Discussion Papers in Economics

# Robust Monetary Rules under Unstructured and Structured Model Uncertainty

By

Paul Levine
(University of Surrey)
&
Joseph Pearlman
(London Metropolitan University)

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Department of Economics
University of Surrey
Guildford
Surrey GU2 7XH, UK
Telephone +44 (0)1483 689380
Facsimile +44 (0)1483 689548
Web www.econ.surrey.ac.uk
ISSN: 1749-5075

# Robust Monetary Rules under Unstructured and Structured Model Uncertainty*

Paul Levine
University of Surrey

Joseph Pearlman
London Metropolitan University

October 13, 2007

**Abstract**

This paper compares two contrasting approaches to robust monetary policy design. The first developed by Hansen and Sargent (2003, 2007) assumes unstructured model uncertainty and uses a minimax robustness criterion to design monetary rules. This contrasts with an older literature that structures uncertainty by seeking rules that are robust across competing views of the economy. This paper carries out and compares robust design exercises using both approaches using a standard 'canonical New Keynesian model'. We pay particular attention to a number of issues: First, we distinguish three possible forms of the implied game between malign nature and the policymaker in the Hansen-Sargent procedure. Second, in both approaches, we examine the consequences for robust rules of the zero lower bound (ZLB) constraint on the nominal interest rate, the monetary instrument. Finally, again for both types of robustness exercise we explore the implications of policy design when the policymaker is obliged to use simple Taylor-type interest rate rules.

**JEL Classification**: E52, E37, E58

**Keywords**: robustness, structured and unstructured uncertainty, zero lower bound interest rate constraint

*Presented at the Conference on "Robust Monetary Rules for the Open Economy" at the University of Surrey, September 20-21, 2007. Feedback from the participants at the Conference are gratefully acknowledged, particularly those of the discussant Andy Blake. We also acknowledge financial support for this research from the ESRC, project no. RES-000-23-1126 and from the European Central Bank's Research Visitors Programme for Paul Levine.

# Contents

# 1 Introduction

This paper compares two contrasting approaches to robust monetary policy design in the face of model uncertainty. The first developed by Hansen and Sargent (2003), Hansen and Sargent (2007) assumes unstructured uncertainty and uses a minimax robustness criterion to design monetary rules. It has three key ingredients that distinguishes it from alternatives. First, it conducts 'local analysis' in the sense that it assumes that the true model is known only up to some local neighbourhood of models that surround the 'approximating' or 'core' model. Second, it uses a minimax criterion without priors in model space. Third, the type of uncertainty is both unstructured and additive being reflected in additive shock processes that are 'chosen' by malevolent nature to feed back on state variables so has to maximize the loss function the policy-maker is trying to minimize.

The Hansen-Sargent minimax criterion for robust design contrasts with an older literature that structures uncertainty in a number of ways. The general feature of this approach is to seek rules that are robust across competing views of the economy. These could be competing and possibly quite different structural models (see, for example, Levin *et al.* (2003), Coenen (2007)) or the same structural model, but with different parameter values. The latter could be draws from an estimated joint distribution of parameters, (see Batini *et al.* (2006)). The latter combines these forms of competing models using Bayesian methods to estimate both the distribution and the model probabilities.

This paper carries out and compares robust design exercises using both approaches using a standard 'canonical New Keynesian model'. We pay particular attention to a number of issues: First, we distinguish three possible forms of the implied game between malign nature and the policymaker in the Hansen-Sargent procedure. Second, in both approaches, we examine the consequences for robust rules of the zero lower bound (ZLB) constraint on the nominal interest rate, the monetary instrument. Finally, again for both types of robustness exercises we explore the implications of policy design when the policymaker is obliged to use simple Taylor-type interest rate rules.

The rest of the paper is organized as follows. Section 2 describes the model. Section 3 sets out the general procedure for Hansen-Sargent robust control and then applies the method to our chosen model. Section 4 address concerns for the interest rate ZLB constraint. Section 5 conducts a parallel rival models exercise and Section 6 concludes.

# 2 Optimal Policy without Model Uncertainty

## 2.1 The Model

The New Keynesian model we employ is now standard in the monetary policy literature. In a linearized form in the vicinity of a no-growth zero-inflation steady state it consists of a Keynes-Ramsey equation for consumption behaviour, (4) below with output equal to consumption the the absence of capital, investment and government spending, and a Phillips curve based on Calvo-type price setting for firms, (5). There are three exogenous shocks: technology, mark-up and preference shocks, (1)–(3). Table 1 provides details of notation.

| | |
|---|---|
| $\pi_t$ | producer price inflation over interval $[t-1,t]$ |
| $i_t$ | nominal interest rate over interval $[t,t+1]$ |
| $mc_t$ | marginal cost |
| $y_t$, $\hat{y}_t$ | output with sticky prices and flexi-prices |
| $l_t$ | employment |
| $r_t$ | expected real interest rate |
| $o_t = \hat{y}_t - y_t$ | output gap |
| $a_{t+1} = \rho_a a_t + \epsilon_{a,t+1}$ | AR(1) process for factor productivity shock, $a_t$ |
| $e_{t+1} = \rho_e e_t + \epsilon_{e,t+1}$ | AR(1) process for mark-up shock, $e_t$ |
| $u_{C,t+1} = \rho_C g_t + \epsilon_{C,t+1}$ | AR(1) process for preference, $u_{C,t}$ |
| $\beta$ | discount parameter |
| $1-\xi$ | probability of a price re-optimization |
| $\sigma$ | risk-aversion parameter |
| $\phi$ | disutility of labour supply parameter |
| $\zeta$ | elasticity of substitution between differentiated goods |

**Summary of Notation (Variables in Deviation Form)**

$$
\begin{aligned}
a_{t+1} &= \rho_a a_t + \epsilon_{a,t+1} &(1)\\
e_{t+1} &= \rho_e e_t + \epsilon_{e,t+1} &(2)\\
u_{C,t+1} &= \rho_C u_{C,t} + \epsilon_{C,t+1} &(3)\\
E_t y_{t+1} &= y_t + \frac{1}{\sigma}(i_t - E_t\pi_{t+1} + E_t u_{C,t+1} - u_{C,t}) &(4)\\
\beta E_t \pi_{t+1} &= \pi_t - \lambda mc_t - e_t &(5)\\
mc_t &= (\sigma+\phi)y_t - (1+\phi)a_t &(6)\\
l_t &= y_t - a_t &(7)\\
r_t &= i_t - E_t\pi_{t+1} &(8)\\
\hat{y}_t &= \left(\frac{1+\phi}{\sigma+\phi}\right) a_t &(9)\\
o_t &= \hat{y}_t - y_t &(10)
\end{aligned}
$$

where

$$\lambda = \frac{(1-\beta\xi)(1-\xi)}{\xi} \tag{11}$$

We choose a loss function that corresponds to the welfare-based quadratic form in Woodford (2003).

$$W_t = \frac{1}{2}\left[(\sigma+\phi)o_t^2 + w_\pi \pi_t^2 + w_i i_t^2\right] + \text{t.i.p} \tag{12}$$

where $w_\pi = \frac{\zeta}{\lambda}$. For a quarterly model, parameter values chosen are $\sigma = 2$, $\beta = 0.99$, $\xi = \frac{2}{3}$ (corresponding to an average price contract of 3 quarters), $\phi = 1.7$, $\zeta = 7.67$ (corresponding to a 15% price mark-up), $\rho_a = \rho_C = 0.7$, $\rho_e = 0.35$, and $sd(\epsilon_i) = 1.0$, $i = a, C, e$. In the absence of any constraint on the nominal interest rate we put $w_i = 0$.

We can write this system in state space form as

$$\begin{bmatrix} \mathsf{z}_{t+1} \\ E_t \mathsf{x}_{t+1} \end{bmatrix} = A \begin{bmatrix} \mathsf{z}_t \\ \mathsf{x}_t \end{bmatrix} + Bi_t + C\epsilon_t \tag{13}$$

$$\mathsf{s}_t = \begin{bmatrix} mc_t \\ y_t \\ l_t \\ r_t \\ o_t \end{bmatrix} = E \begin{bmatrix} \mathsf{z}_t \\ \mathsf{x}_t \end{bmatrix} \equiv E\mathsf{y}_t \tag{14}$$

$$W_t = \frac{1}{2}[\mathsf{y}_t' Q \mathsf{y}_t + 2\mathsf{y}_t' U i_t + R i_t^2] \tag{15}$$

where $\mathsf{z}_t = [a_t,\, e_t,\, u_{C,t}]'$ is a vector of predetermined variables, $\mathsf{x}_t = [y_t,\, \pi_t]'$ is a vector of non-predetermined or 'jump' variables, and $\mathsf{s}_t$ is vector of outputs of interest.

## 2.2 Optimal Policy with and without Commitment

We now examine three monetary policy regimes. The first is the ex ante optimal policy (OP) which is time inconsistent and can only be reached if the policymaker can commit. For the most general linear-quadratic problem, this is found at time $t = 0$ by minimizing with respect to the interest rate path $\{i_t\}$ the inter-temporal conditional welfare loss

$$\Omega_0 = (1-\beta)E_t \left[\sum_{t=0}^{\infty} W_t\right] \tag{16}$$

subject to the model (13) and (14), initial conditions $\mathsf{z}(0)$, terminal conditions for $\mathsf{x}$ and the variance-covariance matrix $\mathrm{cov}(\epsilon_t)$.

To evaluate the discretionary, time-consistent policy (D) we write the expected loss $\Omega_t$ at time $t$ as

$$\Omega_t = E_t \left[(1-\beta)\sum_{\tau=t}^{\infty} \beta^{\tau-t} W_\tau\right] = (1-\beta)W_t + \beta\Omega_{t+1} \tag{17}$$

The dynamic programming solution then seeks a stationary solution of the form $i_t = -Fz_t$, $\Omega_t = \mathsf{z}'S\mathsf{z}$ and $\mathsf{x} = -N\mathsf{z}$ where matrices $S$ and $N$ are now of dimensions $(n-m)\times(n-m)$ and $m \times (n-m)$ respectively, in which $\Omega_t$ is minimized at time $t$ subject to (1) in the knowledge that a similar procedure will be used to minimize $\Omega_{t+1}$ at time $t+1$.[1] Both the instrument $i_t$ and the forward-looking variables $\mathsf{x}_t$ are now proportional to the

[1] See Currie and Levine (1993) and Söderlind (1999).

predetermined component of the state-vector $\mathsf{z}_t$ and the equilibrium we seek is therefore *Markov Perfect*. In Appendix A we set out an iterative process for $F_t$, $N_t$, and $S_t$ starting with some initial values. If the process converges to stationary values independent of these initial values,[2] $F$, $N$ and $S$ say, then the time-consistent feedback rule is $i_t = -Fz_t$.

Our third rule is a Taylor-type simple rule (TR) constrained to be of the form

$$i_t = \rho i_{t-1} + \theta_\pi \pi_t + \theta_y (y_t - \hat{y}_t) \tag{18}$$

The policymaker then maximizes the expected conditional welfare loss $\Omega_0$ with respect to feedback parameters $\rho, \theta_\pi, \theta_y$ given the model, initial conditions $\mathsf{z}(0)$, terminal conditions for $\mathsf{x}$ and the variance-covariance matrix $\text{cov}(\epsilon_t)$. Unlike policy rules OP and D, the optimal form of TR is not certainty equivalence and depends on both $\mathsf{z}(0)$ and $\text{cov}(\epsilon_t)$.

General procedures for calculating these three policy rules are set out in Appendix A. Analytical results for our NK model for the more general robust policy rules are provided in section 3.3. Numerical results, given our calibration, are provided in table 1. This table provides conditional (asymptotic) variances and the expected conditional welfare loss in the vicinity of the steady state; i.e., we put $\mathsf{z}(0) = 0$ and, in effect, only study the stochastic optimization problem, a feature of all our results. The table also gives two further useful properties of these rules: first, the probability of the interest rate hitting the zero lower bound in the vicinity of the steady state, equal to the probability of the standard normal variable $z > \frac{I}{sd(i_t)}$, where $I = \frac{1}{\beta} - 1$ is the steady state interest rate;[3] second, the welfare loss associated with the two sub-optimal policies D and TR measured in terms of a permanent percentage fall in consumption at the steady state given by[4]

$$(\Omega^{OP} - \Omega^i) \times 10^{-2}\,;\; i = D, TR \tag{19}$$

A number of features are worth noting at this stage. First, the optimized Taylor rule comes very close to mimicking the fully optimal policy with only a consumption equivalent cost of $c_e = 0.001\%$. The optimized parameters imply an integral rule with a strong feedback from inflation, but a modest one from the output gap. Second, the gains from commitment measured as the difference between OP and D are small at $c_e = 0.007\%$. The

[2]Indeed we find this is the case in the results reported in the paper.

[3]With $\beta = 0.99$, $I = 1.01\%$ or about 4% per year.

[4]See Levine *et al.* (2007a).

reported variances indicate that these costs take the form of higher output gap and inflation volatilities. Third, the asymptotic variance of interest rate is very high in all cases implying a probability hitting the ZLB of just over a quarter for the two commitment rules, rising to almost a third for discretion. ZLB concerns are therefore serious, but we defer a discussion of their implications to a later section.

| | Optimal (OP) | Discretion (D) | Taylor Rule (TR) |
|---|---|---|---|
| var($a_t$) | 1.96 | 1.96 | 1.96 |
| var($u_{C,t}$) | 1.96 | 1.96 | 1.96 |
| var($e_t$) | 1.96 | 1.96 | 1.96 |
| var($o_t$) | 1.91 | 2.23 | 198 |
| var($\pi_t$) | 0.03 | 0.04 | 0.03 |
| var($y_t$) | 2.95 | 3.28 | 2.93 |
| var($l_t$) | 2.05 | 2.38 | 2.16 |
| var($r_t$) | 2.15 | 4.88 | 2.56 |
| var($i_t$) | 2.40 | 4.60 | 2.53 |
| Prob. ZLB | 0.26 | 0.32 | 0.27 |
| $\Omega_0$ | 4.286 | 4.986 | 4.356 |
| $c_e(\%)$ | 0 | 0.007 | 0.001 |

**Table 1. Volatility and Welfare Outcomes with No Model Uncertainty**

**Note**: For the Taylor rule optimal parameter values are $\rho = 1$, $\theta_\pi = 8.96$, $\theta_y = 0.06$.

Figure 1 shows the impulse responses to a mark-up shock. In the absence of a ZLB constraint, the technology and preference shocks are uninteresting because the output gap and inflations are met perfectly and the only variables to respond are the nominal and real interest rates necessary to meet these targets. For the mark-up shock and for all policy rules labour supply and therefore output ($y_t$) falls leading to an increase in the output gap, $o_t = \hat{y}_t - y_t$. The mark-up shock directly increases inflation and the policy response is to raise the nominal interest by considerably more, so that both the ex ante and expected real interest rate rises. Output then is depressed reducing marginal costs and offsetting the effect of the mark-up shock. The figures indicates that the optimized Taylor rule almost exactly mimics the fully optimal rule, but under discretion the responses of the labour supply, the output gap, inflation and the nominal interest rate are all more exaggerated. The reason for this is that in the absence of a commitment mechanism, the promise under commitment to first raise inflation then lower it below the steady state lacks credibility. With discretion, the interest rate then lacks the same ability to influence demand in any one period with the result that the initial hike is much greater.

# 3 Robust Rules with Unstructured Model Uncertainty

## 3.1 The Approximating and Disturbed Models

Our approximating model is the model of section 2. The distorted model adds miss-specification errors $u_{t+1}$ and $v_{t+1}$ to the Phillips and Euler equations respectively and is given by

$$a_{t+1} = \rho_a a_t + \epsilon_{a,t+1} \tag{20}$$

$$e_{t+1} = \rho_e e_t + \epsilon_{e,t+1} \tag{21}$$

$$u_{C,t+1} = \rho_C u_{C,t} + \epsilon_{C,t+1} \tag{22}$$

$$\beta E_t \pi_{t+1} = \pi_t - \lambda mc_t - e_t - u_{t+1} \tag{23}$$

$$E_t y_{t+1} = y_t + \frac{1}{\sigma}(i_t - E_t \pi_{t+1} + E_t u_{C,t+1} - u_{C,t}) - v_{t+1} \tag{24}$$

$$mc_t = (\sigma + \phi) y_t - (1 + \phi) a_t \tag{25}$$

$$l_t = y_t - a_t \tag{26}$$

$$r_t = i_t - E_t \pi_{t+1} \tag{27}$$

$$\hat{y}_t = \left(\frac{1 + \phi}{\sigma + \phi}\right) a_t \tag{28}$$

$$o_t = \hat{y}_t - y_t \tag{29}$$

We can again write this system in state space form as

$$\begin{bmatrix} \mathsf{z}_{t+1} \\ E_t \mathsf{x}_{t+1} \end{bmatrix} = A \begin{bmatrix} \mathsf{z}_t \\ \mathsf{x}_t \end{bmatrix} + Bi_t + C\epsilon_t + D\hat{\mathsf{w}}_t \tag{30}$$

$$\mathsf{s}_t = \begin{bmatrix} mc_t \\ y_t \\ l_t \\ r_t \\ o_t \end{bmatrix} = F \begin{bmatrix} \mathsf{z}_t \\ \mathsf{x}_t \end{bmatrix} \equiv F\mathsf{y}_t \tag{31}$$

$$\Omega_0 = E_t \left[ \frac{1}{2} \sum_{t=0}^{\infty} \beta^t [\mathsf{y}_t' Q \mathsf{y}_t + 2\mathsf{y}_t' U i_t + R i_t^2] \right] \tag{32}$$

$$W_t = \frac{1}{2}[\mathsf{y}_t' Q \mathsf{y}_t + 2\mathsf{y}_t' U i_t + R i_t^2] \tag{33}$$

### 3.2 The Hansen-Sargent Robust Controller

The robust policy is then found by assuming that nature chooses $\hat{w}_{t+1}' = [u_{t+1}\, v_{t+1}]'$ in a malign fashion so as to *maximize* $\Omega_0$ subject to a constraint on the misspecified dynamics given by $\sum_{t=0}^{\infty} \beta^t \hat{w}_{t+1}' \hat{w}_{t+1} \leq \eta$. Following Hansen-Sargent it is more convenient to reformulate this *constraint problem* as a certainty equivalent *multiplier problem* in which nature maximizes

$$\Lambda_0 = \Omega_0 - \frac{1}{2}\Theta \sum_{t=0}^{\infty} \beta^t \hat{\mathsf{w}}_{t+1}' \hat{\mathsf{w}}_{t+1} \tag{34}$$

where $\Theta$ is a positive Lagrange multiplier, subject to $\Theta \in [\bar{\Theta}, \infty)$.[5] The Hansen-Sargent robust controller is then found as a solution to

$$\min_{\{\mathsf{w}_t\}} \max_{\{\hat{\mathsf{w}}_{t+1}\}} \Lambda_0 \tag{35}$$

To solve the deterministic case of this problem define a Hamiltonian

$$\mathcal{H}_t = \beta^t \left[ (\mathsf{y}_t' Q \mathsf{y}_t + 2\mathsf{y}_t' U i_t + \mathsf{w}_t' R i_t) + 2\beta \mathsf{p}_{t+1}' (A\mathsf{y}_t + Bi_t + D\hat{\mathsf{w}}_{t+1} - \mathsf{y}_{t+1}) - \beta\Theta \hat{\mathsf{w}}_{t+1}' \hat{\mathsf{w}}_{t+1}) \right] \tag{36}$$

[5] A simple minimax problem of this form shows that the lower bound on $\Theta$ is necessary to ensure that the inner optimization by nature satisfies the second order condition.

Then the first-order conditions with respect to $i_t$, $\mathsf{y}_t$ and $\hat{\mathsf{w}}_{t+1}$ respectively are

$$\begin{aligned} i_t &= -R^{-1}(\beta B'\mathsf{p}_{t+1} + U'\mathsf{y}_t) && (37) \\ \beta A'\mathsf{p}_{t+1} &= \mathsf{p}_t - (Q\mathsf{y}_t + Ui_t) && (38) \\ \hat{\mathsf{w}}_{t+1} &= \frac{1}{\Theta} D'\mathsf{p}_t && (39) \end{aligned}$$

Together with the original constraint

$$y_{t+1} = Ay_t + Bi_t + D\hat{\mathsf{w}}_{t+1} \tag{40}$$

(37) to (40) describes the *worst-case equilibrium*. By contrast the *approximating equilibrium* is the approximating model (40) with $D = 0$, i.e.,

$$y_{a,t+1} = Ay_{a,t} + Bi_t \tag{41}$$

but under the robust rule given by $\mathsf{w_t}$. Appealing to certainty equivalence the same rules apply when we add white-noise shocks $\epsilon_\mathsf{t}$. Then substituting for $\mathsf{w_t}$ and $\hat{\mathsf{w}}_{t+1}$ from (37) and (39) we arrive at the following system describing both the worst-case and approximating equilibria

$$\begin{bmatrix} I & 0 & \beta BR^{-1}B' \\ 0 & I & \beta BR^{-1}B' - \frac{1}{\Theta}DD' \\ 0 & 0 & \beta(A' - UR^{-1}B') \end{bmatrix} \begin{bmatrix} \mathsf{y}_{a,t+1} \\ \mathsf{y}_{t+1} \\ \mathsf{p}_{t+1} \end{bmatrix} = \begin{bmatrix} A - BR^{-1}U' & 0 & 0 \\ 0 & A - BR^{-1}U' & 0 \\ 0 & -(Q - UR^{-1}U') & I \end{bmatrix} \begin{bmatrix} \mathsf{y}_{a,t} \\ \mathsf{y}_t \\ \mathsf{p}_t \end{bmatrix} + \begin{bmatrix} C \\ C \\ 0 \end{bmatrix} \epsilon_\mathsf{t} \tag{42}$$

To complete the solution we require $3n$ boundary conditions for (42). Specifying $\mathsf{z}_{a,0} = \mathsf{z}_0$ gives us $2(n - m)$ of these conditions. The initial condition for an optimum for both the policymaker and nature is

$$\mathsf{p}_{2,0} = 0 \tag{43}$$

where $\mathsf{p}_t' = \left[\mathsf{p}_{1,t}' \, \mathsf{p}_{2,t}'\right]$ is partitioned so that $\mathsf{p}_{1,t}$ is of dimension $(n - m) \times 1$. This gives us $m$ more initial conditions. We seek a stable rational expectations solution which imposes $n + m$ terminal conditions on the forward-looking variables $[\mathsf{p}_{1,t}' \, \mathsf{x}_t' \, \mathsf{x}_{a,t}']$, completing the $3n$ boundary conditions. As $\Theta$ becomes large, robust and non-robust rules will converge.

In what follows we treat the parameter $\Theta$ as a parameter whose inverse represents the policymaker's concern for robustness. Note, however, that robust control in the engineering literature, from which the work of Hansen and Sargent stems, adopts a slightly different approach from this section and from what follows. Part of the so-called $H^\infty$ design is that 'malign nature' chooses the value of $\Theta$ as low as possible, subject to two criteria being satisfied: (i) the overall system under control is stable (ii) the optimal feedback as designed to solve the problem yields a stable system when malign nature is switched off. This is similar to the approximating model $y_{a,t}$ being stable. However choosing $\Theta$ 'optimally' can lead to either the system in (i) or that in (ii) being only just stable. This means that in the stochastic case, variances can be very high for one of these systems, and this, as we shall see, raises the probability of the interest rate hitting zero.

### 3.3 Robust Control as a Game

It is useful to characterize the solution to the robust policy problem in terms of a *dynamic game* between the monetary authority and nature. There is a third player in this game, the private sector forming model-consistent expectations based on the model and the monetary rule. As with all multi-person games there are a number of possible equilibria in this game and each leads to a different solution to the robust policy problem. We now consider three possible games, the first corresponding to the above robust control solution.

**Game 1**. This is a two-player zero-sum game with payoffs defined by (35). Each player simultaneously commits to sequences $\{i_t\}$ and $\{\hat{w}_{t+1}\}$ at time $t = 0$ taking the other players sequence of moves as given. Although the optimal solution can be expressed as feedback rules $i_t = D\mathsf{y}_t$ and $\hat{w}_t = \hat{D}\mathsf{y}_t$, the game is actually of an open-loop character because this feed-back is not taken into account in the first-order conditions. Since the players move simultaneously there is no leadership by the monetary authority although there is commitment with respect to the private sector. We refer to the second game as the *time inconsistent open-loop Nash game.*

**Game 2.** In this paper we wish to study robust simple commitment rules that provide the best response of the policymaker to the worst possible outcome. The Hansen-Sargent robust control rule above does not achieve this. A robust rule responds to the worst possible shock from the viewpoint of the authority which requires that nature chooses a sequence for $\{\hat{w}_{t+1}\}$ that is optimal ex ante at time $t = 0$. The monetary authority anticipates the response of nature to its commitment rule and so exercises leadership. It then faces an economy under the malign influence of nature of the form

$$\begin{bmatrix} I & 0 \\ 0 & \beta A' \end{bmatrix} \begin{bmatrix} \mathsf{y}_{t+1} \\ \mathsf{p}_{t+1} \end{bmatrix} = \begin{bmatrix} A & \frac{DD'}{\Theta} \\ -Q & I \end{bmatrix} \begin{bmatrix} \mathsf{y}_t \\ \mathsf{p}_t \end{bmatrix} + \begin{bmatrix} B \\ -U \end{bmatrix} i_t + \begin{bmatrix} C \\ 0 \end{bmatrix} \epsilon_t \quad (44)$$

This is in the standard form for designing ex ante optimal policy, discretionary policy and an optimized Taylor rule as in section 2 and described in general in Appendix A. We call the first of these cases, the *time inconsistent Stackelberg game.*

**Game 3.** In this paper we again study the best response of the policymaker to the worst possible outcome and the monetary authority anticipates the response of nature to policy so exercises leadership. However the policymaker cannot commit and exercises discretion, optimizing in each period on the assumption that a re-optimizations will occur in each subsequent period. We refer to this game the *time consistent discretionary game.*

### 3.4 Application to the Canonical Keynesian Model

First consider nature's problem. Again by an appeal to certainty equivalence we can first consider the deterministic problem and assume that the same policy rule applies to the stochastic case. The former is calculated by setting up the Lagrangian

$$\begin{aligned} \mathcal{L} = & \frac{1}{2}\sum_{t=0}^{\infty}\beta^t \Bigg[ -(\sigma+\phi)o_t^2 - w_\pi \pi_t^2 - w_i i_t^2 + \Theta(u_{t+1}^2 + v_{t+1}^2) \\ & + 2\hat{\mu}_{1,t+1}(\rho_a a_t - a_{t+1}) + 2\hat{\mu}_{2,t+1}(\rho_e e_t - e_{t+1}) \\ & + 2\hat{\mu}_{3,t+1}(\rho_C u_{C,t} - u_{C,t+1}) + 2\hat{\mu}_{4,t+1}\left(\pi_t + \lambda[(1+\phi)a_t - (\sigma+\phi)y_t] - \beta\pi_{t+1} - e_t - u_{t+1}\right) \\ & + 2\hat{\mu}_{5,t+1}\left( y_t + \frac{1}{\sigma}\left(i_t - \pi_{t+1} + (\rho_C - 1)u_{C,t}\right) - y_{t+1} - v_{t+1}\right)\Bigg] \end{aligned} \quad (45)$$

Nature then chooses $\{u_{t+1}, v_{t+1}\}$ to maximize (45), given $\{i_t\}$. The first order conditions for this problem are:

$$u_{t+1} : \Theta u_{t+1} - \hat{\mu}_{4,t+1} = 0 \quad (46)$$

$$v_{t+1} : \Theta v_{t+1} - \hat{\mu}_{5,t+1} = 0 \quad (47)$$

$$\pi_t : -w_\pi \pi_t + \hat{\mu}_{4,t+1} - \hat{\mu}_{4,t} - \frac{1}{\sigma\beta}\hat{\mu}_{5,t} = 0 \quad (48)$$

$$y_t : (\sigma+\phi)o_t - \lambda(\sigma+\phi)\hat{\mu}_{4,t+1} + \hat{\mu}_{5,t+1} - \frac{1}{\beta}\hat{\mu}_{5,t} = 0 \quad (49)$$

plus foc for the shocks which we do not need, with initial conditions $\hat{\mu}_{4,0} = \hat{\mu}_{5,0} = 0$. We can now eliminate $\hat{\mu}_{4,t+1}$ and $\hat{\mu}_{5,t+1}$ from (46) and (47) to obtain the following processes for nature's worst shocks

$$\Theta\left[u_{t+1} - u_t - \frac{1}{\sigma\beta}v_t\right] = w_\pi \pi_t \tag{50}$$

$$\Theta\left[v_{t+1} - \frac{1}{\beta}v_t - \lambda\left(\sigma+\phi\right)u_{t+1}\right] = -(\sigma+\phi)o_t \tag{51}$$

We can express this system as

$$K\begin{bmatrix} u_{t+1} \\ v_{t+1} \end{bmatrix} = L\begin{bmatrix} u_t \\ v_t \end{bmatrix} + M\begin{bmatrix} o_t \\ \pi_t \end{bmatrix} \tag{52}$$

This then expresses the worst-case shocks $u_t$ and $v_t$ of nature as a reaction to past outcomes $[o_{t-1}, o_{t-2}, \cdot\ \cdot; \pi_{t-1}, \pi_{t-2}, \cdot\ \cdot]$.

### 3.4.1 Robust Control: Game 1

In this open-loop Nash game the policymaker's problem is the mirror-image of that of nature: to choose $\{i_t\}$ to minimize the welfare loss for which the Lagrangian is

$$\begin{aligned} L &= \frac{1}{2}\sum_{t=0}^{\infty}\beta^t\Bigg[(\sigma+\phi)o_t^2 + w_\pi\pi_t^2 + w_i i_t^2 + \Theta(u_{t+1}^2 + v_{t+1}^2) \\ &+ 2\mu_{1,t+1}(\rho_a a_t - a_{t+1}) + 2\mu_{2,t+1}(\rho_e e_t - e_{t+1}) \\ &+ 2\mu_{3,t+1}(\rho_C u_{C,t} - u_{C,t+1}) + 2\mu_{4,t+1}\left(\pi_t + \lambda[(1+\phi)a_t - (\sigma+\phi)y_t] - \beta\pi_{t+1} - e_t - u_{t+1}\right) \\ &+ 2\mu_{5,t+1}\left(y_t + \frac{1}{\sigma}\left(i_t - \pi_{t+1} + (\rho_C - 1)u_{C,t}\right) - y_{t+1} - v_{t+1}\right)\Bigg] \end{aligned} \tag{53}$$

given $\{u_{t+1}\}, \{v_{t+1}\}$.

The foc for this problem are

$$i_t \ :\ w_i i_t + \frac{1}{\sigma}\mu_{4,t+1} = 0 \tag{54}$$

$$\pi_t \ :\ w_\pi\pi_t + \mu_{4,t+1} - \mu_{4,t} - \frac{1}{\sigma\beta}\mu_{5,t} = 0 \tag{55}$$

$$y_t \ :\ -(\sigma+\phi)o_t - \lambda\left(\sigma+\phi\right)\mu_{4,t+1} + \mu_{5,t+1} - \frac{1}{\beta}\mu_{5,t} = 0 \tag{56}$$

Comparing these foc with those of nature, (48) and (49), we immediately see that $\mu_{4,t} = \hat{\mu}_{4,t}$ and $\mu_{5,t} = \hat{\mu}_{5,t}$. Hence the *worst-case equilibrium* for this game can be written in state-space form as

$$
\begin{aligned}
a_{t+1} &= \rho_a a_t + \epsilon_{a,t+1} && (57)\\
e_{t+1} &= \rho_e e_t + \epsilon_{e,t+1} && (58)\\
u_{C,t+1} &= \rho_C u_{C,t} + \epsilon_{C,t+1} && (59)\\
\mu_{4,t+1} &= \mu_{4,t} + \frac{1}{\sigma\beta} + \mu_{5,t} - w_\pi \pi_t && (60)\\
\mu_{5,t+1} &= \frac{1}{\beta}\mu_{5,t} + \lambda\left(\sigma+\phi\right)\mu_{4,t+1} + (\sigma+\phi)o_t && (61)\\
\beta E_t \pi_{t+1} &= \pi_t - \lambda mc_t - e_t - u_{t+1} && (62)\\
E_t y_{t+1} &= y_t + \frac{1}{\sigma}(i_t - E_t\pi_{t+1} + E_t u_{C,t+1} - u_{C,t}) - v_{t+1} && (63)\\
i_t &= -\frac{1}{w_i\sigma}\mu_{4,t+1} && (64)\\
mc_t &= (\sigma+\phi)y_t - (1+\phi)a_t && (65)\\
u_{t+1} &= -\frac{1}{\Theta}\mu_{4,t+1} && (66)\\
v_{t+1} &= -\frac{1}{\Theta}\mu_{5,t+1} && (67)\\
r_t &= i_t - E_t\pi_{t+1} && (68)\\
\hat{y}_t &= \left(\frac{1+\phi}{\sigma+\phi}\right)a_t && (69)\\
o_t &= \hat{y}_t - y_t && (70)
\end{aligned}
$$

It is of interest to note that after eliminating $\mu_{5,t}$ and $\mu_{4,t}$ from (60) and (61), the interest rate rule (64) can be expressed as

$$
i_t = \left[\frac{(\beta+1)\sigma + \lambda(\sigma+\phi)}{\sigma\beta}\right] i_{t-1} - \frac{1}{\beta} i_{t-2} - \frac{1}{w_i\sigma}\left[(\sigma+\phi)(o_t - o_{t-1}) - w_\pi\lambda(\sigma+\phi)\pi_t\right] \quad (71)
$$

so given $\{u_{t+1}\}, \{v_{t+1}\}$, the interest rate adjusts gradually, responding negatively to $\Delta o_t$ and positively to $\pi_t$.

The approximating equilibrium is the undisturbed model with the interest rate rule designed for the worst case. This is given by

$$\beta E_t \pi^a_{t+1} = \pi^a_t - \lambda mc^a_t - e_t \quad (72)$$

$$E_t y^a_{t+1} = y^a_t + \frac{1}{\sigma}(i_t - E_t \pi^a_{t+1} + E_t u_{C,t+1} - u_{C,t}) \quad (73)$$

$$mc^a_t = (\sigma + \phi) y^a_t - (1+\phi) a_t \quad (74)$$

$$r^a_t = i_t - E_t \pi^a_{t+1} \quad (75)$$

$$\hat{y}_t = \left(\frac{1+\phi}{\sigma+\phi}\right) a_t \quad (76)$$

$$o^a_t = \hat{y}_t - y^a_t \quad (77)$$

where the interest rate rule is given by (64) in the worst-case equilibrium. An immediate problem with this solution of the approximating equilibrium now emerges: it cannot be saddle-path stable and in fact it is stable, but indeterminate.[6] We return to this problem in the context of our preferred game 2. Table 2 displays the properties of the worst-case equilibrium as $\Theta$ decreases and the concern for robustness increases.

| | $\Theta = \infty$ | $\Theta = 100$ | $\Theta = 50$ | $\Theta = 25$ | $\Theta = 20$ |
|---|---|---|---|---|---|
| $\text{var}(o_t)$ | 1.91 | 2.25 | 2.68 | 4.08 | 5.25 |
| $\text{var}(\pi_t)$ | 0.03 | 0.04 | 0.05 | 0.07 | 0.08 |
| $\text{var}(y_t)$ | 2.95 | 3.29 | 3.73 | 5.13 | 6.29 |
| $\text{var}(l_t)$ | 2.05 | 2.39 | 2.82 | 4.22 | 5.39 |
| $\text{var}(r_t)$ | 2.15 | 2.38 | 2.65 | 3.44 | 4.02 |
| $\text{var}(i_t)$ | 2.40 | 2.66 | 2.98 | 3.89 | 4.56 |
| Prob. ZLB | 0.26 | 0.27 | 0.28 | 0.31 | 0.32 |
| $\Omega_0$ | 4.286 | 5.030 | 5.990 | 9.020 | 11.53 |
| $c_e(\Theta)(\%)$ | 0 | 0.007 | 0.017 | 0.047 | 0.072 |

**Table 2. worst-case Equilibrium in Game 1**

**Note:** $c_e$ is the consumption equivalent cost of robustness as a percentage of steady state consumption.

[6] If one is only interested in deterministic impulse responses this poses no problems as the numerical solution as a two-point boundary value problem can still be computed. Our stochastic problem however requires staddle-path stability.

In Table 2 we define $c_e(\Theta)$ as the cost of planning for the worst case scenario compared with just designing a rule for the approximating model. Let the expected loss in the general case of $\Theta \neq \infty$ be $\Omega_0(\Theta)$. Then in percentage terms we have that

$$c_e(\Theta) = (\Omega_0(\Theta) - \Omega_0(\infty)) \times 10^{-2} \tag{78}$$

Three main results emerge from Table 2. First as concern for robustness increases then so do the volatilities of all variables in the economy under the robust rule increase, including that of the nominal interest rate. Thus in contrast to the result of Brainard (1967), the robust rule responds more aggressively in this environment of unstructured uncertainty than in the case of no model uncertainty. Second, the welfare cost associated with the worst-case compared to optimal policy without uncertainty rises as $\Theta$ falls to $c_e = 0.072$ when $\Theta = 20$. Finally robust control has a further cost: the frequency of hitting the ZLB increases from just over a quarter with the unrobust rule to almost one third with the robust rule and $\Theta = 20$.

### 3.4.2 Robust Control: Game 2

Consider now the design of a robust rule by the monetary authority given (52) and the perturbed model (79) where $u_{t+1}$ and $v_{t+1}$ are given by (52). We can write the worst-case perturbed model in state-space form as

$$\begin{bmatrix} \mathsf{z}_{t+1} \\ u_{t+1} \\ v_{t+1} \\ E_t\mathsf{x}_{t+1} \end{bmatrix} = A^*(\Theta) \begin{bmatrix} \mathsf{z}_t \\ u_t \\ v_t \\ \mathsf{x}_t \end{bmatrix} + B^* i_t + C^* \begin{bmatrix} \epsilon_{a,t+1} \\ \epsilon_{e,t+1} \\ \epsilon_{C,t+1} \end{bmatrix} \tag{79}$$

$$\begin{bmatrix} y_t \\ l_t \\ r_t \\ o_t \end{bmatrix} = F \begin{bmatrix} \mathsf{z}_t \\ \mathsf{x}_t \end{bmatrix} \tag{80}$$

Optimized robust rules can now be found using the same method as for optimized rules without uncertainty.

The approximating equilibrium is again given by (72)-(77) as before and again is not saddle-path stable. However we can remedy this feature in a simple way. By modifying the policymaker's loss function

$$(1-\omega)W_t + \omega W_t^a = (1-\omega)((\sigma+\phi)o_t^2 + w_\pi \pi_t^2) + \omega((\sigma+\phi)o_t^{a\,2} + w_\pi \pi_t^{a\,2}) + w_i i_t^2 \quad (81)$$

a small value of $\omega$ imposes a concern for stability of the approximating equilibrium and makes the composite worst-case, approximating model saddle-path stable. Then in the following table to compare the outcome with the previous game we take $\omega = 0$ for the worst-case equilibrium and $\omega = 0.1$ to calculate the approximating equilibrium.

| | $\Theta = \infty$ | $\Theta = 100$ | $\Theta = 50$ | $\Theta = 25$ | $\Theta = 20$ |
|---|---|---|---|---|---|
| var$(o_t)$ | 1.91 | 2.35 | 2.80 | 3.51 | 3.72 |
| var$(o_t^a)$ | 1.91 | 2.08 | 2.10 | 1.76 | 1.51 |
| var$(\pi_t)$ | 0.03 | 0.03 | 0.03 | 0.04 | 0.05 |
| var$(\pi_t^a)$ | 0.03 | 0.02 | 0.02 | 0.11 | 0.20 |
| var$(y_t)$ | 2.95 | 3.39 | 3.84 | 4.55 | 4.76 |
| var$(y_t^a)$ | 2.95 | 3.12 | 3.14 | 2.80 | 2.56 |
| var$(l_t)$ | 2.05 | 2.49 | 2.94 | 3.65 | 3.86 |
| var$(l_t^a)$ | 2.05 | 2.21 | 2.24 | 1.90 | 1.66 |
| var$(r_t)$ | 2.15 | 2.61 | 2.93 | 3.20 | 3.94 |
| var$(r_t^a)$ | 2.15 | 2.45 | 2.73 | 2.79 | 2.80 |
| var$(i_t)$ | 2.40 | 2.83 | 3.08 | 3.37 | 4.39 |
| Prob. ZLB | 0.26 | 0.28 | 0.29 | 0.29 | 0.32 |
| $\Omega_0(\Theta)$ | 4.286 | 4.983 | 5.737 | 7.254 | 8.007 |
| $\Omega_0^a(\Theta)$ | 4.286 | 4.347 | 4.423 | 5.674 | 7.345 |
| $c_e$ (%) | 0 | 0.007 | 0.014 | 0.030 | 0.037 |
| $c_e^a$ (%) | 0 | 0.001 | 0.001 | 0.014 | 0.031 |

**Table 3. Robust Control in Game 2 (Commitment)**

From Table 3 by comparing the expected welfare loss $\Omega_0(\Theta)$ with that in Table 2 we see immediately the sub-optimal nature of the policymaker's response to nature in game 2. The cost of achieving robustness, $c_e$ is now almost half that of game 2 at when concern for robustness is set at $\Theta = 20$. The cost in the approximating equilibrium rises to a similar level. As in game 1, the frequency of hitting the ZLB rises as the rule becomes more robust.

Figure 2, as with Figure 1, shows the impulse response to a mark-up shock but now compares optimal policy without model uncertainty with the robust control worst-case and approximating equilibria. In the worst-case equilibrium, nature adds to the inflationary pressures resulting in the hump-shape as the effect of the mark-up shock declines. In anticipation of this malign intervention, interest rate policy is more active than the non-robust policy in the absence of model uncertainty, rising by more in the short-run and falling by more in the medium term. This adds to the volatility of the interest rate seen as a feature with all shocks in Table 3. In the approximating equilibrium the output gap and other real variables including the expected real interest rate, $r_t = i_t - E_t\pi_{t+1}$ are almost the same for the approximating equilibrium and the model under non-robust optimal control without model uncertainty. As $r_t$ returns to its steady state of zero the expected inflation rate in the approximating equilibrium must therefore follow the interest rate path designed for the worst-case equilibrium and fall well below zero before gradually returning to the steady state. The approximating equilibrium then experiences a much lower volatility of the output gap but a much higher volatility of inflation for this shock, a feature again seen for all shocks in Table 3. In the welfare loss the former effect slightly outweighs the latter as can be seen from $\Omega_0(\Theta)^a < \Omega_0(\Theta)$ in Table 3.

As with optimal policy without model uncertainty, we now ask the question: can an optimized simple Taylor-type rule mimic the fully optimal policy? We require a rule that is saddle-path stable for both the worst-case and approximating models. A rule of the form (18) will achieve saddle-path stability of the latter but not the former. This requires a rule that responds to nature's malign shocks: i.e.,

$$i_t = \rho i_{t-1} + \theta_\pi \pi_t + \theta_y(y_t - \hat{y}_t) + \theta_u u_{t+1} + \theta_v v_{t+t} \tag{82}$$

where $u_{t+1}$ and $v_{t+1}$ is given by (52). Note that since we are assuming that the current inflation rate and the output gap is observable at time $t$, $u_{t+1}$ and $v_{t+1}$ as functions of outcomes $[o_t, o_{t-1}, \cdot\ \cdot; \pi_t, \pi_{t-1}, \cdot\ \cdot]$ can be calculated at time $t$.

Table 4 shows how the optimized rule changes as concern for robustness increases and the outcomes under the rule. Regarding the former we see that the optimized rule proceeds from an integral rule in the absence of model uncertainty to a rule with far less

| | $\Theta = \infty$ | $\Theta = 100$ | $\Theta = 50$ | $\Theta = 25$ | $\Theta = 15$ |
|---|---|---|---|---|---|
| $[\rho,\ \theta_\pi,\ \theta_y,$ $\theta_u,\ \theta_v]$ | $[1.0, 8.96, 0.06,$ $0, 0]$ | $[0.91, 8.21, 0.22,$ $5.86, 10.0]$ | $[0.64, 10.0, 0.42,$ $7.13, 10.0]$ | $[0.48, 10.0, 0.0$ $7.16, 6.37]$ | $[0.47, 10.0, 0.0.$ $6.27, 5.35]$ |
| $\text{var}(o_t)$ | 1.98 | 2.21 | 2.77 | 3.46 | 3.79 |
| $\text{var}(o_t^a)$ | 1.98 | 1.86 | 2.13 | 2.26 | 2.27 |
| $\text{var}(\pi_t)$ | 0.03 | 0.06 | 0.03 | 0.05 | 0.07 |
| $\text{var}(\pi_t^a)$ | 0.03 | 0.05 | 0.03 | 0.02 | 0.02 |
| $\text{var}(y_t)$ | 2.93 | 3.16 | 3.76 | 4.51 | 4.86 |
| $\text{var}(y_t^a)$ | 2.93 | 2.78 | 3.08 | 3.22 | 3.24 |
| $\text{var}(l_t)$ | 2.16 | 2.38 | 2.93 | 3.61 | 3.92 |
| $\text{var}(l_t^a)$ | 2.16 | 2.04 | 2.31 | 2.43 | 2.45 |
| $\text{var}(r_t)$ | 2.56 | 2.03 | 2.73 | 2.92 | 3.38 |
| $\text{var}(r_t^a)$ | 2.56 | 1.64 | 2.37 | 2.74 | 2.81 |
| $\text{var}(i_t)$ | 2.53 | 2.45 | 2.98 | 3.20 | 3.82 |
| $\text{var}(i_t^a)$ | 2.53 | 1.98 | 2.57 | 2.82 | 2.88 |
| Prob. ZLB | 0.27 | 0.26 | 0.28 | 0.29 | 0.31 |
| Prob. ZLB (a) | 0.27 | 0.24 | 0.27 | 0.27 | 0.28 |
| $\Omega_0(\Theta)$ | 4.356 | 5.330 | 5.859 | 7.4570 | 8.417 |
| $\Omega_0^a(\Theta)$ | 4.356 | 4.488 | 4.505 | 4.688 | 4.722 |
| $c_e$ | 0 | 0.010 | 0.015 | 0.031 | 0.041 |
| $c_e^a$ | 0 | 0.001 | 0.001 | 0.003 | 0.004 |

**Table 4. Robust Control in Game 2 with Optimized Taylor Rule:**

$$i_t = \rho i_{t-1} + \theta_\pi \pi_t + \theta_y (y_t - \hat{y}_t) + \theta_u u_{t+1} + \theta_v v_{t+1}$$

$$i_t^a = \rho i_{t-1}^a + \theta_\pi \pi_t^a + \theta_y (y_t^a - \hat{y}_t) + \theta_u u_{t+1}^a + \theta_v v_{t+1}^a$$

interest-rate smoothing for the most robust case. The cost of robustness $c_e$ in the worst-case equilibrium is rather higher than under the fully optimal rule, but interestingly considerably lower in the approximating model. In a sense then, the simple rule is more robust in that it trades a slightly worse performance in the worst-case equilibrium for a much better outcome if the model is unperturbed. This is achieved by a rule that responds to the observed current output gap and inflation rates in the worst-case scenario, $o_t$ and $\pi_t$, but to the observed counterparts $o_t^a = (\hat{y}_t - y_t^a)$ and $\pi_t^a$ if the economy is not perturbed.

In the two states of the world, the rule takes the form

$$i_t = \rho i_{t-1} + \theta_\pi \pi_t + \theta_y (y_t - \hat{y}_t) + \theta_u u_{t+1} + \theta_v v_{t+1} \tag{83}$$

$$\Theta u_{t+1} = \Theta \left[ u_t + \frac{1}{\sigma\beta} v_t \right] + w_\pi \pi_t \tag{84}$$

$$\Theta v_{t+1} = \Theta \left[ \frac{1}{\beta} v_t + \lambda (\sigma + \phi) u_{t+1} \right] + (\sigma + \phi)(y_t - \hat{y}_t) \tag{85}$$

for the worst-case equilibrium, with the *same rule* responding to the undisturbed outcomes

$$i_t^a = \rho i_{t-1}^a + \theta_\pi \pi_t + \theta_y (y_t^a - \hat{y}_t) + \theta_u u_{t+1}^a + \theta_v v_{t+1}^a \tag{86}$$

$$\Theta u_{t+1}^a = \Theta \left[ u_t^a + \frac{1}{\sigma\beta} v_t^a \right] + w_\pi \pi_t^a \tag{87}$$

$$\Theta v_{t+1}^a = \Theta \left[ \frac{1}{\beta} v_t^a + \lambda (\sigma + \phi) u_{t+1}^a \right] + (\sigma + \phi)(y_t^a - \hat{y}_t) \tag{88}$$

for the approximating model. This contrasts with the fully optimal policies that are designed as the *same interest rate path* conditional on initial displacements for both the worst-case and approximating equilibria.

### 3.4.3 Robust Control: Game 3

In our final robustness game we consider the case where the policymaker cannot commit and is forced to pursue a time consistent discretionary policy. Table 5 sets out the results that correspond to the commitment case in Table 3.

Two features of these results are worth highlighting. First, in the case of a very high setting of $\Theta$ we do *not* revert to the time consistent solution without model uncertainty of Table 1 and in fact the outcome is considerable better than for that case. This result highlights the point raised by Blake and Kirsanova (2007) that it is possible to have multiple discretionary equilibria. In our case this second equilibrium arose because we expanded the state-space to include worst-case shocks $[u_t, v_t]$. Even when $\Theta \to \infty$, in which case these shocks are purely exogenous processes, the higher-order state space creates a new Markov-perfect equilibrium.

| | $\Theta = \infty$ | $\Theta = 100$ | $\Theta = 50$ | $\Theta = 25$ | $\Theta = 20$ |
|---|---|---|---|---|---|
| $\text{var}(o_t)$ | 1.89 | 2.40 | 2.98 | 4.17 | 4.72 |
| $\text{var}(o_t^a)$ | 1.89 | 2.97 | 2.90 | 2.33 | 2.03 |
| $\text{var}(\pi_t)$ | 0.04 | 0.04 | 0.03 | 0.01 | 0.02 |
| $\text{var}(\pi_t^a)$ | 0.04 | 0.08 | 0.04 | 0.01 | 0.02 |
| $\text{var}(y_t)$ | 2.93 | 3.44 | 4.03 | 5.21 | 5.76 |
| $\text{var}(y_t^a)$ | 2.93 | 4.02 | 3.94 | 3.37 | 3.08 |
| $\text{var}(l_t)$ | 2.04 | 2.54 | 3.13 | 4.31 | 4.86 |
| $\text{var}(l_t^a)$ | 2.04 | 3.11 | 3.04 | 2.47 | 2.18 |
| $\text{var}(r_t)$ | 2.67 | 3.31 | 3.86 | 3.88 | 3.41 |
| $\text{var}(r_t^a)$ | 2.67 | 2.69 | 3.12 | 3.34 | 3.22 |
| $\text{var}(i_t)$ | 2.91 | 3.52 | 3.98 | 3.70 | 3.16 |
| Prob. ZLB | 0.28 | 0.30 | 0.31 | 0.30 | 0.29 |
| $\Omega_0(\Theta)$ | 4.436 | 5.220 | 6.097 | 7.991 | 9.042 |
| $\Omega_0^a(\Theta)$ | 4.436 | 7.313 | 6.293 | 4.538 | 4.484 |
| $c_e$ | 0.001 | 0.009 | 0.018 | 0.037 | 0.048 |
| $c_e^a$ | 0.001 | 0.030 | 0.020 | 0.003 | 0.002 |

**Table 5. Robust Control in Game 3 (Discretion)**

The second noteworthy feature of Table 5 concerns the costs of discretion when combined with a concern for robustness. At $\Theta = 20$ comparing Tables 3 and 5 and we see that commitment when combined with a concern for robustness raises welfare in the worst-case equilibrium by a consumption equivalent of $0.048 - 0.037 = 0.011\%$ which although still small is significantly higher that the commitment gain of $c_e = 0.007\%$ reported in Table 1 without model uncertainty. This outcome is reached at a higher frequency of hitting the interest rate ZLB constraint. The cost of discretion in the approximating equilibrium is less clear cut as it rises for low levels of concern for robustness, reaches a peak somewhere between $\Theta = 100$ and $\Theta = 50$ and then falls. This is an interesting phenomenon requiring further investigation. But for low levels of concern for robustness, Table 5 suggests a new result regarding commitment when combined with unstructured model uncertainty: that *robustness concerns increase the welfare gains from commitment*.

## 4 Imposing the Interest Rate Zero Lower Bound

In one respect the modest consumption equivalent costs reported up to now are misleading, especially for the discretionary policy. The reason for this is to be seen for the unconditional variances reported in these which are very large and rise further when we introduce robustness concerns. Such high variances imply that the interest rate under these optimized or optimal rules will hit the interest rate zero lower bound frequently.[7] We now address this design fault in the rules.

We modify our interest-rate rules to approximately impose an interest rate ZLB so that this event hardly ever occurs. As in Woodford (2003), chapter 6, this is implemented by increasing the weight on the interest rate variance $w_i$ in the single period welfare loss (12). Then following Levine *et al.* (2007a), the policymaker's optimization problem is to choose $w_i$ and the unconditional distribution for $i_t$ (characterized by the steady state variance) shifted to the right about a new non-zero steady state inflation rate and a higher nominal interest rate, such that the probability, $p$, of the interest rate hitting the lower bound is very low. This is implemented by calibrating the weight $w_i$ *for each of our policy rules* so that $z_0(p)\sigma_r < R_n$ where $z_0(p)$ is the critical value of a standard normally distributed variable $Z$ such that prob $(Z \leq z_0) = p$, $I = \frac{1}{\beta} - 1 + \pi^*$ is the steady state nominal interest rate, $\sigma_i^2 = \text{var}(i)$ is the unconditional variance and $\pi^*$ is the new steady state inflation rate. Given $\sigma_i$, the steady state positive inflation rate that will ensure $i_t \geq 0$

[7] As Primiceri (2006) has pointed out, optimal rules with this feature are 'not operational'.

with probability $1 - p$ is given by[8]

$$\pi^* = \max[z_0(p)\sigma_r - \left(\frac{1}{\beta} - 1\right) \times 100, 0] \tag{89}$$

In our linear-quadratic framework we can now write the inter-temporal expected welfare loss at time $t = 0$ as the sum of stochastic plus deterministic components, $\Omega_0 = \tilde{\Omega}_0 + \bar{\Omega}_0$. Given $w_i$, denote the expected inter-temporal loss (stochastic plus deterministic components) at time $t = 0$ by $\Omega_0(w_i)$. This includes a term penalizing the variance of the interest rate which does not contribute to utility loss as such, but rather represents the interest rate lower bound constraint. Actual utility, found by subtracting the interest rate term, is given by $\Omega_0(0)$. Since in the new steady state the real interest rate is unchanged, the steady state involving real variables are also unchanged, so from (12) we can write $\bar{\Omega}_0(0) = \frac{1}{2} w_\pi \pi^{*2}$. Both the ex-ante optimal and the optimal time-consistent deterministic welfare loss that guide the economy from a zero-inflation steady state to $\pi = \pi^*$ differ from $\bar{\Omega}_0(0)$ (but not by much because the steady-state contributions by far outweighs the transitional one).

By increasing $w_i$ we can lower $\sigma_i$ thereby decreasing $\pi^*$ and reducing the deterministic component, but at the expense of increasing the stochastic component of the welfare loss. By exploiting this trade-off, we can optimize over $w_i$ and $\pi^*$ to then arrive at the optimal policy that, in the vicinity of the steady state, imposes the ZLB constraint, $i_t \geq 0$ with probability $1 - p$.

[8]If the inefficiency of the steady-state output is negligible, then $\pi^* \geq 0$ is a credible new steady state inflation rate. Note that in our LQ framework, the zero interest rate bound is very occasionally hit. Then interest rate is allowed to become negative, possibly using a scheme proposed by Gesell (1934) and Keynes (1936). Our approach to the ZLB constraint (following Woodford, 2003) in effect replaces it with a nominal interest rate variability constraint which ensures the ZLB is hardly ever hit. By contrast the work of a number of authors including Adam and Billi (2007), Coenen and Wieland (2003), Eggertsson and Woodford (2003) and Eggertsson (2006) study optimal monetary policy with commitment in the face of a non-linear constraint $i_t \geq 0$ which allows for frequent episodes of liquidity traps in the form of $i_t = 0$.

| Rule | $\Theta$ | $\pi^*$ (%) | $\Omega_0 = \tilde{\Omega}_0(0) + \bar{\Omega}_0(0)$ | $c_e$ (%) |
|---|---|---|---|---|
| Optimal | $\infty$ | 0.08 | 5.4 | 0 |
| Discretion | $\infty$ | 2.17 | 134 | 1.29 |
| Optimal | 50 | 0.66 | 29 | 0.24 |
| Discretion | 50 | 2.73 | 177 | 1.72 |

**Table 6. Summary of Welfare Outcome of Rules in the Worst-Case Equilibrium with a Nominal Interest Rate ZLB Imposed.**

Figure 5 and Table 6 show the results of this optimization procedure for the optimal commitment discretionary rules respectively for the worst-case equilibrium. We choose $p = 0.025$. The steady-state inflation rate, $\pi^*$, that will ensure the lower bound is reached only with probability $p = 0.025$ is computed using (89). Given $\pi^*$, we can then evaluate the deterministic component of the welfare loss, $\bar{\Omega}_0(0)$.

Comparing Table 6 with a ZLB constraint with Tables 3 and 4 without the constraint two results stand out: first, without robustness concerns ($\Theta = \infty$) the gains from commitment rise substantially from a very small value of $c_e = 0.007\%$ in Table 1 to the substantial $c_e = 1.29\%$ in Table 6. This confirms the result obtained by Levine *et al.* (2007a) using an empirical DSGE model fitted to Euro-data. Second, the ZLB constraint substantially increases the cost of achieving robustness with commitment in the face of unstructured uncertainty from $c_e = 0.014\%$ in Table 3 to $c_e = 0.24\%$ in Table 6, and from $c_e = 0.018\%$ in Table 5 to $c_e = 1.72\%$ in Table 6 under discretion. *The combination of worst-case robustness, lack of commitment and the interest rate ZLB constraint creates a substantial welfare cost equivalent to a* 1.72% *permanent increase in steady state consumption.* Finally, for the approximating equilibrium a very similar result holds because the steady state inflation rate required to satisfy the ZLB constraint is so high under discretion and is the same for the worst-case and approximating equilibria.

# 5 Robust Rules with Structured Model Uncertainty

## 5.1 A Rival Model Approach to Robustness

In this section we consider model uncertainty in the form of uncertain estimates of the non-policy parameters of the model, $\Gamma = (\beta, \xi, \phi, \sigma, \zeta, \rho_a, \rho_e, \rho_C, \zeta, \sigma^2_{a,t}, \sigma^2_{e,t}, \sigma^2_{C,t})$. Suppose the state of the world $s$ is described by a model with $\Gamma = \Gamma^s$ expressed in state-space form as

$$\begin{bmatrix} \mathsf{z}^s_{t+1} \\ E_t\mathsf{x}^s_{t+1} \end{bmatrix} = A^s \begin{bmatrix} \mathsf{z}^s_t \\ \mathsf{x}^s_t \end{bmatrix} + B^s i^s_t + C^s \begin{bmatrix} \epsilon_{gt+1} \\ \epsilon_{at+1} \end{bmatrix} \tag{90}$$

$$o^s_i = E^s \begin{bmatrix} \mathsf{z}^s_t \\ \mathsf{x}^s_t \end{bmatrix} \tag{91}$$

where $\mathsf{z}^s_t = [a^s_t, e^s_t, u^s_{C,t}, i_{t-1}]$ is a vector of predetermined variables at time $t$ and $\mathsf{x}_t = [c^s_t, \pi^s_t]$ are non-predetermined variables in state $s$ of the world. In (90) and (91) it is important to stress that variables are in deviation form about a zero-inflation steady state of the model in state s. For example output in deviation form is given by $y^s_t = \frac{Y^s_t - \bar{Y}^s}{\bar{Y}_s}$ where $\bar{Y}^s$ is the steady state of the model in state s defined by parameters $\Theta^s$ and $i^s_t = i_t - \bar{i}^s$ where the natural rate of interest in model s, $\bar{i}_s = \frac{1}{\beta^s} - 1$.

Because each model is linearized about a possibly different steady state, we must now set up the model in state s in terms of the *actual* interest rate, not the deviation about the steady state. Then augmenting the state vector to become $\mathsf{z}^s_t = [1, a^s_t, e^s_t, u^s_{C,t}, i_{t-1}]$ we still have a state-space form (90) and (91) and we minimize

$$\Omega_0 = \frac{1}{2}\sum_{t=0}^{\infty}\beta^t \sum_{s=1}^{n} p_s[\mathsf{y}'_t Q \mathsf{y}_t + 2\mathsf{y}'_t U i_t + R i^2_t] \tag{92}$$

where $p_s$ is the weight or probability attached to model $s$. This we refer to as *model-robustness*. For *parameter-robustness* (92) is replaced with the average expected utility loss across a large number of draws from all models constructed using both the estimated posterior model probabilities and the posterior parameter distributions for each model found by Bayesian estimation (see Batini *et al.* (2006) and Levine *et al.* (2007b)).

With model uncertainty even in the absence of initial displacements $\mathsf{z}_0$, there is still a deterministic component of policy arising from differences in the natural rate of interest compatible with zero inflation in the steady state, $\bar{i}^s = \frac{1}{\beta^s} - 1$. A non-integral rule specifying $i_t = \bar{i}^s$ in the long-run will only result in zero inflation in model s. From the consumers' Euler equation in model r with $\beta^r > \beta^s$, implementing the rule designed for model s with $\bar{i} = \bar{i}^s = \frac{1}{\beta^s} - 1$ gives a steady state inflation rate $\bar{\pi}^r$ that is no longer zero but given by

$$\frac{\beta^r(1+\bar{i}^s)}{(1+\bar{\pi}^r)} = \frac{\beta^r}{\beta^s(1+\bar{\pi}^r)} = 1 \quad \text{i.e., } \bar{\pi}^r = \frac{\beta^r}{\beta^s} - 1 > 0 \tag{93}$$

Our robust non-integral rule designed for any model specifies a natural zero inflation rate of interest $\bar{i}_R$, corresponding to a discount factor $\beta_R = \frac{1}{1+\bar{i}_R}$ to result in an expected long-run inflation rate across models of zero. This implies $\beta_R$ is determined by

$$\sum_{s=1}^{n} p_s \left[ \frac{\beta_s}{\beta_R} - 1 \right] = 0 \;\Rightarrow\; \beta_R = \sum_{s=1}^{n} p_s \beta_s \tag{94}$$

That is, $\beta_R$ is the expected value of $\beta_s$ across the model variants. The need to specify a natural rate of interest, $\bar{i}_R$, only applies to non-integral rules. By contrast, a further benefit of integral rules is that the economy is automatically driven to a zero-inflation steady state whatever the state of the world without having to specify $\bar{i}_R$.

As in section 4 we impose the ZLB constraint by varying the weight $w_i$. For Bayesian-robust commitment rules the interest rate volatility is not great and the shift in the steady state inflation rate needed to impose the ZLB constraint is small (as can be confirmed by the $\Theta = \infty$ results in Table 6). We therefore confine ourselves to the case where steady-state inflation is zero ($\pi^* = 0$). For each of the $n$ models (for M-robustness) or draws (for P-robustness), we calculate the equilibrium steady state variance of the interest rate. Then for each draw we use the variance of the interest rate to calculate the probability of hitting the zero lower bound; once again the average of these appears as Prob ZLB in the table and the average of these is included in the last row of tables 7 and 9 below as $\sigma_i^2$. Thus with an equilibrium interest rate of 1% per quarter (4% per annum), the latter are given by

$$\sigma_i^2 = \frac{1}{n} \sum_{j=1}^{n} \sigma_i^2(j) \tag{95}$$

$$Prob\ ZLB = \frac{1}{n} \sum_{j=1}^{n} Z\left( -\frac{1}{\sigma_i(j)} \right) \tag{96}$$

where $Z(x)$ is the probability that a standard normal random variable has a value less than $x$.

## 5.2 Application to the Canonical Keynesian Model

We now apply this procedure to the canonical model. We limit the structural uncertainty to the important parameter $\xi$ that captures the degree of price stickiness in the model. Other parameters remain unchanged, including $\beta$ so that $\beta_R = \beta$ in (94). Four model variants are considered with this parameter taking values $\xi = 0.001$, 1/2, 2/3, 3/4 corresponding to near flexible prices and an average price contract length of 2, 3 and 4 quarters respectively. We consider optimal policy and a simple current inflation rule of the form

$$i_t = \rho i_{t-1} + \theta_\pi \pi_t \tag{97}$$

The form of the optimal rule has be shown to take the form:

$$\begin{aligned} i_t &= \left[\frac{(\beta+1)\sigma + \lambda(\sigma+\phi)}{\sigma\beta}\right] i_{t-1} - \frac{1}{\beta} i_{t-2} - \frac{1}{w_i \sigma}\left[(\sigma+\phi)(o_t - o_{t-1}) - w_\pi \lambda(\sigma+\phi)\pi_t\right] \\ &= \rho_1 i_{t-1} - \rho_2 i_{t-2} - \theta_{\Delta o}(o_t - o_{t-1}) + \theta_\pi \pi_t \end{aligned} \tag{98}$$

say. Woodford (2003), page 584, describes this rule as 'robustly optimal' in the sense that it is independent of the exogenous processes in the model. It is optimal from a 'timeless perspective' in the sense that the optimal rule of the form (98) does become sub-optimal over time.[9] The coefficients have the property that $\rho_1 - \rho_2 > 1$ (the rule is 'super-inertial') and it satisfies the modified Taylor principle for rules with inertia, that $\theta_\pi > 1 - \rho_1 + \rho_2$. Numerical values for $[\rho_1, \rho_2, \theta_{\Delta o}, \theta_\pi]$ are given in table 7 for the 4 model variants with the M-robust rule in the final row. Regarding the latter it is super-inertial and satisfies the Taylor principle. Moreover, unlike the Hansen-Sargent robust rule, the M-robust rule does not call for a more aggressive policy than any of the non-robust rules, but neither does it exhibit the Brainard property that uncertainty calls for more policy caution.

[9] This does not mean there is no time-inconsistency problem - the rule itself does not remain optimal with the passage of time.

| Rule | $\xi$ | $[\rho_1, \rho_2, \theta_{\Delta o}, \theta_\pi]$ | $w_i$ | $\sigma_i^2$ | $\tilde{\Omega}_0(0)$ |
|---|---|---|---|---|---|
| OPT(1) | $10^{-3}$ | [1867, 1.01, 463, 461580] | 0.004 | 0.24 | 0.0002 |
| OPT(2) | $\frac{1}{2}$ | [2.953, 1.01, 0.370, 0.187] | 5 | 0.25 | 1.041 |
| OPT(3) | $\frac{2}{3}$ | [2.328, 1.01, 0.116, 0.020] | 16 | 0.25 | 7.321 |
| OPT(4) | $\frac{3}{4}$ | [2.171, 1.01, 0.053, 0.005] | 35 | 0.24 | 23.87 |
| M-Robust | Aggregate | [2.161, 0.96, 0.055, 0.692] | 15 | 0.25 | 7.56 |

**Table 7. Optimal Commitment with $\pi^* = 0$ and Interest Rate ZLB Imposed.**[10]

Table 8 shows the welfare and ZLB outcomes when each rule designed for model $i$ is implemented in model $j$, $i = 1, \cdots 4$, $j = 1, \cdots 4$. These outcomes are compared with those under the M-robust rule in the last row of table 7. The table shows that non-robustness can take one of two forms. In the off-diagonal cells above the diagonal the welfare losses are below the optimal values but at a cost of severe violations of the ZLB constraint. In the most extreme case, the optimal rule for model 1 implemented in model 4 the probability of hitting the ZLB is 0.41 per period and this is compensated by only a small reduction of welfare loss. The other form of non-robustness shows itself in off-diagonal losses below the diagonal which are substantially higher than the optimal values. Thus in the case of the rule designed for model 4 implemented in model 2, the welfare loss is over three times that of the optimal value with the compensation that the ZLB probability is almost zero.

The final row of table 8 provides provides the cost of robustness analogous to (78). For the M-robust rule this is defined as follows. Let $\Omega_0(i)$ be the minimum welfare loss for model $i$ under optimal policy designed for $i$. Let $\Omega_0^M(i)$ be the welfare loss under the M-robust rule given in the penultimate row. Then we the cost of robustness is given by

$$c_e = (\Omega_0^M(i) - \Omega_0(i)) \times 10^{-2} \tag{99}$$

in consumption equivalent percentage units.

[10]For the M-robust rule in tables 7 and 9, $\sigma_i^2$ and $\tilde{\Omega}_0(0)$ are averages over the 4 model variants.

| Rule OPT(i) | Model 1 | Model 2 | Model 3 | Model 4 |
|---|---|---|---|---|
| OPT(1) | 0.0002 | 0.6028 | 5.320 | 20.87 |
| | (0.021) | (0.17) | (0.33) | (0.41) |
| OPT(2) | 0.002 | 1.041 | 4.837 | 15.72 |
| | (0.000) | (0.023) | (0.17) | (0.32) |
| OPT(3) | 0.003 | 2.231 | 7.321 | 18.42 |
| | (0.000) | (0.000) | (0.023) | (0.15) |
| OPT(4) | 0.003 | 3.213 | 10.16 | 23.87 |
| | (0.000) | (0.000) | (0.000) | (0.021) |
| Robust Rule | 0.001 | 1.462 | 7.335 | 21.442 |
| | (0.009) | (0.006) | (0.023) | (0.064) |
| $c_e$ (%) | $10^{-5}$ | 0.004 | 0.0001 | $-0.02$ |

**Table 8. Optimal Commitment with Model Uncertainty.**

**Note**: OPT($i$) is the optimal rule designed for model $i$. Cell $ij$ contains the welfare loss under OPT($i$) in the model $j$. Values in brackets are ZLB violation probabilities.

| Model | $\xi$ | Rule $[\rho, \theta_\pi]$ | $w_i$ | $\sigma_i^2$ | $\tilde{\Omega}_0(0)$ |
|---|---|---|---|---|---|
| INF(1) | $10^{-3}$ | [1, 0.6203] | 0.25 | 0.25 | 0.001 |
| INF(2) | $\frac{1}{2}$ | [1, 0.7618] | 15 | 0.24 | 1.892 |
| INF(3) | $\frac{2}{3}$ | [1, 0.5653] | 30 | 0.25 | 9.936 |
| INF(4) | $\frac{3}{4}$ | [1, 0.4280] | 50 | 0.25 | 29.47 |
| M-robust | Aggregate | [1, 0.5517] | 25.5 | 0.25 | 9.909 |

**Table 9. Optimal Current Inflation Rule with $\pi^* = 0$ and Interest Rate ZLB Imposed.**

Tables 9 and 10 repeat this exercise for the optimized inflation rule of the form (97). A number of features stand out. First unlike the Hansen-Sargent robust rule, the M-robust rule does come close to exhibiting the Brainard property that uncertainty calls for more policy caution in that the robust rule is less aggressive and all but the final non-robust rule. Second, examining the off-diagonal welfare losses and ZLB probabilities in table 10, by comparison with those for the optimal rule in table 8 they show far less variation. The proportional drop in welfare below the diagonal are far less and indeed in many cells the absolute welfare loss for the optimized (but sub-optimal) simple rule are less than their 'optimal' counterparts. Above the diagonal the ZLB constraint violations are far less serious than the optimal rule. As with the latter a robust rule can be designed that on average across models satisfies the ZLB constraint and reduces the welfare loss variations. As in (78) we calculate the cost of robustness in the final row and here we see that these costs are very similar to (and in fact slightly greater than) those for the optimal counterpart in Table 8.

We conclude that simple rules designed for one model implemented in the wrong model are far more robust than the optimal counterpart. However when both types of rules are designed to M-robust, the penultimate rows of tables 8 and 10 indicate that the costs of robustness for the optimal rules are slightly lower.

| Rule | Model 1 | Model 2 | Model 3 | Model 4 |
|---|---|---|---|---|
| INF(1) | 0.001 | 2.214 | 9.532 | 26.13 |
| | (0.023) | (0.013) | (0.030) | (0.064) |
| INF(2) | 0.001 | 1.892 | 8.678 | 24.36 |
| | (0.029) | (0.021) | (0.05) | (0.093) |
| INF(3) | 0.002 | 2.374 | 9.936 | 26.95 |
| | (0.021) | (0.011) | (0.023) | (0.053) |
| INF(4) | 0.002 | 2.908 | 11.21 | 29.47 |
| | (0.011) | (0.004) | (0.009) | (0.025) |
| M-Robust | 0.002 | 2.418 | 10.05 | 27.17 |
| | (0.019) | (0.009) | (0.023) | (0.051) |
| $c_e$ (%) | $10^{-5}$ | 0.005 | 0.001 | $-0.02$ |

**Table 10. Optimal Current Inflation Rule with Model Uncertainty.**

**Note**: INF($i$) is the optimized current inflation rule designed for model $i$. Cell $ij$ contains the welfare loss under INF($i$) in the model $j$. Values in brackets are ZLB violation probabilities.

## 6 Conclusions

In this paper we have carried out two robust policy exercises for interest rate rules using a work-horse New Keynesian model, one following a Hansen-Sargent minimax approach with unstructured model uncertainty, and the other adopting an older tradition where model uncertainty takes the form of rival models.

For the Hansen-Sargent approach a number of results are worth highlighting. First, robust policy in this case calls for a more aggressive monetary response to shocks than that in the absence of model uncertainty. This is not a new result (see, for example, Giannoni (2002) and Tetlow and von zur Muehlen (2001)), but we pursue an important consequence of this feature that has not appeared in the literature. A high interest rate variability in both the worst-case and approximating equilibria means that, in both scenarios, the robust rule leads to a serious violation of the ZLB constraint. The latter can be taken into account by choosing a steady state inflation rate sufficiently large, but then the costs of achieving robustness are substantial.

Second, Hansen-Sargent robust control can be seen as a non-cooperative game between malign nature and the policymaker. As in any game, the equilibrium concept needs close attention. In Hansen and Sargent (2003) the latter is an open-loop Nash equilibrium. We argue that this is not always a minimax solution as the policymaker who can commit can do better in the face of the worst environment by anticipating nature's strategy and acting as a leader. However if commitment is not possible and the policymakers exercises discretion, the worst-case equilibrium deteriorates sharply and with it the cost of robustness. The corresponding result for the approximating equilibrium is less straightforward in that we have found that as a concern for robustness increases, the cost under discretion is hump-shaped, an interesting result that merits further research. Taking these two points together, a combination of an inability to commit and the ZLB constraint imposes a substantial welfare cost mainly driven by a high steady state inflation rate for both the worst-case and approximating equilibria.

Because of the high volatility of the interest rate the ZLB constraint then results in a high cost of achieving Hansen-Sargent robustness. There are other problems with this approach to robustness. As Svensson (2000) has argued, the worst-case outcome is going to be a very low probability event and from any Bayesian perspective it is inappropriate to design policy that is so heavily influenced by it. More fundamentally, we do have information about the structure of uncertainty in the form of an assessment of the forecasting properties of the approximating model, those of rival models and estimates of parameter uncertainty gleaned from various estimation methods.

In the final section of the paper we have set out a general Bayesian framework for using the information available for the design of commitment interest rate rules. Again we incorporate a ZLB constraint in construction of our robust rules, but in notable contrast with Hansen-Sargent robustness the Bayesian approach does not result in aggressive monetary responses to shocks and a high interest rate volatility. It follows that the steady state inflation rate required to impose the ZLB in an optimal fashion is very low, and in fact we confine ourselves to slightly sub-optimal rules where it remains at zero.

In our Bayesian exercise we have confined ourselves to a very simple form of structured uncertainty in the form of uncertainty surrounding an important parameter capturing the degree of price stickiness. We have compared a robust interest rate rule of the form that is optimal in the absence of any model uncertainty, with a simple rule feeding back on current inflation. We find that simple rules designed for one model implemented in the wrong model are far more robust than its optimal counterpart. This in a sense is an additional argument for simple rules to be considered alongside their transparency and ease of implementation. However when both types of rules are designed to be robust across the possible views of the world, this advantage of simplicity disappears. Nevertheless this exercise suggests that some types of rule may be more robust than others and that robust design using a Bayesian approach should investigate a range of rules with this in mind.

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# A Details of Policy Rules

First consider the purely deterministic problem. In general policy involving several (for example monetary and fiscal) instruments starts with a model in state-space form:

$$\begin{bmatrix} \mathsf{z}_{t+1} \\ \mathsf{x}^e_{t+1,t} \end{bmatrix} = A \begin{bmatrix} \mathsf{z}_t \\ \mathsf{x}_t \end{bmatrix} + B\mathsf{w}_t \tag{A.1}$$

where $\mathsf{z}_t$ is an $(n-m)\times 1$ vector of predetermined variables including non-stationary processed, $\mathsf{z}_0$ is given, $\mathsf{w}_t$ is a vector of policy variables, $\mathsf{x}_t$ is an $m\times 1$ vector of non-predetermined variables and $\mathsf{x}^e_{t+1,t}$ denotes rational (model consistent) expectations of $\mathsf{x}_{t+1}$ formed at time $t$. Then $\mathsf{x}^e_{t+1,t} = \mathsf{x}_{t+1}$ and letting $\mathsf{y}'_t = [\mathsf{z}'_t \ \mathsf{x}'_t]$ (A.1) becomes

$$\mathsf{y}_{t+1} = A\mathsf{y}_t + B\mathsf{w}_t \tag{A.2}$$

Define target variables $\mathsf{s}_t$ by

$$\mathsf{s}_t = M\mathsf{y}_t + H\mathsf{w}_t \tag{A.3}$$

and the policy-maker's loss function at time $t$ by

$$\Omega_t = \frac{1}{2}\sum_{i=0}^{\infty} \beta^t[\mathsf{s}'_{t+i}Q_1\mathsf{s}_{t+i} + \mathsf{w}'_{t+i}Q_2\mathsf{w}_{t+i}] \tag{A.4}$$

which we can rewrite as

$$\Omega_t = \frac{1}{2}\sum_{i=0}^{\infty} \beta^t[\mathsf{y}'_{t+i}Q\mathsf{y}_{t+i} + 2\mathsf{y}'_{t+i}U\mathsf{w}_{t+i} + \mathsf{w}'_{t+i}R\mathsf{w}_{t+i}] \tag{A.5}$$

where $Q = M'Q_1M$, $U = M'Q_1H$, $R = Q_2 + H'Q_1H$, $Q_1$ and $Q_2$ are symmetric and non-negative definite, $R$ is required to be positive definite and $\beta \in (0,1)$ is discount factor. The procedures for evaluating the three policy rules are outlined in the rest of this appendix (or Currie and Levine (1993) for a more detailed treatment).

## A.1 The Optimal Policy with Commitment

Consider the policy-maker's *ex-ante* optimal policy at $t = 0$. This is found by minimizing $\Omega_0$ given by (A.5) subject to (A.2) and (A.3) and given $z_0$. We proceed by defining the Hamiltonian

$$\mathcal{H}_t(y_t, y_{t+1}, \mu_{t+1}) = \frac{1}{2}\beta^t(\mathsf{y}'_tQ\mathsf{y}_t + 2\mathsf{y}'_tU\mathsf{w}_t + \mathsf{w}'_tR\mathsf{w}_t) + \mu_{t+1}(A\mathsf{y}_t + B\mathsf{w}_t - \mathsf{y}_{t+1}) \tag{A.6}$$

where $\mu_t$ is a row vector of costate variables. By standard Lagrange multiplier theory we minimize

$$\mathcal{L}_0(y_0, y_1, \ldots, w_0, w_1, \ldots, \mu_1, \mu_2, \ldots) = \sum_{t=0}^{\infty} \mathcal{H}_t \tag{A.7}$$

with respect to the arguments of $L_0$ (except $z_0$ which is given). Then at the optimum, $\mathcal{L}_0 = \Omega_0$.

Redefining a new costate column vector $\mathsf{p}_t = \beta^{-t}\mu_t'$, the first-order conditions lead to

$$\mathsf{w}_t = -R^{-1}(\beta B'\mathsf{p}_{t+1} + U'\mathsf{y}_t) \tag{A.8}$$

$$\beta A'\mathsf{p}_{t+1} - \mathsf{p}_t = -(Q\mathsf{y}_t + U\mathsf{w}_t) \tag{A.9}$$

Substituting (A.8) into (A.2)) we arrive at the following system under control

$$\begin{bmatrix} I & \beta BR^{-1}B' \\ 0 & \beta(A' - UR^{-1}B') \end{bmatrix} \begin{bmatrix} \mathsf{y}_{t+1} \\ \mathsf{p}_{t+1} \end{bmatrix} = \begin{bmatrix} A - BR^{-1}U' & 0 \\ -(Q - UR^{-1}U') & I \end{bmatrix} \begin{bmatrix} \mathsf{y}_t \\ \mathsf{p}_t \end{bmatrix} \tag{A.10}$$

To complete the solution we require $2n$ boundary conditions for (A.10). Specifying $z_0$ gives us $n-m$ of these conditions. The remaining condition is the 'transversality condition'

$$\lim_{t\to\infty} \mu_t' = \lim_{t\to\infty} \beta^t \mathsf{p}_t = 0 \tag{A.11}$$

and the initial condition

$$\mathsf{p}_{20} = 0 \tag{A.12}$$

where $\mathsf{p}_t' = [\mathsf{p}_{1t}'\, \mathsf{p}_{2t}']$ is partitioned so that $\mathsf{p}_{1t}$ is of dimension $(n-m) \times 1$. Equation (A.3), (A.8), (A.10) together with the $2n$ boundary conditions constitute the system under optimal control.

Solving the system under control leads to the following rule

$$\mathsf{w}_t = -F \begin{bmatrix} I & 0 \\ -N_{21} & -N_{22} \end{bmatrix} \begin{bmatrix} \mathsf{z}_t \\ \mathsf{p}_{2t} \end{bmatrix} \equiv D \begin{bmatrix} \mathsf{z}_t \\ \mathsf{p}_{2t} \end{bmatrix} = -F \begin{bmatrix} \mathsf{z}_t \\ \mathsf{x}_{2t} \end{bmatrix} \tag{A.13}$$

where

$$\begin{bmatrix} \mathsf{z}_{t+1} \\ \mathsf{p}_{2t+1} \end{bmatrix} = \begin{bmatrix} I & 0 \\ S_{21} & S_{22} \end{bmatrix} G \begin{bmatrix} I & 0 \\ -N_{21} & -N_{22} \end{bmatrix} \begin{bmatrix} \mathsf{z}_t \\ \mathsf{p}_{2t} \end{bmatrix} \equiv H \begin{bmatrix} \mathsf{z}_t \\ \mathsf{p}_{2t} \end{bmatrix} \tag{A.14}$$

$$N = \begin{bmatrix} S_{11} - S_{12}S_{22}^{-1}S_{21} & S_{12}S_{22}^{-1} \\ -S_{22}^{-1}S_{21} & S_{22}^{-1} \end{bmatrix} = \begin{bmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{bmatrix} \tag{A.15}$$

$$\mathsf{x}_t = -\begin{bmatrix} N_{21} & N_{22} \end{bmatrix} \begin{bmatrix} \mathsf{z}_t \\ \mathsf{p}_{2t} \end{bmatrix} \tag{A.16}$$

where $F = -(R + B'SB)^{-1}(B'SA + U'), G = A - BF$ and

$$S = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} \tag{A.17}$$

partitioned so that $S_{11}$ is $(n-m) \times (n-m)$ and $S_{22}$ is $m \times m$ is the solution to the steady-state Ricatti equation

$$S = Q - UF - F'U' + F'RF + \beta(A - BF)'S(A - BF) \tag{A.18}$$

The cost-to-go for the optimal policy (OP) at time $t$ is

$$\Omega_t^{OP} = -\frac{1}{2}(\mathrm{tr}(N_{11}Z_t) + \mathrm{tr}(N_{22}\mathsf{p}_{2t}\mathsf{p}_{2t}')) \tag{A.19}$$

where $Z_t = \mathsf{z}_t\mathsf{z}_t'$. To achieve optimality the policy-maker sets $\mathsf{p}_{20} = 0$ at time $t = 0$. At time $t > 0$ there exists a gain from reneging by resetting $\mathsf{p}_{2t} = 0$. It can be shown that $N_{11} < 0$ and $N_{22} < 0$.[11], so the incentive to renege exists at all points along the trajectory of the optimal policy. This is the time-inconsistency problem.

### A.1.1 Implementation

The rule may also be expressed in two other forms: First as

$$\mathsf{w}_t = D_1\mathsf{z}_t + D_2H_{21}\sum_{\tau=1}^{t}(H_{22})^{\tau-1}\mathsf{z}_{t-\tau} \tag{A.20}$$

where $D = [D_1\ D_2]$ is partitioned conformably with $\mathsf{z}_t$ and $\mathsf{p}_{2t}$. The rule then consists of a feedback on the lagged predetermined variables with geometrically declining weights with lags extending back to time $t = 0$, the time of the formulation and announcement of the policy.

The final way of expressing the rule is express the process for $\mathsf{w}_t$ in terms of the target variables only, $\mathsf{s}_t$, in the loss function. This in particular eliminates feedback from the exogenous processes in the vector $\mathsf{z}_t$. Since the rule does not require knowledge of these

[11] See Currie and Levine (1993), chapter 5.

processes to design, Woodford (2003) refers to this as "robust" in describing it as the *Robust Optimal Explicit* rule.

### A.1.2 Optimal Policy from a Timeless Perspective

Noting from (A.16) that long the optimal policy we have $\mathsf{x}_t = -N_{21}\mathsf{z}_t - N_{22}\mathsf{p}_{2t}$, the optimal policy "from a timeless perspective" proposed by Woodford (2003) replaces the initial condition for optimality $p_{20} = 0$ with

$$J\mathsf{x}_0 = -N_{21}\mathsf{z}_0 - N_{22}\mathsf{p}_{20} \tag{A.21}$$

where $J$ is some $1 \times m$ matrix. Typically in New Keynesian models the particular choice of condition is $\pi_0 = 0$ thus avoiding any once-and-for-all initial surprise inflation. This initial condition applies only at $t = 0$ and only affects the deterministic component of policy and not the stochastic, stabilization component.

## A.2 The Dynamic Programming Discretionary Policy

The evaluate the discretionary (time-consistent) policy we rewrite the cost-to-go $\Omega_t$ given by (A.5) as

$$\Omega_t = \frac{1}{2}[\mathsf{y}_t'Q\mathsf{y}_t + 2\mathsf{y}_t'U\mathsf{w}_t + \mathsf{w}_t'R\mathsf{w}_t + \beta\Omega_{t+1}] \tag{A.22}$$

The dynamic programming solution then seeks a stationary solution of the form $w_t = -Fz_t$ in which $\Omega_t$ is minimized at time $t$ subject to (1) in the knowledge that a similar procedure will be used to minimize $\Omega_{t+1}$ at time $t+1$.

Suppose that the policy-maker at time $t$ expects a private-sector response from $t+1$ onwards, determined by subsequent re-optimization, of the form

$$\mathsf{x}_{t+\tau} = -N_{t+1}\mathsf{z}_{t+\tau}, \ \tau \geq 1 \tag{A.23}$$

The loss at time $t$ for the *ex ante* optimal policy was from (A.19) found to be a quadratic function of $\mathsf{x}_t$ and $\mathsf{p}_{2t}$. We have seen that the inclusion of $\mathsf{p}_{2t}$ was the source of the time inconsistency in that case. We therefore seek a lower-order controller

$$\mathsf{w}_t = -F\mathsf{z}_t \tag{A.24}$$

with the cost-to-go quadratic in $\mathsf{z}_t$ only. We then write $\Omega_{t+1} = \frac{1}{2}\mathsf{z}_{t+1}' S_{t+1} \mathsf{z}_{t+1}$ in (A.22). This leads to the following iterative process for $F_t$

$$\mathsf{w}_t = -F_t \mathsf{z}_t \tag{A.25}$$

where

$$\begin{aligned}
F_t &= (\overline{R}_t + \lambda \overline{B}_t' S_{t+1} \overline{B}_t)^{-1} (\overline{U}_t' + \beta \overline{B}_t' S_{t+1} \overline{A}_t) \\
\overline{R}_t &= R + K_t' Q_{22} K_t + U^{2T} K_t + K_t' U^2 \\
K_t &= -(A_{22} + N_{t+1} A_{12})^{-1} (N_{t+1} B^1 + B^2) \\
\overline{B}_t &= B^1 + A_{12} K_t \\
\overline{U}_t &= U^1 + Q_{12} K_t + J_t' U^2 + J_t' Q_{22} J_t \\
\overline{J}_t &= -(A_{22} + N_{t+1} A_{12})^{-1} (N_{t+1} A_{11} + A_{12})
\end{aligned}$$

$$\begin{aligned}
\overline{A}_t &= A_{11} + A_{12} J_t \\
S_t &= \overline{Q}_t - \overline{U}_t F_t - F_t' \overline{U}' + \overline{F}_t' \overline{R}_t F_t + \beta (\overline{A}_t - \overline{B}_t F_t)' S_{t+1} (\overline{A}_t - \overline{B}_t \overline{F}_t) \\
\overline{Q}_t &= Q_{11} + J_t' Q_{21} + Q_{12} J_t + J_t' Q_{22} J_t \\
N_t &= -J_t + K_t F_t
\end{aligned}$$

where $B = \begin{bmatrix} B^1 \\ B^2 \end{bmatrix}$, $U = \begin{bmatrix} U^1 \\ U^2 \end{bmatrix}$, $A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$, and $Q$ similarly are partitioned conformably with the predetermined and non-predetermined components of the state vector.

The sequence above describes an iterative process for $F_t$, $N_t$, and $S_t$ starting with some initial values for $N_t$ and $S_t$. If the process converges to stationary values, $F, N$ and $S$ say, then the time-consistent feedback rule is $w_t = -F z_t$ with loss at time $t$ given by

$$\Omega_t^{TC} = \frac{1}{2}\mathsf{z}_t' S \mathsf{z}_t = \frac{1}{2}\mathrm{tr}(S Z_t) \tag{A.26}$$

## A.3 Optimized Simple Rules

We now consider simple sub-optimal rules of the form

$$\mathsf{w}_t = D \mathsf{y}_t = D \begin{bmatrix} \mathsf{z}_t \\ \mathsf{x}_t \end{bmatrix} \tag{A.27}$$

where $D$ is constrained to be sparse in some specified way. Rule (A.27) can be quite general. By augmenting the state vector in an appropriate way it can represent a PID (proportional-integral-derivative)controller.

Substituting (A.27) into (A.5) gives

$$\Omega_t = \frac{1}{2}\sum_{i=0}^{\infty}\beta^t \mathsf{y}_{t+i}' P_{t+i} \mathsf{y}_{t+i} \tag{A.28}$$

where $P = Q + UD + D'U' + D'RD$. The system under control (A.1), with $\mathsf{w}_t$ given by (A.27), has a rational expectations solution with $\mathsf{x}_t = -N\mathsf{z}_t$ where $N = N(D)$. Hence

$$\mathsf{y}_t' P \mathsf{y}_t = \mathsf{z}_t' T \mathsf{z}_t \tag{A.29}$$

where $T = P_{11} - N'P_{21} - P_{12}N + N'P_{22}N$, $P$ is partitioned as for $S$ in (A.17) onwards and

$$\mathsf{z}_{t+1} = (G_{11} - G_{12}N)\mathsf{z}_t \tag{A.30}$$

where $G = A + BD$ is partitioned as for $P$. Solving (A.30) we have

$$\mathsf{z}_t = (G_{11} - G_{12}N)^t \mathsf{z}_0 \tag{A.31}$$

Hence from (A.32), (A.29) and (A.31) we may write at time $t$

$$\Omega_t^{SIM} = \frac{1}{2} z_t' V z_t = \frac{1}{2}\mathrm{tr}(VZ_t) \tag{A.32}$$

where $Z_t = z_t z_t'$ and $V$ satisfies the *Lyapunov* equation

$$V = T + H'VH \tag{A.33}$$

where $H = G_{11} - G_{12}N$. At time $t = 0$ the optimized simple rule is then found by minimizing $\Omega_0$ given by (A.32) with respect to the non-zero elements of $D$ given $z_0$ using a standard numerical technique. An important feature of the result is that unlike the previous solution the optimal value of $D$, $D^*$ say, is not independent of $\mathsf{z}_0$. That is to say

$$D^* = D^*(z_0)$$

## A.4 The Stochastic Case

Consider the stochastic generalization of (A.1)

$$\begin{bmatrix} \mathsf{z}_{t+1} \\ \mathsf{x}_{t+1,t}^e \end{bmatrix} = A \begin{bmatrix} \mathsf{z}_t \\ \mathsf{x}_t \end{bmatrix} + B\mathsf{w}_t + \begin{bmatrix} \mathsf{u}_t \\ 0 \end{bmatrix} \tag{A.34}$$

where $\mathsf{u}_t$ is an $n \times 1$ vector of white noise disturbances independently distributed with $\mathrm{cov}(\mathsf{u}_t) = \Sigma$. Then, it can be shown that certainty equivalence applies to all the policy rules apart from the simple rules (see Currie and Levine (1993)). The expected loss at time $t$ is as before with quadratic terms of the form $\mathsf{z}_t' X \mathsf{z}_t = \mathrm{tr}(X \mathsf{z}_t, Z_t')$ replaced with

$$E_t \left( \mathrm{tr} \left[ X \left( z_t z_t' + \sum_{i=1}^{\infty} \beta^t \mathsf{u}_{t+i} \mathsf{u}_{t+i}' \right) \right] \right) = \mathrm{tr} \left[ X \left( z_t' z_t + \frac{\lambda}{1-\lambda} \Sigma \right) \right] \tag{A.35}$$

where $E_t$ is the expectations operator with expectations formed at time $t$.

Thus for the optimal policy with commitment (A.19) becomes in the stochastic case

$$\Omega_t^{OP} = -\frac{1}{2} \mathrm{tr} \left( N_{11} \left( Z_t + \frac{\beta}{1-\beta} \Sigma \right) + N_{22} \mathsf{p}_{2t} \mathsf{p}_{2t}' \right) \tag{A.36}$$

For the time-consistent policy (A.26) becomes

$$\Omega_t^{TC} = -\frac{1}{2} \mathrm{tr} \left( S \left( Z_t + \frac{\beta}{1-\beta} \Sigma \right) \right) \tag{A.37}$$

and for the simple rule, generalizing (A.32)

$$\Omega_t^{SIM} = -\frac{1}{2} \mathrm{tr} \left( V \left( Z_t + \frac{\beta}{1-\beta} \Sigma \right) \right) \tag{A.38}$$

The optimized simple rule is found at time $t = 0$ by minimizing $\Omega_0^{SIM}$ given by (A.38). Now we find that

$$D^* = D^* \left( \mathsf{z}_0 \mathsf{z}_0' + \frac{\beta}{1-\beta} \Sigma \right) \tag{A.39}$$

or, in other words, the optimized rule depends both on the initial displacement $\mathsf{z}_0$ and on the covariance matrix of disturbances $\Sigma$.

# B State-Space Set-up of Model for Robust Control

**Exogenous processes**:

$$a_{t+1} = \rho_a a_t + \epsilon_{a,t+1} \tag{B.1}$$

$$e_{t+1} = \rho_e e_t + \epsilon_{e,t+1} \tag{B.2}$$

**Lags for Interest Rate Rule**:

$$i_t = i_t \tag{B.3}$$

$$i_{t-1} = i_{t-1} \tag{B.4}$$

$$o_t = o_t \tag{B.5}$$

**Worst-Case Specification Errors**

$$\Theta u_{t+1} = \frac{1}{\beta\Theta} v_t + \Theta u_t + w_\pi \pi_t \tag{B.6}$$

$$-\lambda(\sigma+\phi)\Theta u_{t+1} + \Theta v_{t+1} = -(\sigma+\phi)o_t + \frac{\Theta}{\beta} v_t \tag{B.7}$$

**Non-predetermined variables**:

$$E_t y_{t+1} = y_t + \frac{1}{\sigma}(i_t - E_t \pi_{t+1}) - u_{t+1} \tag{B.8}$$

$$E_t y^a_{t+1} = y^a_t + \frac{1}{\sigma}(i_t - E_t \pi^a_{t+1}) \tag{B.9}$$

$$\beta E_t \pi_{t+1} = \pi_t - \lambda mc_t - e_t - v_{t+1} \tag{B.10}$$

$$\beta E_t \pi^a_{t+1} = \pi^a_t - \lambda mc^a_t - e_t \tag{B.11}$$

**Instrument**: The nominal interest rate $i_t$.

For the HS robust rule we have a 'simple rule':

$$i_t = \left[\frac{(\beta+1)\sigma + \lambda(\sigma+\phi)}{\sigma\beta}\right] i_{t-1} - \frac{1}{\beta} i_{t-2} - \frac{1}{w_i \sigma}\left[(\sigma+\phi)(o_t - o_{t-1}) - w_\pi \lambda(\sigma+\phi)\pi_t\right] \tag{B.12}$$

**Outputs**:

$$mc_t = \sigma y_t + \phi l_t - a_t \tag{B.13}$$

$$mc_t^a = \sigma y_t^a + \phi l_t^a - a_t \tag{B.14}$$

$$l_t = y_t - a_t \tag{B.15}$$

$$l_t^a = y_t^a - a_t \tag{B.16}$$

$$\hat{mc}_t = \sigma \hat{y}_t + \phi \hat{l}_t - a_t = 0 \tag{B.17}$$

$$\hat{l}_t = \hat{y}_t - a_t \tag{B.18}$$

$$o_t = \hat{y}_t - y_t \tag{B.19}$$

$$o_t^a = \hat{y}_t - y_t^a \tag{B.20}$$

$$r_t = i_t - E_t \pi_{t+1} \tag{B.21}$$

$$r_t^a = i_t - E_t \pi_{t+1}^a \tag{B.22}$$

**Derived Parameters**:

$$\lambda = \frac{(1-\beta\xi)(1-\xi)}{\xi} \tag{B.23}$$

$$w_\pi = \frac{\zeta}{\lambda} \tag{B.24}$$

**Welfare Losses: (Holds for Large Distortions Case)**

$$W_t = \sigma y_t^2 + \phi l_t^2 + w_\pi \pi_t^2 - 2a_t l_t + w_i i_t^2 = (\sigma + \phi) o_t^2 + w_\pi \pi_t^2 + w_i i_t^2 \tag{B.25}$$

$$W_t^a = (\sigma + \phi)(o_t^a)^2 + w_\pi (\pi_t^a)^2 + w_i i_t^2 \tag{B.26}$$

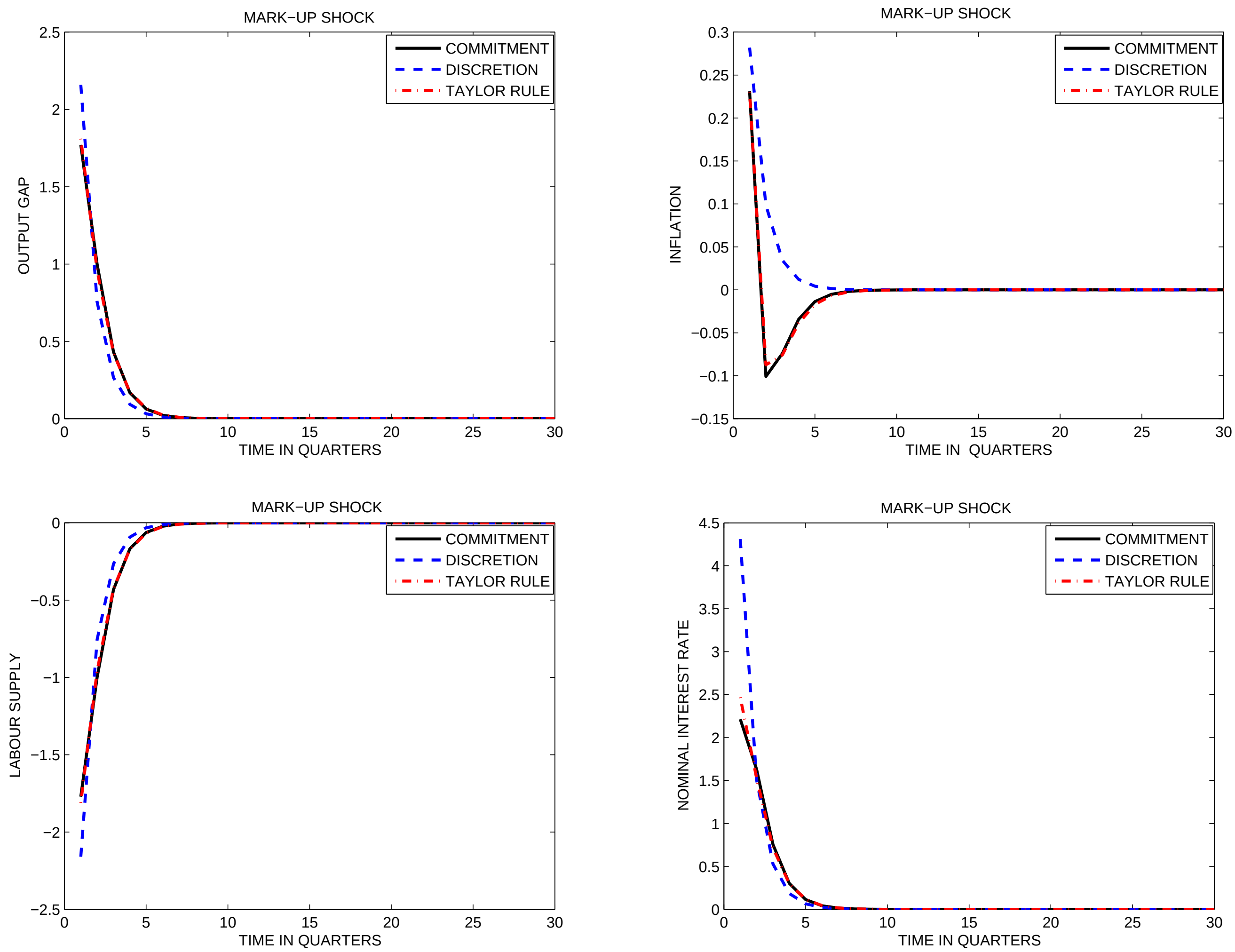


Figure 1: **Optimal Policy without Model Uncertainty**

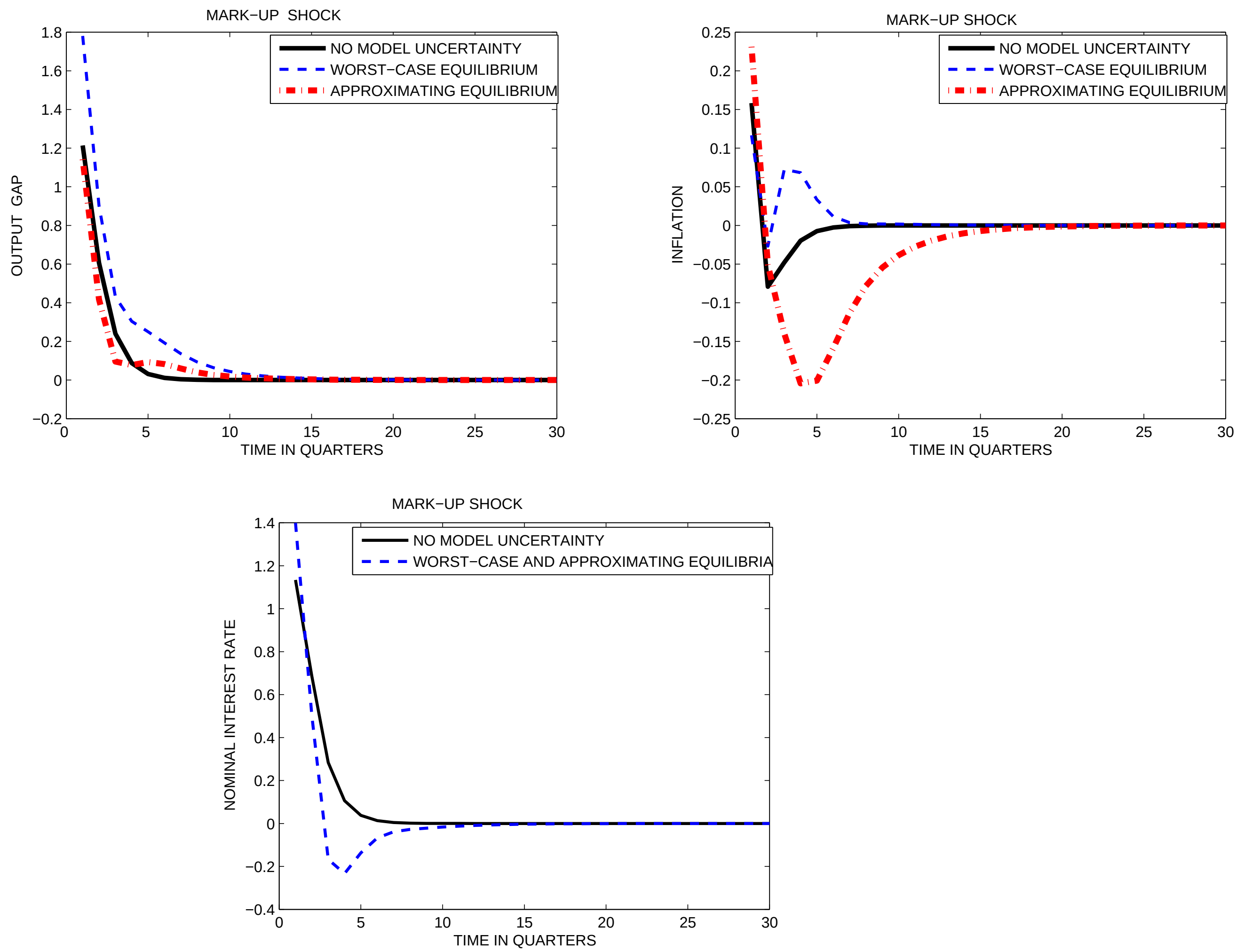


Figure 2: **Optimal Policy with Model Uncertainty**

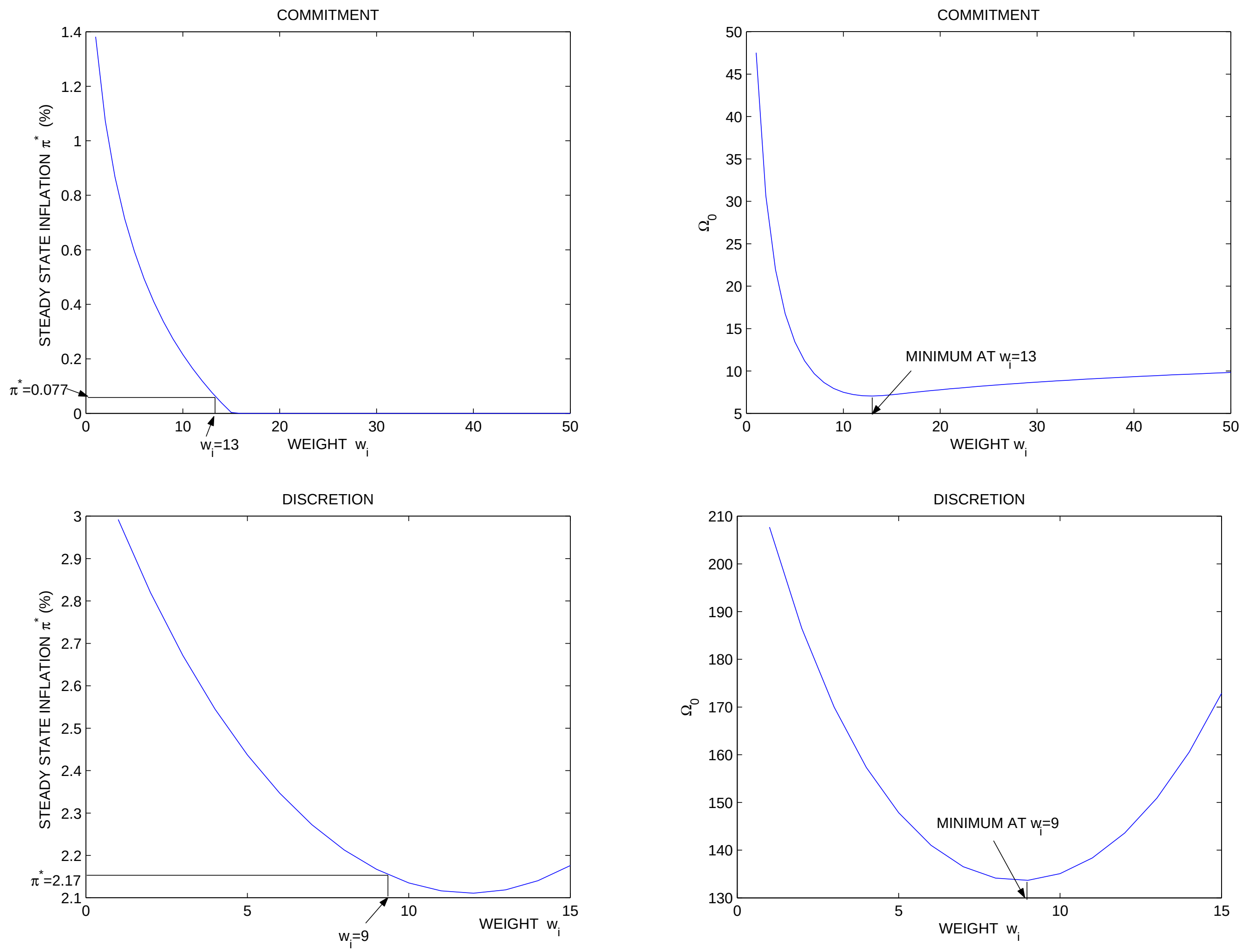


Figure 3: **Optimal Policy without Model Uncertainty with ZLB Constraint**

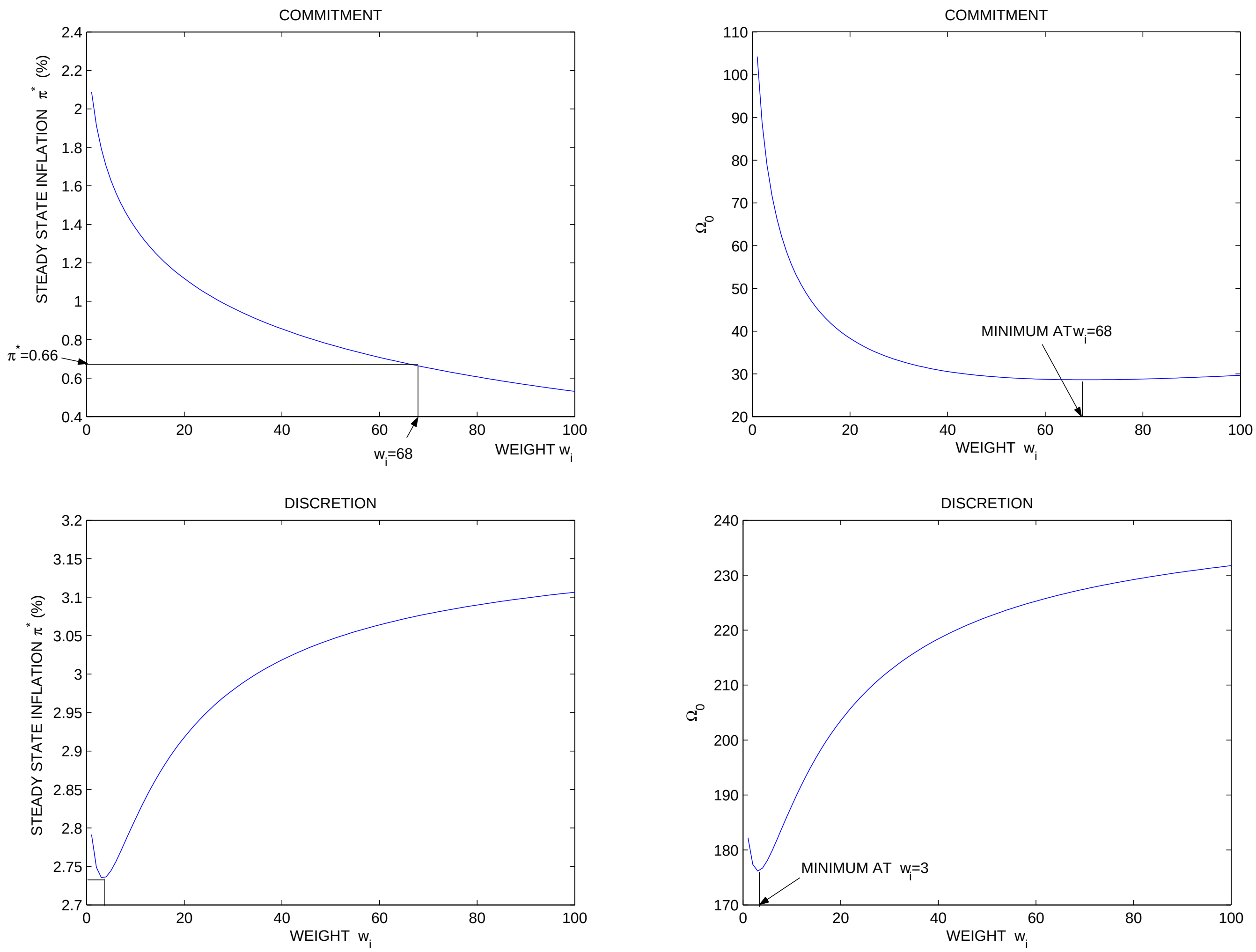


Figure 4: **Optimal Policy with Model Uncertainty and with ZLB Constraint**