\tilde{\mathcal{G}}$ avoiding every $E \in \mathcal{E}$ in their last $n$ vertices and $m$-paths in $\mathcal{G}'_{0}$ avoiding $E \in \mathcal{E}$ in their last $n$ vertices. Thus we obtain a graph with the same exponential growth rate of $m$-paths as $\tilde{\mathcal{G}}$, having $\mathcal{G}_{0}$ as rume. The corresponding rume matrix is $A_{\mathcal{R},0}(x) = A_{0} + x^{1-n} \Delta_{0}$, where $\Delta_{0}$ is a square matrix with ones in the rows corresponding to the (at most $2d$) vertices $E \in \mathcal{E}$, and zeroes elsewhere, and therefore $||\Delta_{0}|| \leq 2d$. Now (4) becomes

$$\det(\tilde{A} - x\tilde{I}) = (-x)^{\#\mathcal{G}'_{0} - \#\mathcal{G}_{0}} \det(A_{\mathcal{R},0}(x) - xI_{0}),$$

where $\tilde{I}$ and $I_{0}$ are the identity matrices of the correct sizes. It follows that $x = \tilde{\rho}$ solves the equation $\det(A_{\mathcal{R},0}(x) - xI_{0}) = 0$, so $\tilde{\rho}$ is an eigenvalue of $A_{\mathcal{R},0}(\tilde{\rho})$. 

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The \((i, j)\)-th entry of \(A_0^k\) indicates the number of maximal subintervals of the \(i\)-th cylinder \(C_i \in \mathcal{P}_n\) that are mapped by \(F_0^k\) into the \(j\)-th cylinder \(C_j \in \mathcal{P}_n\). Therefore, the \(j\)-th column sum of \(A_0^k\) is the total number of maximal intervals that \(F_0\) maps into \(C_j\), and this is bounded by the lapnumber \(l(F_0^k)\). Since the matrix norm \(\|A_0^k\| := \sup_{\|v\|_1 = 1} \|A_0^k v\|_1\) equals the maximal column sum, we have

\[
\|A_0^k\| \leq l(F_0^m) \leq e^{k(h_{\text{top}}(F_0) + \eta_k)} = \rho_0^k e^{\eta_k},
\]

for some \(\eta_k \downarrow 0\) as \(k \to \infty\). Recall that \(A_0\) and \(\rho_0\) depend on the order of the cylinder set partition \(\mathcal{P}_n\), but the above estimate for the lapnumber is independent of \(n\). Hence we can apply Lemma 13 below with \(U_n = A_0\), \(V_n = \bar{\rho}^{-1} \Delta_0\). Indeed, if \(v\) is the positive right eigenvector of \(A_{\mathcal{R},0}(\bar{\rho})\), corresponding to leading eigenvalue \(\bar{\rho}\) and normalized so that \(\|v\|_1 := \sum_i |v_i| = 1\), then

\[
\bar{\rho} = \|\bar{\rho} v\|_1 = \|A_{\mathcal{R},0}(\bar{\rho}) v\|_1^{1/k} = \|(A_0 + \bar{\rho}^{-1} \Delta_0)^k v\|_1^{1/k} \leq \|(A_0 + \bar{\rho}^{-1} \Delta_0)^j\|_1^{1/k} \leq \rho_0(1 + e^{\tilde{\eta}_n})^{1/k} \to \rho_0 e^{\tilde{\eta}_n} \quad \text{as } k \to \infty,
\]

where \(\tilde{\eta}_n\) comes from Lemma 13.}

Recall that \(J\) is a neighborhood of \(c\) and \(N\) is such that \(F^N(J) \supset H\). By shrinking \(J\) slightly if necessary, we can assume that \(F(\partial J) \in \bigcup_{i=0}^{N-1} (F^{-i}\text{Crit})\). The map \(F_1\) is obtained by first flattening \(F\) at \(J\), so that \(F_1(J) = \bar{F}(J) = F(\partial J)\), and then lifting \(F\) locally near the other critical points (\(i.e., the ones outside \(J\)\)) to make \(\mathcal{P}_n\) a Markov partition, precisely in the same way \(\bar{F}\) was created from \(F\). Let \(\mathcal{G}_1\) be the resulting transition graph, with transition matrix \(A_1\) and leading eigenvalue \(\rho_1\).

Then

\[
h_{\text{top}}(\bar{F}) \leq h_{\text{top}}(F_1) = \log \rho_1.
\]

Add to \(\mathcal{G}_1\) an \(N\)-path from each \(C \in \mathcal{P}_n \cap J\) to each \(C' \in \mathcal{P}_n\) if \(\bar{F}^N(C) \supset C'\). Call the resulting graph \(\mathcal{G}'_1\); it has the same exponential growth rate of paths as \(\mathcal{G}\) by the argument given above, and \(\mathcal{G}_1\) is a rime of \(\mathcal{G}'_1\).

Then the transition matrix \(A'_1\) of \(\mathcal{G}'_1\) satisfies \((A'_1)^N \geq A_1^N + \Delta\), where \(\Delta\) has some 1s on rows corresponding to cylinder sets \(C \in \mathcal{P}_n \cap J\) such that each column contains at least one 1, and zeroes elsewhere. Let \(v'\) be the left eigenvector corresponding to the leading eigenvalue \(\rho_1\) of \(A_1\). Then

\[
v'(A'_1)^N \geq v'(A_1^N + \Delta) \geq (\rho_1^N + \kappa)v'
\]

component-wise, where \(\kappa = \min_{C \in \mathcal{P}_n \cap J} v'_C / \max_{C \in \mathcal{P}_n} v'_C\). It follows that

\[
\frac{1}{\rho_1^m} (\rho_1^N + \kappa)^m v' \leq \frac{1}{\rho_1^m} v'(A'_1)^m \to \alpha v
\]

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where $\alpha \leq 1$ and $\bar{v}$ is the left eigenvector of the leading eigenvalue $\bar{\rho}$ of $A'_1$. Hence

$$\bar{\rho} \geq (\rho_1^N + \kappa)^{1/N} > \rho_1,$$

(6)

where the argument below Lemma 14 shows that $\kappa$ is bounded away from 0 uniformly in $n$.

Finally, combining (5) and (6) we obtain

$$(\rho_1^N + \kappa)^{1/N} \leq \rho_0 e^{\tilde{\eta}_n}.$$ 

This is true for every $n$ and since $\tilde{\eta}_n \to 0$, there is $n$ such that the corresponding values satisfy $\rho_1 < \rho_0$. But $h_{\text{top}}(\tilde{F}) \leq \rho_1$ and $\rho_0 \leq h_{\text{top}}(F)$, so the strict inequality $h_{\text{top}}(\tilde{F}) < h_{\text{top}}(F)$ follows.

Next we prove the two technical lemmas used in the previous proof.

**Lemma 13.** Let $\{U_n\}_{n \in \mathbb{N}}$, $\{V_n\}_{n \in \mathbb{N}}$ be positive square matrices such that $\rho_n \geq 1$ is the leading eigenvalue of $U_n$. Assume that there exist $M < \infty$, $\tau \in (0, 1)$ and a sequence $\{\eta_k\}_{k \in \mathbb{N}}$ with $\eta_k \downarrow 0$ as $k \to \infty$ such that for all $n$

$$\|U_n\| \leq M, \quad \|U_n^k\| \leq \rho_n^k e^{\kappa \eta_k} \text{ and } \|V_n\| \leq M \tau^n,$$

Then there exists a different sequence $\{\tilde{\eta}_n\}_{n \in \mathbb{N}}$ with $\tilde{\eta}_n \to 0$ as $n \to \infty$ such that

$$\|(U_n + V_n)^k\| \leq (1 + e^{k \tilde{\eta}_n})\rho_n^k.$$ 

In particular, the leading eigenvalue of $\frac{1}{\rho_n}(U_n + V_n)$ tends to 1 as $n \to \infty$.

**Remark:** Here we take $\|U\| = \sup_{\|v\|_1 = 1} \|Uv\|_1$, and note that we do not assume that all $U_n$ have the same size (although $U_n$ and $V_n$ have the same size for each $n$).

**Proof.** Note that $U_n + V_n$ is a positive matrix and so its leading eigenvalue is equal to the growth rate $\lim_{k \to \infty} \frac{1}{k} \log \|(U_n + V_n)^k\|$. We have

$$(U_n + V_n)^k = \sum_{|p| + |q| = k} U_n^{p_1} V_n^{q_1} \ldots U_n^{p_t} V_n^{q_t},$$

where $p = (p_1, \ldots, p_t)$, $q = (q_1, \ldots, q_t)$ and $|p| = \sum p_i$ and $|q| = \sum q_i$. More precisely, the sum runs over all $t \in \{1, \ldots, [k/2]\}$ and distinct vectors $p, q$ with $p_i, q_i > 0$ (except that possibly $p_1 = 0$ or $q_t = 0$). Let us split the above sum into two parts.

(i) If $|q| > \varepsilon k$, then each of the above terms can be estimated in norm by

$$\|U_n\|^{p_i} \|V_n\|^{q_i} \leq M^k (\tau^n)^{\varepsilon k} = (M \tau^{\varepsilon n})^k.$$
Since there are at most $2^k$ such terms, this gives
\[ \left\| \sum_{|p| + |q| = k \atop |q| \leq \epsilon k} U_{p}^q V_n \right\| \leq (2M\tau^{\epsilon k})^k. \] (7)

(ii) If $(q_1, \ldots, q_t)$ satisfies $|q| \leq \epsilon k$, then there are at most $t-1 \leq \epsilon k$ indices $i$ with $p_i \leq N$ and at least one index $i$ with $p_i > N$, where $N < 1/(2\epsilon)$ is to be determined later. The norm of each of these terms can be estimated by $\|U_{p_i}^q\| \|M|q|^\tau^{n|q|}$, where the factors
\[ \|U_{p_i}^q\| \leq \begin{cases} \rho_i^q \epsilon N p_i & \text{if } p_i > N, \\ MN & \text{if } p_i \leq N. \end{cases} \]

So the product of all these factors is at most $\rho^k N \epsilon N k M^{\epsilon k N}$. Using Stirling’s formula, we can derive that there are at most
\[ \sum_{t=0}^{\lfloor \epsilon k \rfloor} \binom{k}{t} \leq \epsilon k \binom{k}{\lfloor \epsilon k \rfloor} \leq \sqrt{\epsilon k} \frac{1}{\epsilon} \left( \frac{1}{1-\epsilon} \right)^{(1-\epsilon)k} \leq e^{\sqrt{\epsilon k}} 
\]
possible terms of this form. Combining all this gives an upper bound of this part of
\[ \left\| \sum_{|p| + |q| = k \atop |q| \leq \epsilon k} U_{p}^q V_n \right\| \leq e^{\sqrt{\epsilon k}} \rho_i^q \epsilon N k M^{\epsilon k N} M|q|^\tau^{n|q|}. \] (8)

Adding the estimates of (7) and (8), we get
\[ \|(U_n + V_n)^k\| \leq (2M\tau^{\epsilon k})^k + e^{\sqrt{\epsilon k}} \rho_i^q \epsilon N k M^{\epsilon k N} M|q|^\tau^{n|q|}. \]

Now take $N = \sqrt[4]{n}$ and $\epsilon = 1/\sqrt{n}$ (so indeed $N < 1/(2\epsilon)$) and $n$ so large that $M\tau^n \leq 2M\tau^{\sqrt{n}} \leq 1$. Then we get
\[ \|(U_n + V_n)^k\| \leq \rho_i^k \left( 1 + e^{k(n^{-1/4} + \eta_{1/4} + n^{-1/4} \log M)} \right). \]

The lemma follows with $\tilde{\eta}_n = (n^{-1/4} + \eta_{1/4} + n^{-1/4} \log M)$.

The second technical lemma is to show that $\kappa$ from (6) is bounded away from 0 uniformly in $n$. To this end, we need some more notation. Let $W$ be a transition matrix of a graph $G$. Given a vertex $g \in G$, we can represent the 2-paths from $g$ by splitting $g$ as follows (for simplicity, we assume that the first row/column in $W$ represents arrows from/to $g$):

- If $g \to b_1, g \to b_2, \ldots, g \to b_m$ are the outgoing arrows, replace $g$ by $m$ vertices $g_1, \ldots, g_m$ with outgoing arrows $g_1 \to b_1, g_2 \to b_2, \ldots, g_m \to b_m$ respectively.
- Replace all incoming arrows $c \to g$ by $m$ arrows $c \to g_1, c \to g_2, \ldots, c \to g_m$. 


• If \( g \rightarrow g \) was an arrow in the old graph, this means that \( g_1 \) will now have \( m \) outgoing arrows: \( g_1 \rightarrow g_1, g_1 \rightarrow g_2, \ldots, g_1 \rightarrow g_m \).

**Lemma 14.** If \( W \) has leading eigenvalue \( \rho \) with left eigenvector \( v = (v_1, \ldots, v_n) \), then the transition matrix \( \tilde{W} \) obtained from the above procedure has again \( \rho \) as leading eigenvalue, and the corresponding left eigenvector is \( \tilde{v} = (v_1, \ldots, v_1, v_2, \ldots v_n) \) \( m \) times.

**Proof.** Write \( W = (w_{i,j}) \) and assume that \( w_{1,1} \neq 0 \), and the other non-zero entries in the first row are at columns \( b_2, \ldots, b_m \). The multiplication \( \tilde{v} \tilde{W} \) for the new matrix and eigenvector becomes

\[
\begin{pmatrix}
 w_{1,1} & \cdots & w_{1,1} \\
 0 & \cdots & 0 \\
 \vdots & \vdots & \vdots \\
 0 & \cdots & 0 \\
 \end{pmatrix}
\begin{pmatrix}
 0 & \cdots & 0 & \cdots & 0 \\
 w_{1,2} & \cdots & w_{1,m} & \cdots & 0 \\
 \vdots & \vdots & \vdots & \ddots & \vdots \\
 \vdots & \vdots & \vdots & \vdots & \vdots \\
 w_{n,1} & \cdots & w_{n,1} & \cdots & w_{n,n} \\
\end{pmatrix}
\]

where each indicated \( w_{i,j} = 1 \). A direct computation shows that this equals \( \rho \tilde{v} \). Since \( \tilde{v} \) is positive, it has to belong to the leading eigenvalue, so \( \rho \) is the leading eigenvalue of \( \tilde{W} \) as well. The proof when \( w_{1,1} = 0 \) is similar. \( \square \)

The effect of going from \( P_n \) to \( P_{n'} \) for \( n' > n \) is that by repeatedly applying Lemma 14, the entries \( v_C \) for \( C \in P_n \) have to be replaced by \( \#(C \cap P_{n'}) \) copies of themselves, leading to the new left eigenvector \( \tilde{v} \). As \( F^N(J) \supseteq H \), there is a constant \( K \) such that \( \#P_{n'} \leq K \#(P_{n'} \cap J) \). Thus the cylinder sets in \( J \) are replaced by a number of copies that is no more than \( K \) times the maximal number of copies by which a cylinder set can be replaced with. Therefore

\[
\frac{\min_{C \in P_{n'} \cap J} v_C}{\max_{C \in P_{n'}} v_C} \geq \frac{1}{K} \frac{\min_{C \in P_n \cap J} v_C}{\max_{C \in P_n} v_C},
\]

whence \( \kappa \) is indeed bounded away from zero uniformly in \( n' \).

This completes the proof of Proposition 1.

Although we shall not use the following proposition, we include it for completeness:
Proposition 2. Assume that $F$ and $\tilde{F}$ are topologically conjugate piecewise affine maps with slopes $\pm \theta$. Then $F$ and $\tilde{F}$ are affinely conjugate.

Proof. By Theorem B in [6] such a map $F$ (and $\tilde{F}$) has the property that there exists an interval $I$ and a subset $J$ consisting of intervals such that $I \setminus J$ has zero Lebesgue measure and such that for any component $J'$ of $J$ there exists an iterate of $F$ which maps $J'$ monotonically onto $I$. Since this construction is topological, the transformations induced by $F$ and $\tilde{F}$ are also conjugate (by the same conjugacy). Clearly the multipliers of periodic points of $F$ and $\tilde{F}$ are the same. Hence Corollary 2.9 in [13] implies that the Markov maps associated to $F$ and $\tilde{F}$ are $C^2$ conjugate. It follows that $F$ and $\tilde{F}$ are also $C^2$ conjugate. Since any $C^2$ maps which conjugates $y \mapsto \lambda y$ to itself is affine, the $C^2$ conjugacy between $F$ and $\tilde{F}$ is affine near periodic points. It follows that the conjugacy is affine everywhere. 

5.2 Semi-conjugating maps in $S$ to piecewise linear maps with constant slopes.

Let us now relate maps $T \in S$ to piecewise affine maps with constant slope, and describe their non-wandering set.

Proposition 3. Assume that $T \in S$ has $h_{\text{top}}(T) > 0$ and let $\theta = \exp(h_{\text{top}}(T))$. Then there exists a continuous, piecewise affine map $F : [0, 1] \to [0, 1]$ with slopes $\pm \theta$, and a continuous, monotone increasing map $\lambda : I \to [0, 1]$ which is a semi-conjugacy between $T$ and $F$, i.e., $\lambda \circ T = F \circ \lambda$. Let $X_i$, $i = 1, \ldots, m$, be the minimal cycles of periodic intervals associated to $F$ as in Proposition 3 and let $\text{Crit}$ be the set of turning points of $F$. Then $T$ has periodic cycles of intervals $P_i$, $i = 1, \ldots, m$, and $Q_{i'}$, $i' = 1, \ldots, m'$, such that

1. for each $i$, $P_i = \lambda^{-1}(X_i)$ and $h_{\text{top}}(T|P_i) = h_{\text{top}}(T)$.
2. Each $Q_{i'}$ is of the form $\lambda^{-1}(p)$ where $p$ is a periodic point of $F$, and $Q_{i'}$ strictly contains a plateau which is not contained in $\cup_i P_i$.
3. If $Q$ is of the form $\lambda^{-1}(x)$ and strictly contains one of the plateaus $Z_i$ then $Q$ is eventually periodic: $\lambda(Z_i)$ is eventually mapped into a periodic point of $F$ and $Q$ is eventually mapped into $\cup_i P_i \cup \cup_{i'} Q_{i'}$.
4. Let $C$ be the set of points $y$ such that $T^n(y) \notin \cup_i P_i \cup \cup_{i'} Q_{i'}$ for all $n \geq 0$. Then $h_{\text{top}}(T|C) < h_{\text{top}}(T)$.

Remark: $P_i$ is not necessarily a minimal cycle of intervals. (For example, $T$ could be infinitely renormalizable.) However, each endpoint of $P_i$ either is an iterate of
Figure 10: The maps $T$ and $\tilde{T}$ (left) and their images $F$ and $\tilde{F}$ under the semi-conjugacy.

a plateau of $T$ or - if the corresponding interval $X_i$ has a periodic point $q$ in its boundary and $\lambda^{-1}(q)$ is a non-trivial interval - a boundary point of $\lambda^{-1}(q)$.

**Proof.** Let $I = [a, b]$ and first assume $h_{\text{top}}(T) > 0$. Define $\lambda(x) = \Lambda(a, x)$ where for any interval $J$,

$\Lambda(J) = \lim_{t \uparrow 1/\theta} \frac{\sum_{n \geq 0} l(T^n|J) t^n}{\sum_{n \geq 0} l(T^n|I) t^n},$  

(9)

where $l(T^n|J)$ is the number of laps of $T^n|J$. As is proved in [17], see also Section II.8 of [14], $\lambda$ semi-conjugates $T$ to a piecewise affine map $F$ with slopes $\pm \theta := \exp(h_{\text{top}}(T))$. From this the first assertion follows (since $h_{\text{top}}(T) \geq h_{\text{top}}(T|P_i) \geq h_{\text{top}}(F|X_i) = \log \theta$ where the second inequality holds because $T$ is semi-conjugate and the last equality is proved in Lemma 12). The intervals $Q_\nu$ correspond to $\lambda^{-1}(p_\nu)$ where $p_\nu$ are the periodic turning points of $F$ which are not contained in $\cup X_i$. The third assertion holds because $T$ has no wandering intervals. The final assertion follows from the corresponding in Lemma 12.  

**Remark:** The semi-conjugacy $\lambda$ can collapse a periodic interval $J$ with $h_{\text{top}}(T|J) = h_{\text{top}}(T)$ and $\lambda$ does not depend continuously on $T$. Indeed let $J_- = [-e, 0]$ and $J_+ = [0, e]$, and take for example a map $T: [-e, e] \to [-e, e]$ such that $T(J_+) \subset J_+$ in such a way that $T: J_+ \to J_+$ has exactly three laps each of them mapping onto $J_+$. In this case, the semi-conjugate map will have slope $\pm 3$ and will have 5 laps, see Figure 10, and neither of $J_\pm$ is collapsed.. On the other hand, now take $\tilde{T}$ near $T$ so that $\tilde{T}|J_+ = T|J_+$ but so that the interior maximum of $\tilde{T}|J_-$ is mapped into $J_+$. The entropy of $\tilde{T}$ is still $\log 3$, but by computing the growth of the lap number of $\tilde{T}$ it is not hard to show that $\lambda(J_+)$ becomes a point and that $\tilde{F}$ will only have 3 laps (each with slope $\pm 3$).  

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5.3 Increasing the entropy of maps in $S$: the construction of $\gamma_t$ and $\Gamma_t$.

As before, $T : [-e, e] \to [-e, e]$ is uniquely determined by the parameters $(\zeta_1, \ldots, \zeta_d)$, and so we can define the norm

$$\text{dist}(T, \tilde{T}) = \max_i |\zeta_i - \tilde{\zeta}_i|. \quad (10)$$

Notice that

$$(\zeta_1, \ldots, \zeta_d) \leq (\tilde{\zeta}_1, \ldots, \tilde{\zeta}_d) \text{ implies } h_{\text{top}}(T) \leq h_{\text{top}}(\tilde{T}). \quad (11)$$

**Construction of $\gamma_t$:** Let $\gamma_t$ linearly increase all parameters: $\gamma_t : \zeta_i \mapsto \zeta_i + 2et$, as long as they do not map onto $\pm e$. For a fixed integer $u \geq 0$, as $t$ increases, the image of $J_{t,i} = [Z_{t,i}, Z_{t,i+1}]$ under the $u$-th iterate of $\gamma_t(T)$ becomes larger while the sizes of plateaus shrink. It follows that no new wandering pairs can be created by the deformation $\gamma_t$ as $t$ increases. Also $\gamma_t(T)$ has no non-trivial blocks of plateaus for $t > 0$. Therefore, if $T \in \mathcal{S}_d^*$ then for any $t > 0$, $\gamma_t(T) \in \mathcal{S}_d^*$.

**Construction of $\Gamma_t$:** The deformation $\Gamma_t$ uses $\gamma_t$ and the following observation. As before define $\text{dist}(T, \tilde{T}) = \max_i |\zeta_i - \tilde{\zeta}_i|$ and write $T < \tilde{T}$ if $\zeta_i < \tilde{\zeta}_i$ for all $i \in \{1, \ldots, d\}$. If $\text{dist}(T, \tilde{T}) < \varepsilon/(2e)$, then $\text{dist}(\gamma_t(T), \gamma_t(\tilde{T})) < \varepsilon$ for any $t > 0$. It follows that

$$\gamma_{t-\varepsilon}(T) < \gamma_t(\tilde{T}) < \gamma_{t+\varepsilon}(T)$$

and so

$$h_{\text{top}}(\gamma_{t-\varepsilon}(T)) \leq h_{\text{top}}(\gamma_t(\tilde{T})) \leq h_{\text{top}}(\gamma_{t+\varepsilon}(T)).$$

Hence the function $t_{\text{max}}(T) := \max\{t : h_{\text{top}}(\gamma_t(T)) = h_0\}$ is continuous in $T$. Therefore

$$\Gamma_t(T) := \gamma_{t_{\text{max}}(T)}(T).$$

is continuous in $t$ and $T$ as well. By construction

$$\Gamma_1 : \cup_{s < h_0} \overline{L_*(s)} \to L_*(h_0) \quad (12)$$

and $\Gamma_1 : \overline{L_*(h_0)} \to \overline{L_*(h_0)}$.

5.4 Decreasing entropy of maps in $S$: the construction of $\delta_t$, $\hat{\delta}_t$ and the retract $r_t$.

**Construction of $\delta_t$:** Define the ‘sign’

$$\text{sgn}(Z_i) = \begin{cases} 0 & \text{if } Z_i \text{ touches another plateau,} \\ 1 & \text{otherwise.} \end{cases} \quad (13)$$
Now deform $T$ according to the flow defined by the system of differential equations on the parameters, where we will now indicate the $t$-dependence by $\zeta_{i,t}(T)$.

$$\frac{d\zeta_{i,t}(T)}{dt} = \begin{cases} -2 \text{sgn}(Z_i)e & \text{if } \zeta_{i,t}(T) \in (-e, e) \\ 0 & \text{otherwise.} \end{cases}$$

$$\zeta_{i,t}(T)|_{t=0} = \zeta_i(T).$$

Let us denote the resulting deformation by $\delta_t$. The deformation $\delta_t$ decreases/increases the height of a plateau if it is local maximum/minimum of $T$ continuously until this plateau touches a neighboring plateau. Note that $\delta_t(T) = \delta_s(T)$ for $s, t \geq 1$ but this might mean that all plateaus of $\delta_t(T)$ touch another plateau, and therefore this is not a guarantee that $h_{\text{top}}(\delta_t(T)) = 0$. Therefore deformation is not sufficient for our purposes (but as we will see in Section 5.9 we do need this deformation.)

We will make use of $\delta_t$ however because it has the useful property that if $T \in S^d_*$ then for $s \in (0, t)$ sufficiently small, $h_{\text{top}}(\gamma_s(\delta_t(T))) \leq h_{\text{top}}(T)$. (This property will fail for the deformation $\hat{\delta}_t$ that we are about to construct, see Figure 11 on page 48.)

Now we will define another, but related deformation. The difference in the deformation $\delta_t$ and the deformation $\hat{\delta}_t$ defined below is that $\delta_t$ decreases the parameter $\zeta_i$ until the corresponding plateau $Z_i$ touches another plateau (or reaches $e$); blocks (consisting of more than one plateau) do not move under $\delta_t$. The deformation $\hat{\delta}_t$ also moves blocks of plateaus, provided they form a local extremum, which happens whenever the block consists of an odd number of plateaus. Blocks of an even number of plateaus are not moved by $\hat{\delta}_t$ (unless an extra plateau joins the block).

**Construction of $\hat{\delta}_t$:** A natural variant is the deformation $\hat{\delta}_t$ which will increase the width of plateaus to decrease entropy to 0. This may introduce new wandering pairs. Define the ‘sign’ for plateaus that are part of a block of plateaus:

$$\hat{\text{sgn}}(Z_i) = \begin{cases} 0 & \text{if } Z_i \text{ is part of a block of an even number of plateaus,} \\ 1 & \text{if } Z_i \text{ touches no other plateau or is the } 2k+1 \text{-th plateau in a block of an odd number of plateaus,} \\ -1 & \text{if } Z_i \text{ is the } 2k \text{-th plateau in a block of an odd number of plateaus.} \end{cases}$$

Notice that $\hat{\text{sgn}}(Z_i)$ depends not only on $i$ but also on $T$, and $\hat{\text{sgn}}(Z_i) = \pm 1$ means that $T$ has a local extremum at the block of plateaus that $Z_i$ is part of. We deform $T$ according to the flow defined by the system of differential equations on the parameters, where we will now indicate the $t$-dependence by $\zeta_{i,t}(T)$.

$$\frac{d\zeta_{i,t}(T)}{dt} = \begin{cases} -2 \hat{\text{sgn}}(Z_i)e & \text{if } \zeta_{i,t}(T) \in (-e, e) \\ 0 & \text{otherwise.} \end{cases}$$

$$\zeta_{i,t}(T)|_{t=0} = \zeta_i(T).$$
This means that if the \( Z_i \) (or the block of touching plateaus touching \( Z_i \)) represents a local maximum/minimum, then the deformation decreases/increases it, while if it corresponds to an inflection plateau or is already completely flat (i.e., has value equal to \( \pm \epsilon \)), then it remains unchanged. During the deformation blocks can collide, and then the combined larger blocks are deformed according to the same rule. (As a result \( \text{sgn}(Z_i) \) can change during the deformation.) Obviously this defines a continuous deformation \( \hat{\delta}_t \) with the property that \( t \mapsto \hat{\delta}_t(T) \) (not necessarily strictly) decreases the topological entropy.

If \( d \) is odd and \( t \geq 1 \), then all plateaus of \( \hat{\delta}_t(T) \) will touch and the map \( \hat{\delta}_t(T) \) is constant \( \pm \epsilon \). If \( d \) is even and \( t \geq 1 \), then each map \( \hat{\delta}_t(T) \) will be monotone (with some blocks of touching plateaus). More precisely, if \( t \geq 1 \) then \( \hat{\delta}_t(T) \in \Sigma^d_t \) where

\[
\Sigma^d_t = \begin{cases} 
\{ T_0(x) \equiv \pm \epsilon \} & \text{if } d \text{ is odd and } \epsilon = \mp; \\
\{ \text{monotone maps in } \mathcal{S}^d_t \} & \text{if } d \text{ is even.}
\end{cases}
\]

**Construction of the retract \( r_t \):** Since \( \Sigma^d_t \) is a singleton in the first case and a simplex in the second case, there exists a continuous retract \( r_t : \Sigma^d_t \to \Sigma^d_t \) with \( r_0 = id \) and \( r_1 \equiv T_0 \) with \( T_0 \) some map in \( \Sigma^d_t \). So if we apply first \( \delta_t \) for \( t \in [0, 2] \) and then \( r_t \) we obtain a retract of \( \mathcal{S}^d \) to a point. Of course this is insufficient for our purposes as this is not a retract of \( \mathcal{S}^d_t \).

5.5 The case \( h_0 = 0 \).

In the case that \( h_0 = 0 \) we can now complete the proof using the following lemma.

**Lemma 15.** Let \( T \in \mathcal{S}^d \) with \( h_0 = h_{\text{top}}(T) = 0 \) and assume that \( t > 0 \). Then there exists \( k_0 < \infty \) such that all periodic attractors of \( \hat{\delta}_t(T) \) are of the form \( 2^k \), \( k \leq k_0 \).

**Proof.** It is well-known that every interval map of zero entropy and finite modality has only periodic points of period \( 2^k \) for \( k \in \mathbb{N} \). Take \( T \in \mathcal{S}^d \) such that \( h_{\text{top}}(T) = 0 \). If \( Z_i \) is a plateau with an infinite orbit, then \( T \) must be infinitely renormalizable, i.e., there exists a sequence of periodic intervals \( K_u \), \( u \in \mathbb{N} \), with period \( 2^u \) such that \( \cap_u K_u \supset Z_j \) for some \( j \), and \( \omega(Z_i) = \omega(Z_j) \). In fact, \( \text{orb}(K_u) \) can contain more plateaus, but since the period of \( K_u \) tends to infinity as \( u \to \infty \), and there are only \( d \) plateaus, we can assume (by an appropriate choice of \( K_u \)) that there exists \( n_u \to \infty \) as \( u \to \infty \) such that \( T^n(K_u) \) does not intersect any plateau for \( 0 < n < n_u \). Therefore \( |T(K_u)| \to 0 \) as \( u \to \infty \).

Since \( T \in \mathcal{S}^d \) there exists \( t_1 < t \) and \( \delta \) such that all plateaus of \( T \) move at \( \delta \) when \( t' \) moves from 0 to \( t_1 \) (for \( t' \) small \( \delta_{t'} \) agrees with \( \hat{\delta}_{t'} \)). For \( u \) so large that \( |T(K_u)| < \delta \), this means that \( K_u \) is no longer invariant under \( \hat{\delta}_{t_1}(T) \) and so this map...
is not infinitely renormalizable anymore. Instead, there is $k_0$ such that every plateau of $\hat{\delta}_1(T)$ is (eventually) periodic with period $2^k$ for some $k \leq k_0$. If we increase $t'$ further from $t_1$ to $t$, each periodic attractor remains but can undergo period halving bifurcations. So all periodic orbits of $\hat{\delta}_1(T)$ are of the form $2^k$, $k \leq k_0$.

Now define a retract $R_t$ of the zero-entropy level set of $S^d_*$ as follows:

$$R_t = \begin{cases} \hat{\delta}_{2t} & \text{for } t \in [0, 1/2], \\ r_{2t-1} \circ \hat{\delta}_1 & \text{for } t \in [1/2, 1]. \end{cases}$$

The previous lemma implies that under $\delta_{2t}(T)$ (resp. $r_{2t-1} \circ \hat{\delta}_1(T)$), each point is either periodic or eventually mapped into the basin of one of the periodic attractors of $\hat{\delta}_{2t}(T)$ (resp. $r_{2t-1} \circ \hat{\delta}_1(T)$). This follows from the general structure of maps with only a finite number of periodic orbits, see for example [14]. Hence $\hat{\delta}_1(T), r_t \circ \hat{\delta}_1(T) \in S^d_*$. Thus we have defined a retract of $S^d_*$ and proved Theorem 7 in this case.

The remainder of this section will deal with the case that $h_0 > 0$. In that case we shall encounter several additional difficulties.

### 5.6 The retract $R_t$ when $h_0 > 0$ and the trouble with $\hat{\delta}_t$.

If we only had to construct a retract of an entropy level set of $S^d$ then we would could finish the construction as follows. The deformation $R_t$ defined by $\Gamma_{3t}$ for $t \in [0, 1/3]$, $\Gamma_1 \circ \hat{\delta}_{3t-1}$ for $t \in [1/3, 2/3]$ and $\Gamma_1 \circ r_{3t-2} \circ \hat{\delta}_1$ for $t \in [2/3, 1]$ would form the required retract of the space $S^d$ to a point.

However, we need to construct a retract of an entropy level set of $S^d_*$ (so the deformation is not allowed to leave the space $S^d_*$). So the hurdle we have to overcome is that if $T \in S^d_*$ then $\hat{\delta}_t(T)$ need no longer be in $S^d_*$ for $t > 0$ because the deformation $t \mapsto \hat{\delta}_t(T)$ can create wandering pairs $(Z_i, Z_j)$. To resolve this issue, the aim is to ensure that the deformation $\Gamma_t$ (or a similar deformation) will be able to ‘undo’ these wandering pairs. In view of (12) we will construct a deformation $\beta_t$ with the property that when $T \in S^d_*$ then $\beta_1(T) \in S^d_*$. It does this by deforming $T$ in such a way that $\beta_1(T)$ never eventually maps an interval of the form $[Z_i, Z_{i+1}]$ into the interior of another plateau. However, (12) only applies to maps $T \in S^d_*$ with topological entropy $h_0$. For this reason we need to define a more subtle way of ‘lowering’ and ‘increasing’ the map while the topological entropy is equal to $h_0$. These analogues of $\hat{\delta}$ and $\Gamma_1$ may act on some of the plateaus while leaving some others alone. The challenge will be to define this as a continuous deformation. To summarize, we will introduce three additional deformations.

$\hat{\gamma}_t$: To increase the topological entropy more carefully by increasing some (but
possibly not all) ζ’s, so we essentially increase each ζ ‘as far as possible’. This is the purpose of \( \hat{\gamma}_t \) defined in Section 5.8.

\( \Delta_t \): To decrease the topological entropy more carefully in such a way that if \( \hat{\gamma}_t(T') \) or \( \Gamma_t(T') \) does not move certain plateaus (because otherwise the entropy would be too large), then we ‘can assume’ that \( T' \in S^d_* \). This is the purpose of \( \Delta_t \) defined in Section 5.9.

\( \beta_t \): Finally, we want to ensure that we only need to apply \( \hat{\gamma}_s \) to maps \( T' \in S^d \) with the property that a convex hull \([Z_i, Z_{i+1}]\) is never eventually mapped into the interior of another plateau (i.e., only to maps with \( T' \in S^d_* \)). This means that \( \gamma_s(T') \in S^d \) for every \( s > 0 \). Unfortunately, this may not be enough because it may be that \( h_{\text{top}}(\hat{\gamma}_s(T')) > h_0 \) for any \( s > 0 \). The deformation \( \beta_t \), defined in Section 5.7, will not change the entropy.

### 5.7 The construction of \( \beta_t \).

Consider a map \( T' = \hat{\delta}_t(T) \) so that the convex hull \([Z_i, Z_{i+1}]\) of two neighboring plateaus are eventually mapped into the interior of another plateau. If this happens then for \( s > 0 \) small, \( \gamma_s(T') \) will still have this property and so in general \( \gamma_s(T') \notin S^d_* \). To overcome this problem we define another deformation \( \beta_t \). This transformation does not change topological entropy, and only moves plateaus which are mapped into other plateaus.

Given a map \( T \in S^d \), let \( I(T) \) be the collection of integers \( i \) such that \((Z_i, Z_{i+1})\) is a wandering pair for which there exists \( n > 0 \) so that

1. the convex hull \( J_i = [Z_i, Z_{i+1}] \) is mapped in \( n \) steps into a plateau \( Z_{k(n)} \);
2. \( T^s(J_i) \cap Z_{k(n)} = \emptyset \) for each \( 0 \leq s < n \);
3. \( T^s(J_i) \) does not intersect a periodic plateau for \( 0 \leq s < n \).

That is, \( J_i \) is contained in a first entry domain into a plateau, and if \( Z_i \) is periodic then \( i \notin I(T) \). If \( J_i \) only maps into blocks of plateaus (and never into a single plateau), then \( i \notin I(T) \). It is worth emphasizing that, as before, plateaus are closed. If \( i \in I(T) \) then let \( n_i > 0 \) be the largest integer as above and \( Z_{k(i)} \) the corresponding plateau. So \( Z_{k(i)} \) either is a periodic plateau, or a plateau which is never mapped into another plateau.

The purpose of \( \beta_t \) is to make sure that \( \beta_1(T) \in \overline{S}^d \). The deformation \( \beta_t \) will not change the parameters \( \zeta_{k(i)}, i \notin I(T) \), but it will potentially increase some other \( \zeta_j \)'s. Let us now determine which plateaus we need to shrink.
Note that either \( T(J_{i-1}) \subset T(J_i) \) or \( T(J_i) \subset T(J_{i-1}) \). Let us say that the latter holds; if \( i, i-1 \in I(T) \) and \( T^{n_i-1}(J_{i-1}) \subset Z_{k(i-1)} \) then \( T^{n_i-1}(J_i) \subset Z_{k(i)} \) and, by maximality of \( n_{i-1} \), later iterates of \( J_i \) do not map into (new) plateaus. It follows that
\[
i - 1, i \in I(T) \text{ implies } n_{i-1} = n_i, Z_{k(i)} = Z_{k(i-1)}.
\]

Let the subset \( I^o(T) \) of \( I(T) \) consist of those \( i \in I(T) \) for which \( T^{n_i}(J_i) \subset \text{int}(Z_{k(i)}) \). Note that \( T^u(J_i) \) can meet another plateau for \( u < n_i \) and so the endpoints of \( T^{s_i}(J_i) \) do not have to be iterates of \( Z_i \) and \( Z_{i+1} \); this is why we formulate the above condition in terms of \( J_i \) (rather than in terms of \( Z_i, Z_{i+1} \)). Moreover, for \( i \in I^o(T) \) define \( I_i(T) \) to be the set of integers \( a \in \{1, \ldots, d\} \) for which \( Z_a \) is a non-periodic plateau which intersects \( T^u(J_i) \) for some \( 0 < u < n_i \).

**Construction of \( \beta_i \):** Let us define a deformation \( \beta_i(T) \) of \( T \) by continuation. To a one-parameter family \( T_t \), we associate \( Z_{t,i}, J_{t,i}, I(t), I^o(T), I(T) \) and the parameters \( (\zeta_{t,1}, \ldots, \zeta_{t,d}) \). Given \( T_t \), we will define \( \varepsilon > 0 \) and \( T_s \) for \( s \in [t, t + \varepsilon] \) as follows.

- If \( j \in I^o(T) \) or \( j - 1 \in I^o(T) \), then take
  \[
  \zeta_{s,j} = \zeta_{t,j} + (s - t)2e.
  \]  

- For each other \( j \), take
  \[
  \zeta_{s,j} = \zeta_{t,j} \text{ for all } s \geq 0.
  \]

In particular, periodic plateaus and plateaus which never map into other plateaus do not change in size.

Next choose the maximal \( \varepsilon(T) > 0 \) so that all of the sets \( I^o(T_t) \) and \( I_i(T_t) \), \( i \in I^o(T_t) \) remain constant for \( t \in [0, \varepsilon(T)] \), and define \( \beta_i(T) = T_t \) for \( t \in [0, \varepsilon(T)] \). (Claim 1 below implies that indeed \( \varepsilon(T) > 0 \).) Next define \( \beta_t(T) = \beta_{t-\varepsilon(T)}(\beta_{\varepsilon(T)}(T)) \) for \( t \in [\varepsilon(T), \varepsilon(\beta_{\varepsilon(T)}(T))] \). By repeating this given \( T \), we get that \( \beta_t(T), t \geq 0 \) satisfies the semi-flow property \( \beta_{t_1 + t_2}(T) = \beta_{t_1} \circ \beta_{t_2}(T), t_1, t_2 \geq 0 \).

Let us analyse in more detail what happens as \( t \) increases. For convenience, let us say that the endpoint of a non-degenerate interval \( A_t \) moves *outwards with speed \( b \) if the right/left endpoint of \( A_t \) increases/decreases with speed \( b \).

**Claim 1:** If \( a \in I_i(T) \) then \( a \in I_i(\beta_i(T)) \) for \( t > 0 \) small.

**Proof of Claim 1:** Let us first analyse what happens if \( \partial T^u(J_i) \cap \partial Z_a \neq \emptyset \) for some \( u < n_i \) and that \( u \) is minimal with this property. Let \( J_{t,i} \) and \( Z_{t,a} \) be the corresponding objects for \( \beta_i(T) \) and see with what speed these vary with \( t \). Since \( u \) is minimal and the slope of \( T \) is in \( \{-\lambda, \lambda, 0\} \), the chain rule implies that the images of the endpoints of \( J_{t,i} \) under the iterate \( (\beta_i(T))^u \) move outwards with speed
λu2e. At the same time the endpoints of the plateau \( Z_{t,a} \) move at most with speed \( 2e/λ \). Hence the \( u \)-th iterate of \( J_{t,i} \) ‘overtakes’ and then intersects the interval \( Z_{t,a} \) for \( t > 0 \). Therefore \( a \in I_i(β_t(T)) \) for \( t > 0 \) small and \((β_t(T))^{u+1}(J_{t,i})\) intersects \( β_t(Z_o) \) for \( t > 0 \) small. In particular, the images of the endpoints of \( J_{t,i} \) under \((u+1)\)-iterates still move outwards at least with speed \( 2e \). It follows that for any \( u < n_i \), the endpoints of \((β_t(T))^{u}(J_{t,i})\) move outwards with at least speed \( 2e \), and the claim follows. We also proved

**Claim 2:** If \( i \in I^o(T) \) then the endpoints of \((β_t(T))^{n_i}(J_{t,i})\) move outwards with speed at least \( 2e \). In particular, the set \( I^o(β_t(T)) \) can only get smaller as \( t \) increases (By the construction, if \( i \in I^o(T) \) then \( i \in I(β_t(T)) \) for each \( t \geq 0 \).)

**Claim 3:** For some \( t < 1 \), \( I^o(β_t(T)) = \emptyset \).

**Proof of Claim 3:** By Claims 1 and 2 and since each plateau has size \( \leq 2e \), for each \( i \in I^o(T) \) there exists \( t \in (0, 1) \) so that \( i \in I(β_t(T)) \setminus I^o(β_t(T)) \) for \( s \geq t \).

It follows that \( I^o(β_t(T)) \) and \( I_i(β_t(T)) \) only change at most \( 2d \) times. By construction, \( β_t(T) \) has the property that for no \( i \in \{1, \ldots, d-1\} \), the interval interval \( J_i = [Z_i, Z_{i+1}] \) is mapped into the interior of some plateau.

Let us now prove that the deformation \((T, t) \mapsto β_t(T)\) is continuous. First, we will show that the set of times when \( β_t(T) \) is deformed according to (15) while \( β_t(T) \) is deformed according to (16) (or vice versa) consist of at most \( 2d \) small intervals (provided \( T \) and \( T \) are close). For this it suffices to show that \( β_t(T) \) depends continuously on \( t \) and \( T \) for \( t > 0 \) small, and consider the case that \( T \) is deformed as in (15) and arbitrarily near to \( T \) are maps \( T \) which are deformed as in (16), or vice versa. Let us show that there exists \( t > 0 \) small \((t \) can be chosen arbitrarily close to zero by taking \( T \) and \( T \) sufficiently close\) such that for each \( j \), \( β_t(T) \) and \( β_t(T) \) are both deformed according to the rule (15) or both according to the rule (16) for \( t \). For this it is enough to consider the following cases:

**Case 1:** There exists \( j \) such that \( j \in I(T) \setminus I^o(T) \) and \( j \in I^o(T) \setminus I^o(\tilde{T}) \) for some arbitrarily nearby maps \( T \) and \( T \). If this holds, then \( T^u(J_j) \) has a common boundary point with \( Z_{k(j)} \) and \( I^o(\tilde{T}) \subset \text{int}(Z_{k(j)}) \). It follows from Claim 2 above (applied to \( T \)) that \( j \notin I^o(β_t(T)) \) for some \( t > 0 \) with \( t \) close to zero and with \( β_t(T) \) and \( β_t(T) \) close together, if \( T \) and \( T \) are close together.

**Case 2:** There exists \( j \) such that \( j \in I(T) \setminus I^o(T) \), but \( j \notin I(T) \setminus I^o(\tilde{T}) \) for some arbitrarily nearby maps \( T \) and \( T \). This implies that \( T^u(J_j) \) has a boundary point in common with \( Z_j \). It follows from Claim 2 that for any nearby \( T \) there exists \( t > 0 \) so that \( β_s(T)^u(J_j) \) intersects \( Z_{s,a} \) for \( s \geq t \).

It follows that if \( T, T \) are close and different deformation rules are applied to \( T \) and \( T \), then there exists \( t > 0 \) small so that for \( β_t(T) \) and \( β_t(T) \) the same deforma-
tion rules are applied and that \( \beta_t(T), \beta_t(\tilde{T}) \) are also close. From this continuity of \((t, T) \mapsto \beta_t(T)\) follows. Note that \( \beta_t(T) \) coincides with \( T \) except for intervals that are mapped into plateaus, and hence cannot contribute to the exponential growth rate of lapnumbers. Recall from [15] that \( \lim_{n \to \infty} \frac{1}{n} \log l(f^n) = h_{\text{top}}(T) \), so it follows that \( \beta_t \) does not change entropy.

By construction the convex hull of plateaus of \( \beta_1(T) \) is never mapped into the interior of another plateau.

### 5.8 Increasing entropy of maps more carefully: the construction of \( \hat{\gamma}_t \).

We use the entropy increasing deformation \((t, T) \mapsto \gamma_t(T)\) until \( h_{\text{top}}(\gamma_t(T)) = h_0 \). But it is possible that only part of the phase space is responsible for reaching this entropy bound, while in other parts (namely in renormalization cycles, see definition below), plateaus have not been lifted “sufficiently” yet. Thus in the presence of renormalization intervals, we may need to lift some plateaus faster than others. This subsection explains how this is done.

An interval \( K \) is a renormalization interval for \( T \in S^d \) if \( T^n(K) \subset K \) and \( f^n(\partial K) \subset \partial K \). The set \( \text{orb}(K) = K \cup T(K) \cup \cdots \cup T^{n-1}(K) \) is called a renormalization cycle. Note that \( \partial K \) consists of (pre)periodic points which do change with \( T \) (unless they disappear in a saddle node bifurcation).

**Remark:** If the period of some renormalization interval \( \tilde{K}_i(T) \) containing \( Z_i \) is \( m \), then \( h_{\text{top}}(T|\text{orb}(\tilde{K}_i)) < \frac{d}{m} \log 2 \). So if we assume that \( h_{\text{top}}(T|\text{orb}(K_i(T))) = h_0 \), then this implies that the period of \( K_i(T) \) cannot be too large.

Fix \( h_0 \in (0, \log(d+1)] \). Given a plateau \( Z_i \) which is part of a wandering pair with convex hull \( J = [Z_i, Z_j] \), let \( K_i(T) \) be the smallest renormalization interval for \( T \) which contains \( Z_i \) and such that \( h_{\text{top}}(T|\text{orb}(K_i(T))) \geq h_0 \). Since \( h_0 > 0 \), the above remark shows that there is indeed such a smallest renormalization interval. Let

\[
L_i(h_0) = \text{clos}\{T \in S^d : h_{\text{top}}(T|\text{orb}(K_i(T))) = h_0\},
\]

and

\[
L_i(h_0^+) = \text{clos}\{T \in S^d : h_{\text{top}}(T|\text{orb}(K_i(T))) > h_0\}.
\]

Consequently, \( L_i(h_0) \cap L_i(h_0^+) \) contains maps whose entropy restricted to \( \text{orb}(K_i(T)) \) is \( h_0 \) but can be increased by an arbitrarily small change in parameter \( \zeta_j \) for some \( Z_j \subset \text{orb}(K_i(T)) \).

Define

\[
\Psi_j(T) = \text{dist}(T, L_j(h_0) \cap L_j(h_0^+))
\]
where dist is as in (10), and we set dist}(T, \emptyset) = 1. If 0 < h_0 < \log(d + 1), define the following deformation of maps \( T \in \mathcal{S}^d \):

\[
\hat{\gamma}_t(\zeta_1, \ldots, \zeta_d) = (\min(\zeta_1 + \Psi_1(T)t, e), \ldots, \min(\zeta_d + \Psi_d(T)t, e)).
\]

If \( h_0 = \log(d + 1) \), i.e., \( \zeta_i = e \) for all \( i \), then define \( \hat{\gamma}_t = id \) for \( t \in [0,1] \).

**Lemma 16.** Assume \( h_0 > 0 \), define \( \hat{\gamma}_t \) as above and let \( T \in \mathcal{S}^d \) be such that \( h_{top}(T) \leq h_0 \). Then the following hold:

1. The deformation \((t, T) \mapsto \hat{\gamma}_t(T)\) is continuous in \( T \) and \( t \) and \( t \mapsto h_{top}(\hat{\gamma}_t(T)) \) is non-decreasing.

2. \( h_{top}(\hat{\gamma}_t(T)) \leq h_0 \) for all \( 0 \leq t \leq 1 \).

3. If \( T \in \overline{\mathcal{S}^d} \) and \( T \in L_j(h_0) \cap L_j(h_0^+) \) implies that \( Z_j \) is contained in the closure of a component of the basin of a periodic attractor, then \( \hat{\gamma}_t(T) \in \mathcal{S}^d \) for each \( t > 0 \).

**Proof.** Continuity and monotonicity of statement 1. are obvious.

The difficult part in this proof is statement 2. Take \( t \in [0,1] \) and \( T \in \mathcal{S}^d \) with \( h_{top}(T) \leq h_0 \). Let \( j_1 \) be such that \( m_1 := \Psi_{j_1}(T) \) is maximal among \( \{\Psi_1(T), \ldots, \Psi_d(T)\} \), and let \( M_1 \) be the open \( d \)-dimensional \( m_1 \)-cuboid centered at \( T \), parallel to the coordinate axes and of length \( 2m_2 \) to the side. (In fact, \( M_1 \) is a genuine cube, but the sets \( M_i \) below are only cuboids.) Then \( M_1 \) is disjoint from \( L_{j_1}(h_0) \cap L_{j_1}(h_0^+) \), and in particular disjoint from \( L_{j_1}(h_0^+) \). Therefore \( h_{top}(\{ \text{orb}(K_{j_1}) \}) \leq h_0 \) for any \( T' \in M_1 \) and in particular for \( \hat{\gamma}_t(T) \).

Now let \( j_2 \) be such that \( m_2 := \Psi_{j_2}(T) \) is second largest among \( \{\Psi_1(T), \ldots, \Psi_d(T)\} \). The corresponding \( m_2 \)-cuboid \( M_2 \) is the set of \( T' \) with parameters \( \{\zeta_1, \ldots, \zeta_d\} \) such that \( |\zeta_j - \zeta_j(T)| < m_2 \) for all \( j \neq j_1 \) and \( |\zeta_{j_1} - \zeta_j(T)| < m_1 \). (This is the Cartesian product of an \( d - 1 \)-dimensional cube and an arc of length \( 2m_1 \) in the \( \zeta_{j_1} \)-direction.) The set \( M_2 \) disjoint from \( L_{j_2}(h_0) \cap L_{j_2}(h_0^+) \), and from \( L_{j_2}(h_0^+) \). For \( T' \in M_2 \), there are two cases:

(i) \( \text{orb}(K_{j_2}(T')) = \text{orb}(K_{j_1}(T')) \). Since the \( m_2 \)-cuboid is contained in the \( m_1 \)-cuboid, \( h_{top}(T'|\text{orb}(K_{j_2}(T'))) = h_{top}(T'|\text{orb}(K_{j_1}(T'))) \leq h_0 \).

(ii) \( \text{orb}(K_{j_2}(T')) \neq \text{orb}(K_{j_1}(T')) \). In this case, the parameter \( \zeta_{j_1}(T') \) has no influence on \( h_{top}(T'|\text{orb}(K_{j_2}(T'))) \) and since the \( m_2 \)-cuboid is disjoint from \( L_{j_2}(h_0^+) \), \( h_{top}(T'|\text{orb}(K_{j_2}(T'))) \leq h_0 \).

Therefore both \( h_{top}(T'|\text{orb}(K_{j_2}(T'))) \leq h_0 \) and \( h_{top}(T'|\text{orb}(K_{j_1}(T'))) \leq h_0 \), and this holds in particular for \( T' = \hat{\gamma}_t(T') \).

Continuing inductively, we see that if \( T' \in \cap_k M_k \), then \( h_{top}(T'|\text{orb}(K_{j}(T'))) \leq h_0 \) for each \( j \), and this holds in particular for \( T' = \hat{\gamma}_t(T) \). If \( K_j(\hat{\gamma}_t(T)) = [-e, e] \) for
some \( j \), then this proves statement 2. If, however, every plateau belongs to some renormalization cycle and the entropy of \( \hat{\gamma}_t(T) \) is carried by the Cantor set of points that never enter these renormalization cycles, then we argue as follows.

Assume by contradiction that \( h_{\text{top}}(\hat{\gamma}_t(T)) > h_0 \). Take \( t_0 < t \) maximal such that \( h_{\text{top}}(T') \leq h_0 \) for \( T' = \hat{\gamma}_{t_0}(T) \). Clearly \( t_0 < t \) and \( T' \in \cap_k M_k \). The latter, and the fact that \( \cap_k M_k \) is open imply that \( T' \) has a \( \varepsilon \)-neighborhood \( U_\varepsilon \subset \cap_k M_k \) such that \( h_{\text{top}}(T'')|\text{orb}(K_j(T'')) \leq h_0 \) for every \( T'' \in U_\varepsilon \). (Note that this is regardless of whether \( K_j(T''') = K_j(T'') \) or not.) This means that for all \( t_1 \in (t_0, t_0 + \varepsilon) \) and \( T'' = \hat{\gamma}_{t_1}(T) \), \( K_j(T''') \neq [-e, e] \) for all \( j \). So then \( h_{\text{top}}(T''|\text{orb}(K_j(T'''))) \leq h_0 < h_{\text{top}}(T''') \). But this means that moving \( t_1 \in [t_0, t_0 + \varepsilon) \) has no effect on the topological entropy, which contradicts that \( t_0 \) is maximal. (Here we use the following observation. The set of parameters for which \( T_s \) has some interval as a renormalization interval is closed. So either there exists some set of the form \([t_0, t_0 + \varepsilon]\) such that \( T_s \) has a renormalization interval of period \( \geq 2 \) containing \( Z_j \) for each \([t_0, t_0 + \varepsilon]\) or there exist parameters for which the only renormalization interval containing \( Z_j \) is equal to \([-e, e]\).) Now for \( t = 1 \), continuity of entropy ensures that \( h_{\text{top}}(\hat{\gamma}_1(T)) = \lim_{t\to1} h_{\text{top}}(\hat{\gamma}_t(T)) \leq h_0 \).

Finally, statement 3. follows essentially from the definition of \( \hat{\gamma}_t \) and from the fact that by increasing the parameters \((\zeta_1, \ldots, \zeta_d)\) no new wandering pairs can be created.

\[ \square \]

### 5.9 Decreasing the entropy more carefully: the construction of \( \Delta_t \).

Take \( T \in \mathcal{S}_d^4 \) with \( h_{\text{top}}(T) = h_0 > 0 \). Even though \( t \mapsto h_{\text{top}}(\hat{\delta}_t(T)) \) is non-increasing, it is possible that for fixed \( t > 0 \), \( h_{\text{top}}(\gamma_s \circ \hat{\delta}_t(T)) > h_0 \) for all \( s > 0 \). The reason is that (in the notation of Section 5.4) \( \text{sgn}(Z_i) \) can change from 1 to \(-1\) (or vice versa) during the deformation. To explain what can happen, let us discuss the example from Figure 11 on page 48. Although \( T \in \mathcal{S}_d^4 \), the map \( T' = \hat{\delta}_t(T) \) has a wandering pair that does not map into a periodic plateau (so \( T' \) is no longer in \( \mathcal{S}_d^4 \)). There is a periodic interval \( \hat{K} \) (here of period 1) and \( T' \) maps the convex hull \( \mathcal{J} = [Z_2', Z_4'] \) into \( \partial \hat{K} \). (Note that \( \hat{\delta}_t(T) \) first decreases \( \zeta_3 \) and then increases it again, after the plateaus \( Z_2 \) and \( Z_4 \) touch \( Z_3 \). Even though \( \gamma_t \) initially is ‘the inverse’ of the deformation \( \hat{\delta}_t \), the map \( T' = \hat{\delta}_t(T) \) will have some touching plateaus.) Because the entropy within the renormalization interval \( \hat{K} \) is \( \leq h_0 \), the movement of plateaus \( Z_2, Z_3, Z_4 \) under \( \hat{\delta}_t \) will have no effect on the global entropy. Therefore \( T' \) has the same entropy as \( T \), whereas \( h_{\text{top}}(\gamma_s(T')) > h_{\text{top}}(T) \) for any \( s > 0 \). The effect is that in the deformation \( \hat{\gamma}_s \), the deformation \( \gamma_s \) will not be applied at all, and hence it will not be able to resolve the wandering pair created by \( \hat{\delta}_t \).
Figure 11: The maps $T$ and $T' := \hat{\delta}_t(T)$ (in dotted lines). For $t > 0$ small, $t \mapsto \hat{\delta}_t(T)$ decreases the height of the plateau $Z_2$ and increases those of $Z_1, Z_3$. Once they merge, this deformation lowers all of them together. In this example, the plateaus $Z_1, Z_2$ and $Z_3$ are mapped into $\partial \hat{K}$, i.e., $T'(Z_1) = T'(Z_2) = T'(Z_3) \in \partial \hat{K}$. The map is constructed so that $T'|\hat{K}$ is unimodal with entropy $h_{\text{top}}(T'|\hat{K}) < h_{\text{top}}(T)$. In this case, $h_{\text{top}}(\gamma_s(T')) > h_{\text{top}}(T)$ for any $s > 0$ (because points near $Z_3$ will then map outside $\hat{K}$). So $\hat{\delta}_1(T) \in L_3(1) \cap L_3(h_0^+)$ where $h = h_{\text{top}}(T)$ and $\not\in S'_{\epsilon}^d$ unless the left boundary point of $\hat{K}$ is in the boundary of a component of the basin of a periodic attractor.

To deal with this issue we will first apply the deformation $\delta_t$ from Section 5.4 and subsequently a modification of the deformation $\hat{\delta}_t$, which allows some of the plateaus (namely those within renormalization intervals of ‘low entropy’) to move before others, as to render these renormalization intervals incapable of sustaining non-degenerate wandering pairs.

Fix $h_0 > 0$. As before, we say that $K$ is a renormalization interval for $T \in S$ if $T^n(K) \subset K$ and $f^n(\partial K) \subset \partial K$. Take $i \in \{1, \ldots, d\}$, and define $K_i(T)$ to be the smallest renormalization interval for $T$ which contains $Z_{i,T}$. If there exists no such smallest renormalization interval (because there exists an infinite nested sequence of renormalization intervals) then define $K_i(T) = \emptyset$.

Let $\omega(Z_{i,T})$ be the $\omega$-limit set of $Z_{i,T}$ (under iterations of $T$). Let $y_{i,T} := \max(\omega(Z_{i,T}) \cap K_i(T))$ and $\hat{K}_i(T)$ be the maximal renormalization interval with $y_{i,T} \in \hat{K}_i(T) \subset K_i(T)$ and $\hat{K}_i(T) \neq K_i(T)$; therefore $\omega(Z_{i,T}) \subset \text{orb}(\hat{K}_i(T))$. If $\hat{K}_i(T)$ does not exist, then set $\hat{K}_i(T) = \emptyset$.

Let us first show that $\beta_1 \circ \delta_t$ does not take us out of the space $S'_{\epsilon}^d$.

**Lemma 17.** Assume $T \in S'_{\epsilon}^d$ and let $T' = \beta_1 \circ \delta_t(T)$. If $T' \in L_j(1) \cap L_j(h_0^+)$ then $Z_j$ is contained in the closure of a component of the basin of a periodic attractor of $T'$. 

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Proof. We can assume $t > 0$. For $0 \leq s \leq t$, write $T_s = \beta_1 \circ \delta_s(T)$. First we remark that the parameter $\zeta_j$ of $T_s$ strictly decreases with $s$, except possibly when $Z_j$ is eventually mapped into another plateau (due to the effect of $\beta_1$).

Let $M = \text{orb}(K_j(T_t))$. Let us first assume that for $s < t$ close to $t$, the map $T_s$ also preserves $M$. If $T_t \in L_j(h_0)$ then Proposition 1 implies that for each $s \leq t$, some iterate of $T_s$ maps $Z_j$ into a renormalization interval $\hat{K}_j(T_t)$ of $T_t$ contained in $K_j(T_t)$. Moreover, if $T_t \in L_j(h_0^+)$ then $Z_j$ is eventually mapped by $T_t$ into the boundary of this renormalization interval. By the remark above, $Z_j$ is mapped into another plateau $Z_k$ for some $s < t$, and for $s = t$ this plateau is equal to $\hat{K}_j(T_t)$. Since $\hat{K}_j(T_t)$ is not mapped into $Z_k$ for $s < t$ close to $t$, the lemma follows in this case.

The other possibility is that for $s < t$ close to $t$, the map $T_s$ does not preserve $M$. Note that the set of parameters for which $T_s$ has some fixed interval as a renormalization interval is closed. So either there exists some interval $[t', t]$ such that $T_s$ has a renormalization interval of period $\geq 2$ containing $Z_j$ for each $s \in [t', t]$ or there exist parameters for which the only renormalization interval containing $Z_j$ is equal to $[-e, e]$. In the latter case, Proposition 1 implies that $h_{\text{top}}(T_t) < h_0$ and in the former case we argue as before.

Let us now define the modification $\Delta_t$ of $\hat{\delta}_t$. Given a renormalization interval $\hat{K}$, say of period $m$, let $\Xi_{\text{triv}}(\hat{K})$ be the interior of the space of maps $T \in S^d$ such that $T^m(\hat{K}) \subset \hat{K}$ and $\hat{K}$ is the closure of finitely many components of basins of periodic attractors of $T$ of period $\leq 2m$. If $\hat{K}$ is not a renormalization interval, then we set $\Xi_{\text{triv}}(\hat{K}) = \emptyset$.

Next we define $\Xi_i(K, \hat{K})$ as the set of $T \in S^d$ satisfying

- $T \in \delta_1(S^d \cap L_i(h_0))$ (the subscript 1 in $\delta_1$ is taken so large that the action of $\delta_s$ has ended);
- $K = K_i(T)$ and $\hat{K} = \hat{K}_i(T)$;
- $h_{\text{top}}(T|_{\text{orb}(K)}) = h_0$;
- $T \notin \Xi_{\text{triv}}(\hat{K}_i(T))$.

**Lemma 18.** For each pair of intervals $K, \hat{K}$, the set $\Xi_i(K, \hat{K})$ is a closed subset of $S^d$ (possibly empty) and

1. If $(K, \hat{K}) \neq (K', \hat{K}')$ then $\Xi_i(K, \hat{K}) \cap \Xi_i(K', \hat{K}') = \emptyset$.
2. There exists at most finitely many $K$’s for which $\Xi_i(K, \hat{K}) \neq \emptyset$ for some $\hat{K}$.
Proof. The first statement is obvious from the definition, so let us prove the second statement. Assume that the period of $K$ is $m$. Since the intervals $K, \ldots, T^{m-1}(K)$ are disjoint, the first return map of $T$ to $\text{orb}(K)$ has at most $2^d$ branches. It follows that $0 < h_0 = h_{\text{top}}(T|\text{orb}(K)) \leq (d/m) \log 2$. This gives an upper bound on $m$, and since for each fixed $m$, there are only finitely many configurations of period $m$ renormalization cycles, the second statement follows.

Lemma 19. For all pairs $(K, \hat{K})$ there exists open sets $U_i(K, \hat{K}) \supset \Xi_i(K, \hat{K})$ such that for each map $T$ there are at most a finite number of pairs $(K', \hat{K}')$ so that $T \in U_i(K', \hat{K'})$.

Proof. Take $T \in \Xi_i(K, \hat{K})$. We claim there exists $\varepsilon = \varepsilon(T) > 0$ such that the $\varepsilon$-neighborhood $U_\varepsilon(T)$ of $T$ does not intersect $\Xi_i(K', \hat{K}')$ when $(K', \hat{K}') \neq (K, \hat{K})$.

By Lemma 18, there are at most finitely many intervals $K'$ such that $\Xi_i(K', \hat{K}') \neq \emptyset$. So it remains to show the “local finiteness” of the intervals $\hat{K}'$. Take $(K', \hat{K}') \neq (K, \hat{K})$ and assume $T' \in \Xi(K', \hat{K}')$ is close to $T$. We will denote the objects of $T'$ with a '$$-$-accent, but by the finiteness statements on $K$ above, we can assume that $K = K'$. Since $T \notin \Xi_{\text{triv.}}$, the periodic boundary points of $\hat{K}$ cannot disappear by a small perturbation. For simplicity let $m$ be the period of $\hat{K}$ under $T$. By the maximality assumption on $\hat{K}$, there are only two reasons why $\hat{K} \neq \hat{K}'$ is possible, marked as cases (i) and (ii) below. We show that case (i) is impossible (when $T'$ is sufficiently close to $T$), and that case (ii) gives at most a finite overlap of sets $U_i(K, \hat{K})$

Case (i) $(T')^m(\hat{K}) \notin \hat{K}$: Since $T'$ is assumed to be near $T$ one of the plateaus $Z_j$ of $T$ is eventually mapped to the boundary of $\hat{K}$. Take $S \in \mathcal{S}_d^*$ so that $T = \delta_1(S)$ and $h_{\text{top}}(S|\text{orb}(K')) = h_0$. Since $S \in \mathcal{S}_d^*$ and $T = \delta_1(S)$, the plateau $Z_j$ is part of a block $\hat{Z}_j$ of at least three plateaus for $T$. Hence, since $T = \delta_1(S)$, we have $S^m(\hat{K}) \notin \hat{K}$.
and so at least two of the plateaus $Z_{j'}$ corresponding to $\hat{Z}_j$ are mapped outside $\hat{K}$ by $S^m$ and the plateau $Z_j$ in between is mapped inside, see the left panel of Figure 12. But then it follows from Proposition 1 and from the fact that $\hat{K}$ is a maximal renormalization interval inside $K$ that $h_{\text{top}}(T|\text{orb}(K)) < h_0$. Indeed, let $F$ be the piecewise affine map with constant slope which is semi-conjugate to the first return map $R$ of $S$ to $K$. Since $\hat{K}$ is a maximal interval of renormalization of $T$ but not for $S$, the semi-conjugacy between $R$ and $F$ does not collapse all of $\hat{K}$ to a point. In fact, it cannot collapse the convex hull of the three plateaus $[Z_{j'-1}, Z_{j'+1}]$ since the $S^m$-image of this interval contains the boundary point of $K'$. So Proposition 1 implies that $h_{\text{top}}(T|O(K)) < h_{\text{top}}(S|O(K)) \leq h_0$. It follows that in case (i), no $T'$ near $T$ can be in a set of the form $\Xi_i(K, \hat{K}')$.

**Case (ii): $T^m(\hat{K}') \subset \hat{K}$, but $\hat{K}' \supset \hat{K}$:** How this can happen is shown in the example in the right panel of Figure 12. For both $T$ and $T'$, the smaller interval $\hat{K}$ is invariant, but whereas $\hat{K}$ is indeed the largest for $T$, $T'$ has a larger invariant periodic interval $\hat{K}'$. However, for a given pair $(K, \hat{K})$, there can only be a finite number of choices of $\hat{K}' \supset \hat{K}$ corresponds to a non-empty set $\Xi_i(K', \hat{K}')$ (with $K' = K$).

The lemma follows by taking $U_i(K, \hat{K}) = \bigcup_{T \in \Xi_i(K, \hat{K})} U_{\varepsilon(T)}(T)$.

Since $U_i(K, \hat{K})$ is a neighborhood of $\Xi_i(K, \hat{K})$ there exists a continuous function $\rho_i(K, \hat{K}): S^d \to [0, 1]$ so that

$$\rho_i(K, \hat{K})(T) = \begin{cases} 1 & \text{outside } U_i(K, \hat{K}), \\ 0 & \text{on } \Xi_i(K, \hat{K}). \end{cases}$$

Now define the modification $\Delta_t$ of $\hat{\delta}_i$ which moves a maximal block of touching plateaus according to the flow (with $\hat{\text{sgn}}$ defined as in (14))

$$\frac{d\zeta_{i,t}}{dt}(T) = \begin{cases} -\left(\prod_{K,\hat{K}} \rho_i(K, \hat{K})(T)\right) \cdot \hat{\text{sgn}}(Z_i) \cdot d \cdot e & \text{when } \zeta_{i,t}(T) \in (-e, e) \\ 0 & \text{otherwise} \end{cases}$$

$$\zeta_{i,t}(T)|_{t=0} = \zeta_i(T).$$

Since $\delta_i(S^d)$ consists of maps where all plateaus appear in blocks (i.e., touch other plateaus), this flow preserves $\delta_i(S^d)$. Because of the previous lemma, the product above only involves a finite number of terms which are not equal to one, and so the differential equation makes sense. Here $\zeta_i(T)$, $\hat{Z}_i$ and $\hat{\text{sgn}}(Z_i)$ are defined as in (14)

**Lemma 20.** The map

$$\mathbb{R} \times \delta_i(S^d) \ni (t, T) \mapsto \Delta_t(T) \in S^d$$

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associated to the above flow is continuous in \( t \) and \( T \in \delta_1(S^d_*) \) and preserves the set \( \delta_1(S^d_*) \). Moreover, let \( T \in \delta_1(S^d_*) \), \( t > 0 \) and \( T' = \Delta_t(T) \) or \( T' = \beta_1 \circ \Delta_t(T) \), then

\[
T' \in L_j(h_0) \cap L_j(h_0^+) \quad \text{implies that} \quad Z_j \quad \text{is contained in the closure of a component of the basin of a periodic attractor of} \ T'.
\] (17)

Finally, \( \Delta_1(T) \in \Sigma^d \) where \( \Sigma^d \) is the space of monotone maps as in Subsection 5.4.

Proof. The continuity of \( (t, T) \mapsto \Delta_t(T) \in S^d \) and the fact that \( \Delta_t(T) \in \delta_1(S^d_*) \) follows immediately from the definitions. Assume that \( T \in \Xi_i(K, \hat{K}) \). Let us first assume that \( T \notin \Xi_j(K_1, \hat{K}_1) \) for all plateaus \( Z_j \) in \( \hat{K} \) and all \( \hat{K}_1 \subset K_1 \subset K \). Then the plateaus in \( \hat{K} \) are all moving under the flow, and as soon as the return map of \( \Delta_t(T) \) to \( \hat{K} \) becomes ‘trivial’ (i.e., \( \Delta_t(T) \in \Xi_{\text{triv}}(\hat{K}) \)), the plateau \( Z_i \) also starts moving. At some moment \( t_0 \), \( \Delta_{t_0}(T) \) reaches the situation that it either maps \( \hat{K} \) into its boundary (so \( \text{int}(\hat{K}) \) is in the basin of a periodic attractor and for \( t > t_0 \), \( \hat{K} \) is no longer a renormalization interval), or \( Z_i \) is no longer mapped into \( \hat{K} \) by the \( n \)-th iterate of \( \Delta_t(T) \), \( t > t_0 \). The semi-conjugacy between the first return map of \( T \) to \( K \) and the piecewise linear map with constant slope can only collapse preimages of renormalization intervals. So in either of the above cases, the semi-conjugacy does not collapse a neighborhood of \( \hat{K} \). Therefore Proposition 1 implies that either \( Z_i \) is contained in the closure of a component of the basin of a periodic attractor of \( \Delta_t(T) \) or \( h_{\text{top}}(\Delta_t(T)|\text{orb}(K)) < h_0 \). It follows that (17) holds for \( T' = \Delta_t(T) \). The proof for \( T' = \beta_1 \circ \Delta_t(T) \) goes as in Lemma 17.

Of course it is possible that \( T \in \Xi_j(K_1, \hat{K}_1) \) for some plateaus \( Z_j \) in \( \hat{K} \) (for some \( \hat{K}_1 \subset K_1 \subset \hat{K} \)). But then we can repeat the argument and one find at least some \( j' \) so that \( Z_{j'} \subset \hat{K} \) and so that \( T \notin \bigcup_{K', \hat{K}} \Xi_j(K', \hat{K}') \). Therefore the \( j' \)-th plateau is moving under the flow \( \Delta_t \). Repeating this argument brings us to the previous situation.

Since always at least one of the plateaus is moving with speed \( d \), for each \( T \in \delta_1(S^d_*) \) there exists \( t \leq d \) so that \( \Delta_t(T) \in \Sigma^d \). \( \square \)
5.10 The proof of Theorem 7.

Now that we have developed the ingredients of the proof, we can define the retract for a fixed \( h_0 \in (0, \log(d + 1)) \). (The case \( h_0 = 0 \) was dealt with in Section 5.5.)

\[
R_t = \begin{cases} 
\beta_{6t} & \text{for } t \in [0, \frac{1}{6}], \\
\Gamma_{6t-1} \circ \beta_1 & \text{for } t \in [\frac{1}{6}, \frac{2}{6}], \\
\Gamma_1 \circ \delta_{6t-2} \circ \beta_1 & \text{for } t \in [\frac{2}{6}, \frac{3}{6}], \\
\Gamma_1 \circ \hat{\gamma}_1 \circ \beta_1 \circ \delta_{6t-3} & \text{for } t \in [\frac{3}{6}, \frac{4}{6}], \\
\Gamma_1 \circ \hat{\gamma}_1 \circ \beta_1 \circ \delta_{6t-4} \circ \delta_1 & \text{for } t \in [\frac{4}{6}, \frac{5}{6}], \\
\Gamma_1 \circ \hat{\gamma}_1 \circ \beta_1 \circ r_{6t-5} \circ \Delta_1 \circ \delta_1 & \text{for } t \in [\frac{5}{6}, 1].
\end{cases}
\]

Obviously, \( R_0(T) = T \), and since for \( t = 1 \), the retract \( r_{6t-5} \) has been carried out completely, \( R_1(T) \) is the same map for each \( T \in L_*(h_0) \) of the same shape \( \epsilon \). All components of \( R_t \) are continuous in \( t \) and \( T \), so the same holds for \( R_t \).

Let us show that \( R_t \) keeps maps within the space \( S^d_\ast \). First note that the only deformations which take maps outside the space \( S^d_\ast \) are \( \delta_t \) and \( \Delta_t \). Take \( T' \) of the form \( T' = \delta_t(T), T' = \Delta_t \circ \delta_1(T) \) or \( T' = r_t \circ \Delta_1 \circ \delta_1(T) \). The deformation \( \beta_t(T') \) moves plateaus \( Z_i, Z_{i+1} \) whose convex hull is mapped into other other plateaus. It does so in such a way that \( \beta_1(T') \) never eventually maps \( [Z_i, Z_{i+1}] \) into the interior of another plateau and so \( \beta_1(T') \in \overline{S^d_\ast} \).

If \( h_{top}(T') < h_0 \) then \( h_{top}(\beta_1(T')) < h_0 \) and \( \beta_1(T) \in \overline{S^d_\ast} \). Therefore (12) ensures that \( \Gamma_1 \circ \beta_1(T') \in S^d_\ast \). So in particular we are done if \( T' = r_t \circ \Delta_1 \circ \delta_1 \).

The last case is that \( h_{top}(T') = h_0 \). By the third part of Lemma 16, \( \Gamma_1 \circ \hat{\gamma}_1 \circ \beta_1(T') \in S^d_\ast \) provided \( \beta_1(T') \in L_j(h_0) \cap L_j(h_0^+) \) implies that \( Z_j \) is contained in the closure of a component of the basin of a periodic attractor of \( T' \). But in Lemma 17 and Lemma 20 it is shown that any map \( T' \) of the form \( T' = \beta_1 \circ \delta_t(T) \) or of the form \( T' = \beta_1 \circ \Delta_t \circ \delta_1(T) \) with \( t > 0 \) and \( T \in S^d_\ast \) has this property, and so again the resulting map is in \( S^d_\ast \). This concludes the proof of Theorem 7.

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