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NONPARAMETRIC LIKELIHOOD FOR VOLATILITY UNDER HIGH FREQUENCY DATA

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Abstract. We propose a nonparametric likelihood inference method for the integrated volatility under high frequency financial data. The nonparametric likelihood statistic, which contains the conventional statistics such as empirical likelihood and Pearson's χ^2 as special cases, is not asymptotically pivotal under the so-called infill asymptotics, where the number of high frequency observations in a fixed time interval increases to infinity. We show that multiplying a correction term recovers the χ^2 limiting distribution. Furthermore, we establish Bartlett correction for our modified nonparametric likelihood statistic under the constant and general non-constant volatility cases. In contrast to the existing literature, the empirical likelihood statistic is not Bartlett correctable under the infill asymptotics. However, by choosing adequate tuning constants for the power divergence family, we show that the second order refinement to the order $O(n^{-2})$ can be achieved.

1. Introduction

Realized volatility and its related statistics have become standard tools to explore the behavior of high frequency financial data and to evaluate financial theoretical models including stochastic volatility models. This increase in popularity has been propelled by recent developments of probability and statistical theory and by the increasing availability of high frequency financial data (see, e.g., Andersen, Bollerslev and Diebold, 2010, for a review). By employing the asymptotic framework so-called the infill asymptotics, where the number of high frequency observations in a fixed time interval (say, a day) increases to infinity, Jacod and Protter (1998) and Barndorff-Nielsen and Shephard (2002) established laws of large numbers and central limit theorems for realized volatility, which were extended to more general setups and statistics by Barndorff-Nielsen *et al.* (2006). Also, Gonçalves and Meddahi (2009) studied higher order properties of the realized volatility statistic and its bootstrap counterpart.

In this paper, we propose a nonparametric likelihood inference method for the integrated volatility under high frequency financial data. The nonparametric likelihood statistic, which contains the conventional statistics such as Owen's (1988) empirical likelihood and Pearson's χ^2 as special cases, is not asymptotically pivotal under the infill asymptotics. We show that multiplying a correction term recovers the χ^2 limiting distribution. Furthermore, we establish Bartlett correction for our modified nonparametric likelihood statistic under the constant and general non-constant volatility cases. In contrast to the existing literature, the empirical likelihood statistic is not Bartlett correctable under the infill asymptotics. However, by choosing adequate tuning constants for the power divergence family, we show that the second order refinement to the order $O(n^{-2})$ can be achieved.

Our theoretical results also contribute to the literature of empirical likelihood (see, Owen, 2001, for a review). Since DiCiccio, Hall and Romano (1991), many papers reported Bartlett correctability of empirical likelihood in various contexts. Baggerly (1998) showed that in the power divergence family of nonparametric likelihood functions, only empirical likelihood is Bartlett correctable. Our results show that under the infill asymptotics, another nonparametric likelihood statistic achieves Bartlett correction instead of empirical likelihood.

The rest of the paper is organized as follows. Section 2 introduces our basic setup and nonparametric likelihood statistic and derives the first order asymptotic distribution. In Section 3, we conduct the second order analysis for the proposed statistic and establish the Bartlett corrections for the constant volatility case (Section 3.1) and general non-constant volatility case (Section 3.2). Section 4 presents some simulation results.

2. Setup and nonparametric likelihood

Let us consider a scalar continuous time process $\{X_t\}_{t\geq 0}$ (typically a log-price) defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$ that follows

$$dX_t = \mu_t dt + \sigma_t dW_t, \tag{1}$$

where $\{\mu_t\}_{t\geq 0}$ is an adapted predictable locally bounded drift process, $\{\sigma_t\}_{t\geq 0}$ is an adapted cadlag volatility process, and $\{W_t\}_{t\geq 0}$ is a standard Brownian motion. We wish to conduct statistical inference on the integrated volatility $\theta = \int_0^1 \sigma_u^2 du$ over a fixed interval $[0,1]$ (say, a day) based on the high frequency returns $r_i = X_{i/n} - X_{(i-1)/n}$ measured over the period $[(i-1)/n, i/n]$ for $i = 1, \ldots, n$.

As a nonparametric measure of volatility, the integrated volatility θ has been drawing considerable attention from researchers in finance who face to high frequency financial data. One popular estimator of θ is so-called the realized volatility $\hat{\theta} = \sum_{i=1}^n r_i^2$. It is known that under general conditions on the volatility process, $\hat{\theta}$ is consistent for θ and asymptotically normal under the limit $n \to \infty$ for increasingly finely sampled returns over the fixed interval $[0,1]$ (called the infill asymptotics) (e.g., Jacod and Protter, 1998, and Barndorff-Nielsen and Shephard, 2002). As one of the most general setups, we consider the following one employed by Barndorff-Nielsen *et al.* (2006).

Assumption X. *The process $\{X_t\}_{t\geq 0}$ follows (1) and satisfies*

$$\begin{aligned}\sigma_t &= \sigma_0 + \int_0^t a_u^* du + \int_0^t \sigma_{u-}^* dW_u + \int_0^t v_{u-}^* dV_u \\ &\quad + \int_0^t \int_E \phi \circ w(u-, x)(\mu - \nu)(du, dx) + \int_0^t \int_E (w - \phi \circ w)(u-, x)\mu(du, dx),\end{aligned}$$

where a^* is an adapted predictable locally bounded process, σ^* and v^* are adapted cadlag processes, V is a Brownian motion independent of W, μ is a Poisson measure on $(0,\infty) \times E$ independent of W and V with intensity measure $\nu(dt, dx) = dt \otimes F(dx)$, F is a σ-finite measure on the Polish space $(E, \mathcal{E})$, ϕ is an indicator function for a neighborhood of 0, and $w(\omega, u, x)$ is a mapping from $\Omega \times [0,\infty) \times E$ to the space of processes that is $\mathcal{F}_u \otimes \mathcal{E}$-measurable in (ω, x) for all u and *cadlag in u and for some sequence $\{S_k\}$ with increasing stopping time to $+\infty$ and $\{\psi_k\}$ satisfying $\int_E (1 \wedge \psi_k(x)^2) F(dx) < \infty$, it holds $\sup_{\omega\in\Omega, u<S_k(\omega)} |w(\omega,u,x)| \le \psi_k(x)$.*

This assumption is general enough to allow for jumps, intraday seasonality, and correlation between σ_t and W_t (called the leverage effect). Under Assumption X, Barndorff-Nielsen *et al.* (2006) showed the consistency $\hat{\theta} \overset{p}{\to} \theta$ and asymptotic normality

$$\frac{\sqrt{n}(\hat{\theta}-\theta)}{\sqrt{\hat{V}}} \overset{d}{\to} N(0,1), \tag{2}$$

where $\hat{V} = \frac{2n}{3}\sum_{i=1}^n r_i^4$. Based on this result, it is customary to construct a Wald-type confidence interval for θ. Also, Gonçalves and Meddahi (2009) proposed bootstrap inference methods on θ. In this paper, we develop a nonparametric likelihood inference method for θ.

As a general class of nonparametric likelihood functions for the integrated volatility θ, we introduce the power divergence family (Cressie and Read, 1984)

$$L_\gamma(p_1\dots,p_n) = \begin{cases} \frac{2}{\gamma(\gamma+1)}\sum_{i=1}^n\{(np_i)^{\gamma+1}-1\} & \text{if } \gamma \neq -1,0, \\ -2\sum_{i=1}^n \log(np_i) & \text{if } \gamma=-1, \\ 2n\sum_{i=1}^n p_i\log(np_i) & \text{if } \gamma=0. \end{cases}$$

Based on $L_\gamma(p_1\dots,p_n)$, we specify the likelihood function as

$$\ell_{\gamma,\phi}(\theta) = L_\gamma(p_{\phi,1}\dots,p_{\phi,n}), \tag{3}$$

where the weights $p_{\phi,1},\dots,p_{\phi,n}$ solve

$$\min_{p_1,\dots,p_n} L_\phi(p_1\dots,p_n), \quad \text{subject to } \sum_{i=1}^n p_i = 1, \ \sum_{i=1}^n p_i(nr_i^2-\theta)=0. \tag{4}$$

Note that the nonparametric likelihood function $\ell_{\gamma,\phi}(\theta)$ contains two tuning constants, γ and ϕ. In the literature, it is commonly assumed $\gamma=\phi$. For example, Owen's (1988) empirical likelihood corresponds to $\gamma=\phi=-1$ and Pearson's χ^2 corresponds to $\gamma=\phi=-2$. Also Baggerly (1998) showed that in the class of likelihood functions with $\gamma=\phi$, only empirical likelihood is Bartlett correctable. On the other hand, Schennach (2005, 2007) considered the case of $\gamma\neq\phi$ and studied the exponentially tilted empirical likelihood statistic with $\gamma=-1$ and $\phi=0$ from Bayesian and frequentist perspectives. In the current setup where we employ the infill asymptotics, it is crucial to consider the general class of $\ell_{\gamma,\phi}(\theta)$ indexed by γ and ϕ to achieve Bartlett correction. For example, even if the volatility process σ_t is constant over $t\in[0,1]$, the empirical likelihood statistic (i.e., $\ell_{\gamma,\phi}(\theta)$ with $\gamma=\phi=-1$) is not Bartlett correctable under the infill asymptotics, and the constants γ and ϕ need to be chosen separately to achieve Bartlett correction.

By the Lagrange multiplier argument, the solution of (4) is written as (see, Baggerly, 1998)

$$p_{\phi,i} = \frac{1}{n}(1+\eta+\lambda(nr_i^2-\theta))^{\frac{1}{\phi}}, \tag{5}$$

for $\phi\neq 0$ and $p_{\phi,i}=\frac{1}{n}\eta\exp(\lambda(nr_i^2-\theta))$ for $\phi=0$, where η and λ solve

$$\frac{1}{n}\sum_{i=1}^n(1+\eta+\lambda(nr_i^2-\theta))^{\frac{1}{\phi}} = 1, \qquad \frac{1}{n}\sum_{i=1}^n(1+\eta+\lambda(nr_i^2-\theta))^{\frac{1}{\phi}}(nr_i^2-\theta)=0, \tag{6}$$

for $\phi \neq 0$ and solve $\frac{1}{n}\sum_{i=1}^{n} \eta \exp(\lambda(nr_i^2 - \theta)) = 1$ and $\frac{1}{n}\sum_{i=1}^{n} \eta \exp(\lambda(nr_i^2 - \theta))(nr_i^2 - \theta) = 0$ for $\phi = 0$. In practice, we employ the expression in (5) to compute the likelihood function in (3).

The first order asymptotic distribution of $\ell_{\gamma,\phi}(\theta)$ is obtained as follows. Let $R_q = n^{q/2-1}\sum_{i=1}^{n} |r_i|^q$.

Theorem 1. *Under Assumption X, it holds that for each $\gamma, \phi \in \mathbb{R}$,*

$$T_{\gamma,\phi}(\theta) = \frac{3}{2}\left(1 - \frac{R_2^2}{R_4}\right)\ell_{\gamma,\phi}(\theta) \xrightarrow{d} \chi_1^2,$$

as $n \to \infty$.

See Appendix A.1 for the proof. It should be noted that under the infill asymptotics, the nonparametric likelihood statistic $\ell_{\gamma,\phi}(\theta)$ (including empirical likelihood) does not converge to the χ^2 distribution. In other words, the nonparametric likelihood statistic is not internally studentized. This is due to the fact that the asymptotic variance of the term $\frac{1}{\sqrt{n}}\sum_{i=1}^{n}(nr_i^2 - \theta)$ does not match to the limit of $\frac{1}{n}\sum_{i=1}^{n}(nr_i^2 - \theta)^2$ under the infill asymptotics. The correction term $\frac{3}{2}\left(1 - \frac{R_2^2}{R_4}\right)$ is required to recover the studentization. On the other hand, the first order asymptotic distribution of $T_{\gamma,\phi}(\theta)$ does not depend on the tuning constants γ and ϕ. In the next section, we study the second order asymptotic properties of the statistic $T_{\gamma,\phi}(\theta)$ to compare difference choices of γ and ϕ.

3. Second order asymptotics

To investigate the second order asymptotic properties of the nonparametric likelihood statistic, we follow the conventional recipe put forward in DiCiccio, Hall and Romano (1991) and Baggerly (1998), among others. In particular, we first derive the signed root of the nonparametric likelihood statistic, and then evaluate the cumulants of the signed root. Based on these cumulants, we seek values of γ and ϕ at which the third and fourth cumulants vanish at sufficiently fast rates to admit Bartlett correction.

For the second order analysis, we add the following assumption.

Assumption H. *The process $\{X_t\}_{t\geq 0}$ follows (1) with $\mu_t = 0$ and σ_t is independent of W_t and bounded away from zero.*

This assumption is restrictive since it rules out the drift term and leverage effect. Gonçalves and Meddahi (2009, p. 289) imposed a similar but stronger assumption for higher order analysis of the bootstrap inference. Although the drift term μ_t is asymptotically negligible at the first order, it will appear in the higher order terms and complicates our second order analysis. Ruling out the leverage effect (i.e., independence between σ_t and W_t) also simplifies our second order analysis since it allows to condition on the path of σ_t to compute the cumulants of the nonparametric likelihood statistic. Relaxing Assumption H for the second order analysis is beyond the scope of this paper.

Due to independence between σ_t and W_t, throughout this section the symbols such as $E[\cdot]$ and $O_p(\cdot)$ mean the conditional expectation and stochastic order given the path of σ_t, respectively.

Before analyzing Bartlett correctability of the nonparametric likelihood statistic, we introduce further notation. We transform the moment function as $w_i = V^{-1/2}(nr_i^2 - \theta)$ with

$V = E[n^{-1}\sum_{i=1}^n (nr_i^2 - \theta)^2]$ and define

$$\bar{A}_k = \frac{1}{n}\sum_{i=1}^n w_i^k, \qquad \alpha_k = E[\bar{A}_k], \qquad A_k = \bar{A}_k - \alpha_k,$$

for $k = 1, 2, \ldots$. Note that Assumption H implies

$$\alpha_1 = 0, \qquad \alpha_2 = 1, \qquad A_k = O_p(n^{-1/2}),$$

for each $k = 1, 2, \ldots$, where the first equality follows from $E[nr_i^2] = \int_{(i-1)/n}^{i/n} \sigma_u^2 du$, the second equality follows by construction, and the third equality follows from Barndorff-Nielsen *et al.* (2006, Theorem 2).

Based on the above notation, the nonparametric likelihood statistic is rewritten as $\ell_{\gamma,\phi}(\theta) = L_\gamma(p_{\phi,1} \ldots, p_{\phi,n})$, where

$$p_{\phi,i} = \frac{1}{n}(1 + \eta + \tilde{\lambda} w_i)^{\frac{1}{\phi}},$$

and η and $\tilde{\lambda}$ solves

$$\frac{1}{n}\sum_{i=1}^n (1 + \eta + \tilde{\lambda} w_i)^{\frac{1}{\phi}} = 1, \qquad \frac{1}{n}\sum_{i=1}^n (1 + \eta + \tilde{\lambda} w_i)^{\frac{1}{\phi}} w_i = 0.$$

Expansions of these equations around $\eta + \tilde{\lambda} w_i = 0$ and repeated substitutions yield expansions of η and $\tilde{\lambda}$ as follows

$$\begin{aligned}
\eta &= \frac{1}{2}\phi(1+\phi)A_1^2 + \frac{1}{6}\phi(1+\phi)(1-\phi)\alpha_3 A_1^3 - \frac{1}{2}\phi(1+\phi)A_1^2 A_2 \\
&\quad + \frac{1}{2}\phi(1+\phi)A_1^2 A_2^2 - \frac{1}{2}\phi(1-\phi)(1+\phi)\alpha_3 A_1^3 A_2 + \frac{1}{6}\phi(1-\phi)(1+\phi)A_1^3 A_3 \\
&\quad + \frac{1}{8}\phi\left\{(1+\phi)^3 + (1-\phi)^2(1+\phi)\alpha_3^2 - \frac{1}{3}(1-\phi)(1+\phi)(1-2\phi)\alpha_4\right\} A_1^4 + O_p(n^{-5/2}).
\end{aligned}$$

and

$$\begin{aligned}
\tilde{\lambda} &= -\phi A_1 - \frac{1}{2}\phi(1-\phi)\alpha_3 A_1^2 + \phi A_1 A_2 \\
&\quad -\phi A_1 A_2^2 + \frac{3}{2}\phi(1-\phi)\alpha_3 A_1^2 A_2 - \frac{1}{2}\phi(1-\phi)A_1^2 A_3 \\
&\quad -\frac{1}{2}\phi\left\{\phi(1+\phi) + (1-\phi)^2\alpha_3^2 - \frac{1}{3}(1-\phi)(1-2\phi)\alpha_4\right\} A_1^3 + O_p(n^{-2}).
\end{aligned}$$

By inserting these formulae to an expansion of $n^{-1}\ell_{\gamma,\phi}(\theta)$ around $\eta + \tilde{\lambda} w_i = 0$, we obtain

$$\begin{aligned}
& n^{-1}\ell_{\gamma,\phi}(\theta) \\
={}& A_1^2 + \frac{1}{3}(1-\gamma)\alpha_3 A_1^3 - A_1^2 A_2 + A_1^2 A_2^2 - (1-\gamma)\alpha_3 A_1^3 A_2 + \frac{1}{3}(1-\gamma)A_1^3 A_3 \\
& + \left\{\left(\frac{1}{4} + \frac{\gamma}{2} + \frac{\gamma\phi}{2} - \frac{\phi^2}{4}\right) + \left(\frac{1}{4} - \frac{\gamma}{2} + \frac{\gamma\phi}{2} - \frac{\phi^2}{4}\right)\alpha_3^2 + \left(-\frac{1}{12} + \frac{\gamma}{4} + \frac{\gamma^2}{12} - \frac{\gamma\phi}{2} + \frac{\phi^2}{4}\right)\alpha_4\right\} A_1^4 \\
& + O_p(n^{-5/2}). \qquad (7)
\end{aligned}$$

Let $\bar{\sigma}_{q,n} = n^{q/2-1}\sum_{i=1}^{n}\left(\int_{(i-1)/n}^{i/n}\sigma_u^2 du\right)^{q/2}$. For the term $\frac{3}{2}\left(1-\frac{R_2^2}{R_4}\right)$, expansions around $R_4 = 3\bar{\sigma}_{4,n}$ and $R_2 = \theta$ yield

$$\begin{aligned}\frac{3}{2}\left(1-\frac{R_2^2}{R_4}\right) &= \frac{1}{2}\frac{3\bar{\sigma}_{4,n}-\theta^2}{\bar{\sigma}_{4,n}} + \frac{1}{2}\frac{\theta^2}{\bar{\sigma}_{4,n}}\left(\frac{R_4}{3\bar{\sigma}_{4,n}}-1\right) - \frac{1}{2}\frac{\theta^2}{\bar{\sigma}_{4,n}}\left(\frac{R_4}{3\bar{\sigma}_{4,n}}-1\right)^2 \\ &\quad -\frac{V^{1/2}\theta}{\bar{\sigma}_{4,n}}A_1 - \frac{1}{2}\frac{V}{\bar{\sigma}_{4,n}}A_1^2 + \frac{2}{3}\frac{V\theta^2}{\bar{\sigma}_{4,n}^2}A_1^2 + \frac{1}{3}\frac{V^{3/2}\theta}{\bar{\sigma}_{4,n}^2}A_1A_2 + O_p(n^{-3/2}), \qquad (8)\end{aligned}$$

where $\frac{R_4}{3\bar{\sigma}_{4,n}} - 1 = \frac{2}{3}\frac{V^{1/2}\theta}{\bar{\sigma}_{4,n}}A_1 + \frac{1}{3}\frac{V}{\bar{\sigma}_{4,n}}A_2$.

To proceed, Section 3.1 below focuses on the case of constant volatility ($\sigma_t = \sigma$ over $t \in [0,1]$). In Section 3.2 we consider the general non-constant volatility case.

3.1. **Constant volatility case .** Throughout Section 3.1, we assume $\sigma_t = \sigma$ over $t \in [0,1]$. In this case, it holds

$$\sigma^2 = \theta, \qquad \bar{\sigma}_{4,n} = \theta^2, \qquad V = 2\theta^2, \qquad \alpha_3 = 2\sqrt{2}, \qquad \alpha_4 = 15. \qquad (9)$$

Then by (7) and (8), the expansion of the nonparametric likelihood statistic $n^{-1}T_{\gamma,\phi}(\theta)$ is written as

$$\begin{aligned} n^{-1}T_{\gamma,\phi}(\theta) &= A_1^2 - \frac{2\sqrt{2}}{3}\gamma A_1^3 - \frac{2}{3}A_1^2A_2 + \frac{4}{9}A_1^2A_2^2 - \frac{2\sqrt{2}}{9}(19-8\gamma)A_1^3A_2 \\ &\quad + \frac{1}{3}(1-\gamma)A_1^3A_3 + \left(\frac{41}{36}\gamma - 3\gamma\phi + \frac{5}{4}\gamma^2 + \frac{3}{2}\phi^2\right)A_1^4 + O_p(n^{-5/2}).\end{aligned}$$

As in Baggerly (1998), to achieve Bartlett correction, we investigate the conditions of γ and ϕ where the third and fourth cumulants of the signed root of the above expansion vanish at sufficiently fast rates.

First, we consider the third cumulant. After some algebra, the signed root form is obtained as $n^{-1}T_{\gamma,\phi}(\theta) = (S_1+S_2+S_3)^2 + O_p(n^{-5/2})$, where

$$S_1 = A_1, \qquad S_2 = -\frac{1}{3}A_1A_2 - \frac{\sqrt{2}}{3}\gamma A_1^2,$$

and $S_3 = O_p(n^{-3/2})$ is not displayed since it is not used to compute the third cumulant. Based on this form, the third cumulant of $S_1+S_2+S_3$ is obtained as

$$\kappa_3(\gamma,\phi) \;=\; E[S_1^3] + 3E[S_1^2S_2] - 3E[S_1^2]E[S_2] + O(n^{-3}),$$

where by Lemma 2 in Appendix A.2,

$$\begin{aligned} E[S_1^3] &= 2\sqrt{2}n^{-2} + O(n^{-3}), \qquad E[S_1^2S_2] = -\sqrt{2}(\gamma+2)n^{-2} + O(n^{-3}), \\ E[S_1^2]E[S_2] &= -\frac{\sqrt{2}}{3}(\gamma+2)n^{-2} + O(n^{-3}).\end{aligned}$$

Therefore, if $\gamma = -1$, then the dominant term of the third cumulant vanishes and it holds $\kappa_3(-1,\phi) = O(n^{-3})$.

Next, we set $\gamma = -1$ and analyze the fourth cumulant. After some algebra, the signed root form of $n^{-1}T_{\gamma,\phi}(\theta)$ with $\gamma = -1$ is obtained as $n^{-1}T_{-1,\phi}(\theta) = (T_1+T_2+T_3)^2 + O_p(n^{-5/2})$, where

$$T_1 = A_1, \qquad T_2 = -\frac{1}{3}A_1A_2 + \frac{\sqrt{2}}{3}A_1^2,$$

$$T_3 = \frac{1}{6}A_1A_2^2 - \frac{11\sqrt{2}}{9}A_1^2A_2 + \frac{1}{3}A_1^2A_3 + \left(\frac{3}{4}\phi^2 + \frac{3}{2}\phi - \frac{1}{18}\right)A_1^3.$$

Then the fourth cumulant of $T_1 + T_2 + T_3$ is obtained as

$$\begin{aligned}\kappa_4(-1,\phi) \quad = \quad & E[T_1^4] + 4E[T_1^3T_2] + 4E[T_1^3T_3] - 3(E[T_1^2])^2 \\ & +6E[T_1^2T_2^2] - 4E[T_1^3]E[T_2] - 12E[T_1^2T_2]E[T_2] - 6E[T_1^2]E[T_2^2] \\ & +12E[T_1^2](E[T_2])^2 - 12E[T_1^2]E[T_1T_2] - 12E[T_1^2]E[T_1T_3] + O(n^{-4}),\end{aligned}$$

where by Lemma 2 in Appendix A.2,

$$\begin{aligned}& E[T_1^4] = 3n^{-2} + 12n^{-3} + O(n^{-4}), \qquad E[T_1^3T_2] = -\frac{76}{3}n^{-3} + O(n^{-4}), \\ & E[T_1^3T_3] = \left\{\frac{74}{3} + \frac{15}{4}\left(3\phi^2 + 6\phi - \frac{2}{9}\right)\right\}n^{-3} + O(n^{-4}), \qquad (E[T_1^2])^2 = n^{-2}, \\ & E[T_1^2T_2^2] = \frac{16}{3}n^{-3} + O(n^{-4}), \qquad E[T_1^3]E[T_2] = -\frac{4}{3}n^{-3} + O(n^{-4}), \\ & E[T_1^2T_2]E[T_2] = \frac{2}{3}n^{-3} + O(n^{-4}), \qquad E[T_1^2]E[T_2^2] = \frac{4}{3}n^{-3} + O(n^{-4}), \\ & E[T_1^2](E[T_2])^2 = \frac{2}{9}n^{-3} + O(n^{-4}), \qquad E[T_1^2]E[T_1T_2] = -\frac{10}{3}n^{-3} + O(n^{-4}), \\ & E[T_1^2]E[T_1T_3] = \left\{\frac{16}{3} + \frac{3}{4}\left(3\phi^2 + 6\phi - \frac{2}{9}\right)\right\}n^{-3} + O(n^{-4}).\end{aligned}$$

Therefore, if

$$9\phi^2 + 18\phi + 4 = 0,$$

i.e. $\phi = -1 \pm \frac{\sqrt{5}}{3}$, then the dominant term of the fourth cumulant vanishes and it holds $\kappa_4\left(-1, -1 \pm \frac{\sqrt{5}}{3}\right) = O(n^{-4})$.

Finally, by setting $\gamma = -1$ and $\phi = -1 \pm \frac{\sqrt{5}}{3}$, it holds

$$\begin{aligned}& E[T_1^2] = n^{-1}, \qquad E[T_1T_2] = -\frac{10}{3}n^{-2} + O(n^{-3}), \\ & E[T_2^2] = \frac{4}{3}n^{-2} + O(n^{-3}), \qquad E[T_1T_3] = \frac{25}{6}n^{-2} + O(n^{-3}),\end{aligned}$$

and thus the second cumulant used to compute the Bartlett correction factor is obtained as

$$nE[(T_1 + T_2 + T_3)^2] = 1 + 3n^{-1} + O(n^{-2}).$$

Combining these results, we obtain the following theorem. Let $\chi^2_{1,\alpha}$ be the $(1-\alpha)$-th quantile of the χ^2_1 distribution.

Theorem 2. *Suppose Assumptions X and H hold true and $\sigma_t = \sigma$ over $t \in [0,1]$. Then, for $\gamma = -1$ and $\phi = -1 \pm \frac{\sqrt{5}}{3}$, the nonparametric likelihood statistic $T_{\gamma,\phi}(\theta)$ is Bartlett correctable,* *i.e., conditionally on σ,*

$$\Pr\left\{T_{\gamma,\phi}(\theta) \leq \chi^2_{1,\alpha}(1+3n^{-1})\right\} = 1-\alpha+O(n^{-2}).$$

This theorem says that when we choose $\gamma=-1$ and $\phi=-1\pm\frac{\sqrt{5}}{3}$, the nonparametric likelihood test based on $T_{\gamma,\phi}(\theta)$ using the adjusted critical value $\chi^2_{1,\alpha}(1+3n^{-1})$ provides a refinement to the order $O(n^{-2})$ on the null rejection probability error. It should be noted that the empirical likelihood statistic (i.e., $T_{\gamma,\phi}(\theta)$ with $\gamma=\phi=-1$) is not Bartlett correctable because the fourth cumulant of the signed root does not vanish at the order of $O(n^{-4})$. Also note that in the constant volatility case, the Bartlett factor $1+3n^{-1}$ does not contain any unknown object.

3.2. **General case .** In Section 3.2, we drop the assumption of constant volatility and study the second order property of the nonparametric likelihood statistic under the general case. In the general case, the identities in (9) do not apply. Thus the objects such as V, α_3, and α_4 become unknown and need to be estimated. In this case, by (7) and (8), the expansion of the nonparametric likelihood statistic $n^{-1}T_{\gamma,\phi}(\theta)$ is written as

$$\begin{aligned} n^{-1}T_{\gamma,\phi}(\theta) \;=\;& \frac{1}{2}cA_1^2+\frac{1}{6}c^{1/2}\left\{c^{1/2}(1-\gamma)\alpha_3+2d^{3/2}-6d^{1/2}\right\}A_1^3+\frac{1}{6}c(d-3)A_1^2A_2 \\ &+\frac{1}{18}c(9-3d-cd)A_1^2A_2^2+\frac{1}{6}c(1-\gamma)A_1^3A_3 \\ &+\frac{1}{18}c^{1/2}\left\{c^{1/2}(d-9)(1-\gamma)\alpha_3+18d^{1/2}+9cd^{1/2}-6d^{3/2}-4cd^{3/2}\right\}A_1^3A_2 \\ &+\frac{1}{18}\left\{2c^{1/2}d^{1/2}(d-3)(1-\gamma)\alpha_3-9c+12cd-4cd^2+9cf\right\}A_1^4+O_p(n^{-5/2}), \end{aligned}$$

where

$$c=\frac{V}{\bar{\sigma}_{4,n}},\qquad d=\frac{\theta^2}{\bar{\sigma}_{4,n}},$$

$$f=\left(\frac{1}{4}+\frac{\gamma}{2}+\frac{\gamma\phi}{2}-\frac{\phi^2}{4}\right)+\left(\frac{1}{4}-\frac{\gamma}{2}+\frac{\gamma\phi}{2}-\frac{\phi^2}{4}\right)\alpha_3^2+\left(-\frac{1}{12}+\frac{\gamma}{4}+\frac{\gamma^2}{12}-\frac{\gamma\phi}{2}+\frac{\phi^2}{4}\right)\alpha_4.$$

First, we consider the third cumulant. After some algebra, the signed root form is obtained as $n^{-1}T_{\gamma,\phi}(\theta)=(S_1+S_2+S_3)^2+O_p(n^{-5/2})$, where

$$S_1=\frac{\sqrt{2}}{2}c^{1/2}A_1,\qquad S_2=\frac{\sqrt{2}}{12}\left\{c^{1/2}(1-\gamma)\alpha_3-6d^{1/2}+2d^{3/2}\right\}A_1^2+\frac{\sqrt{2}}{12}c^{1/2}(d-3)A_1A_2,$$

and $S_3=O_p(n^{-3/2})$ is not displayed since it is not used to compute the third cumulant. Based on this form, the third cumulant of $S_1+S_2+S_3$ is obtained as

$$\kappa_3(\gamma,\phi) \;=\; E[S_1^3]+3E[S_1^2S_2]-3E[S_1^2]E[S_2]+O(n^{-3}),$$

where by Lemma 1 in Appendix A.2,

$$\begin{aligned}
E[S_1^3] &= \frac{2\sqrt{2}}{15}\left\{c^{3/2}\alpha_3 + 9d^{1/2} - 2d^{3/2}\right\} n^{-2} + O(n^{-3}),\\
E[S_1^2 S_2] &= \frac{\sqrt{2}}{10}\left\{5c^{-1/2}(1-\gamma)\alpha_3 - 2c^{3/2}\alpha_3 - 18d^{1/2} + 4d^{3/2}\right\} n^{-2} + O(n^{-3}),\\
E[S_1^2]E[S_2] &= \frac{\sqrt{2}}{30}\left\{5c^{-1/2}(1-\gamma)\alpha_3 - 2c^{3/2}\alpha_3 - 18d^{1/2} + 4d^{3/2}\right\} n^{-2} + O(n^{-3}).
\end{aligned}$$

Therefore, if we set γ as

$$\gamma^* = 1 - \frac{4}{15}c^2 - \frac{12}{5}\frac{c^{1/2}d^{1/2}}{\alpha_3} + \frac{8}{15}\frac{c^{1/2}d^{3/2}}{\alpha_3}, \tag{10}$$

then it holds $\kappa_3(\gamma^*, \phi) = O(n^{-3})$. Note that under the constant volatility case considered in Section 3.1, the equation (10) reduces to $\gamma^* = -1$. In the general case, however, γ^* depends on unknown objects c, d, and α_3. By replacing these objects with consistent estimators, we propose the data-dependent value of γ:

$$\hat{\gamma} = 1 - \frac{4}{15}\hat{c}^2 - \frac{12}{5}\frac{\hat{c}^{1/2}\hat{d}^{1/2}}{\hat{\alpha}_3} + \frac{8}{15}\frac{\hat{c}^{1/2}\hat{d}^{3/2}}{\hat{\alpha}_3}, \tag{11}$$

where $\hat{c} = \frac{\hat{V}}{\hat{\sigma}_{4,n}}$, $\hat{d} = \frac{\theta^2}{\hat{\sigma}_{4,n}}$, $\hat{\alpha}_3 = \hat{V}^{-3/2}\frac{1}{n}\sum_{i=1}^n (nr_i^2 - \theta)^3$, $\hat{V} = \frac{1}{n}\sum_{i=1}^n (nr_i^2 - \theta)^2$, and $\hat{\sigma}_{4,n} = \frac{1}{3}(\hat{V} + \theta^2)$. Since $\hat{\gamma} - \gamma^* = O_p(n^{-1/2})$, we need to take the estimation error of $\hat{\gamma}$ into account for the second order analysis below.

Next, we rederive the stochastic expansion of $n^{-1}T_{\hat{\gamma},\phi}(\theta)$ with $\hat{\gamma}$ in (11). By expanding $\hat{\gamma}$ around $(\hat{c}, \hat{d}, \hat{\alpha}_3) = (c, d, \alpha_3)$, it holds

$$\hat{\gamma} = \gamma^* + gA_2 + hA_3 + O_p(n^{-1}),$$

where

$$\begin{aligned}
g &= -\frac{8}{15}c^2 + \frac{8}{45}c^3 - \frac{6}{5}\frac{c^{1/2}d^{1/2}}{\alpha_3} + \frac{4}{5}\frac{c^{3/2}d^{1/2}}{\alpha_3} + \frac{4}{15}\frac{c^{1/2}d^{3/2}}{\alpha_3} - \frac{16}{45}\frac{c^{3/2}d^{3/2}}{\alpha_3},\\
h &= \frac{4}{15}(9-2d)\frac{c^{1/2}d^{1/2}}{\alpha_3^2}.
\end{aligned}$$

By using this expansion of $\hat{\gamma}$, we can rewrite the expansion of the nonparametric likelihood statistic as

$$\begin{aligned}
& n^{-1}T_{\hat{\gamma},\phi}(\theta)\\
=\ & \frac{1}{2}cA_1^2 + \frac{1}{6}c^{1/2}\left\{c^{1/2}(1-\gamma^*)\alpha_3 + 2d^{3/2} - 6d^{1/2}\right\} A_1^3 + \frac{1}{6}c(d-3)A_1^2A_2\\
& + \frac{1}{18}c(9 - 3d - cd)A_1^2A_2^2 + \frac{1}{6}c(1 - \gamma^* - h\alpha_3)A_1^3A_3\\
& + \frac{1}{18}c^{1/2}\left\{c^{1/2}\left((d-9)(1-\gamma^*) - 3g\right)\alpha_3 + 18d^{1/2} - 6d^{3/2} + 9cd^{1/2} - 4cd^{3/2}\right\} A_1^3A_2\\
& + \frac{1}{18}\left\{2c^{1/2}d^{1/2}(d-3)(1-\gamma^*)\alpha_3 - 9c + 12cd - 4cd^2 + 9cf\right\} A_1^4 + O_p(n^{-5/2}).
\end{aligned} \tag{12}$$

After some algebra, the signed root form is obtained as $n^{-1}T_{\hat{\gamma},\phi}(\theta) = (T_1+T_2+T_3)^2 + O_p(n^{-5/2})$, where

$$
\begin{aligned}
T_1 &= \frac{\sqrt{2}}{2}c^{1/2}A_1, \qquad T_2 = jA_1^2 + kA_1A_2,\\
T_3 &= lA_1A_2^2 + qA_1^2A_3 + mA_1^2A_2 + \zeta(\phi)A_1^3,
\end{aligned}
$$

and

$$
\begin{aligned}
j &= \frac{\sqrt{2}}{12}\left\{c^{1/2}(1-\gamma)\alpha_3 + 2d^{3/2} - 6d^{1/2}\right\}, \qquad k = \frac{\sqrt{2}}{12}c^{1/2}(d-3),\\
l &= \frac{\sqrt{2}}{36}c^{1/2}(9-3d-cd) - \frac{\sqrt{2}}{144}c^{1/2}(d-3)^2,\\
m &= \frac{\sqrt{2}}{72}\left\{c^{1/2}(d-15)(1-\gamma)\alpha_3 + 18d^{1/2} - 2d^{5/2} + 18cd^{1/2} - 8cd^{3/2} - 6c^{1/2}g\alpha_3\right\},\\
q &= \frac{\sqrt{2}}{12}c^{1/2}(1-\gamma-h\alpha_3),\\
\zeta(\phi) &= \frac{\sqrt{2}}{36}\left\{9c^{1/2}f - 9c^{1/2} + 12c^{1/2}d - 4c^{1/2}d^2 - \frac{1}{4}c^{1/2}(1-\gamma)^2\alpha_3^2 - c^{-1/2}d^3 - 9c^{-1/2}d + 6c^{-1/2}d^2\right\}.
\end{aligned}
$$

By the definition of γ^*, we can show that the third cumulant of $T_1+T_2+T_3$ satisfies $\kappa_3(\hat{\gamma},\phi) = O(n^{-3})$. After lengthy calculations, by using the expectations in Lemma 1 in Appendix A.2, the fourth cumulant

$$
\begin{aligned}
\kappa_4(\hat{\gamma},\phi) &= E[T_1^4] + 4E[T_1^3T_2] + 4E[T_1^3T_3] - 3(E[T_1^2])^2\\
&\quad +6E[T_1^2T_2^2] - 4E[T_1^3]E[T_2] - 12E[T_1^2T_2]E[T_2] - 6E[T_1^2]E[T_2^2]\\
&\quad +12E[T_1^2](E[T_2])^2 - 12E[T_1^2]E[T_1T_2] - 12E[T_1^2]E[T_1T_3] + O(n^{-4})
\end{aligned}
$$

is written in the form of

$$
\kappa_4(\hat{\gamma},\phi) = \xi_1\zeta(\phi) + \xi_2 + O(n^{-4}), \tag{13}
$$

where ξ_1 and ξ_2 are implicitly defined and do not depend on ϕ. Although $\zeta(\phi)$, ξ_1 and ξ_2 contain unknown objects c, d, and α_3, they can be estimated by $\hat{c}$, $\hat{d}$, and $\hat{\alpha}_3$, respectively (denote by $\hat{\zeta}(\phi)$, $\hat{\xi}_1$ and $\hat{\xi}_2$). Then if the solution exists, the ideal value $\hat{\phi}$ is given by a solution of

$$
\hat{\xi}_1\hat{\zeta}(\hat{\phi}) + \hat{\xi}_2 = 0. \tag{14}
$$

It should be noted that in the expansion (12), ϕ appears only in the term f. Therefore, the estimation error $\hat{\phi} - \phi$ is of negligible order $O_p(n^{-5/2})$, and it holds $\kappa_4(\hat{\gamma},\hat{\phi}) = O(n^{-4})$, i.e., the dominant term of the fourth cumulant vanishes if we choose $\hat{\gamma}$ and $\hat{\phi}$ as in (11) and (14), respectively.

Finally, we compute the second cumulant and Bartlett factor. Using the expectations in Lemma 1 in Appendix A.2, we have

$$
\begin{aligned}
E[T_1^2] &= \frac{c}{2}E[A_1^2] = n^{-1}, \qquad E[T_1T_2] = rn^{-2} + O(n^{-3}),\\
E[T_2^2] &= sn^{-2} + O(n^{-3}), \qquad E[T_1T_3] = tn^{-2} + O(n^{-3}),
\end{aligned}
$$

where (recall $\bar{\sigma}_{q,n} = n^{q/2-1}\sum_{i=1}^{n}\left(\int_{(i-1)/n}^{i/n}\sigma_u^2 du\right)^{q/2}$)

$$\begin{aligned}
r &= 4\sqrt{2}c^{1/2}V^{-2}\left\{j\bar{\sigma}_{6,n}V^{1/2} + k(9\bar{\sigma}_{8,n} - 2\theta\bar{\sigma}_{6,n})\right\}, \\
s &= 12j^2\bar{\sigma}_{4,n}^2V^{-2} + 2jk(72\bar{\sigma}_{4,n}\bar{\sigma}_{6,n} - 24\theta\bar{\sigma}_{4,n}^2)V^{-5/2} \\
&\quad + k^2(192\bar{\sigma}_{4,n}\bar{\sigma}_{8,n} + 288\bar{\sigma}_{6,n}^2 - 288\theta\bar{\sigma}_{4,n}\bar{\sigma}_{6,n} + 48\theta^2\bar{\sigma}_{4,n}^2)V^{-3}, \\
t &= 24\sqrt{2}c^{1/2}l(4\bar{\sigma}_{4,n}\bar{\sigma}_{8,n} + 6\bar{\sigma}_{6,n}^2 - 6\theta\bar{\sigma}_{4,n}\bar{\sigma}_{6,n} + \theta^2\bar{\sigma}_{4,n}^2)V^{-3} \\
&\quad + 18\sqrt{2}c^{1/2}q(15\bar{\sigma}_{4,n}\bar{\sigma}_{8,n} - 6\theta\bar{\sigma}_{4,n}\bar{\sigma}_{6,n} + \theta^2\bar{\sigma}_{4,n}^2)V^{-3} \\
&\quad + 6\sqrt{2}c^{1/2}V^{-5/2}\left\{m(6\bar{\sigma}_{4,n}\bar{\sigma}_{6,n} - 2\theta\bar{\sigma}_{4,n}^2) + \zeta(\phi)\bar{\sigma}_{4,n}^2V^{1/2}\right\}.
\end{aligned}$$

Thus, the second cumulant used to compute the Bartlett correction factor is obtained as

$$nE[(T_1 + T_2 + T_3)^2] = 1 + an^{-1} + O(n^{-2}),$$

where $a = 2(r + t) + s$. Combining these results, we obtain the following theorem.

Theorem 3. *Suppose Assumptions X and H hold true. Then, for $\hat{\gamma}$ in (11) and $\hat{\phi}$ in (14) (if the solution exists), the nonparametric likelihood statistic $T_{\hat{\gamma},\hat{\phi}}(\theta)$ is Bartlett correctable, i.e., conditionally on the path of $\{\sigma_t\}$,*

$$\Pr\left\{T_{\hat{\gamma},\hat{\phi}}(\theta) \leq \chi^2_{1,\alpha}(1 + an^{-1})\right\} = 1 - \alpha + O(n^{-2}).$$

This theorem says that even for the general case, the nonparametric likelihood statistic $T_{\hat{\gamma},\hat{\phi}}(\theta)$ with the estimated tuning constants $\hat{\gamma}$ and $\hat{\phi}$ using the adjusted critical value $\chi^2_{1,\alpha}(1 + an^{-1})$ provides a refinement to the order $O(n^{-2})$ on the null rejection probability error. In the general case, the Bartlett factor a needs to be estimated by the method of moments. Gonçalves and Meddahi (2009) obtained the second order refinement by the bootstrap to the order $o(n^{-1})$. In contrast, our Bartlett correction to the nonparametric likelihood statistic yields a refinement to the order $O(n^{-2})$.

4. Simulation

This section conducts simulation studies in order to evaluate finite sample properties of the nonparametric likelihood inference and second-order refinements proposed in the last section.

We adopt simulation designs considered in Gonçalves and Meddahi (2009). In particular, we consider the stochastic volatility model

$$dX_t = \mu_t dt + \sigma_t\left(\rho_1 dW_{1t} + \rho_2 dW_{2t} + \sqrt{1 - \rho_1^2 - \rho_2^2}dW_{3t}\right),$$

where W_{1t}, W_{2t}, and W_{3t} are independent standard Brownian motions.

First, we consider a general case (i.e. with drift and leverage effects) to illustrate the first-order asymptotic theory in Theorem 1 for the nonparametric likelihood statistic $T_{\gamma,\phi}(\theta)$. We consider two different models for the volatility process σ_t. The first model for σ_t is the GARCH(1,1) diffusion

$$d\sigma_t^2 = 0.035(0.636 - \sigma_t^2)dt + 0.144\sigma_t^2 dW_{1t}.$$

The second model is the two-factor diffusion model

$$\sigma_t = f(-1.2 + 0.04\sigma_{1t}^2 + 1.5\sigma_{2t}^2),$$

where $d\sigma_{1t}^2 = -0.00137\sigma_{1t}^2 dt + dW_{1t}$, $d\sigma_{2t}^2 = -1.386\sigma_{2t}^2 dt + (1 + 0.25\sigma_{2t}^2)dW_{2t}$, and

$$f(x) = \begin{cases} \exp(x) & x \leq x_0 \\ \frac{\exp(x_0)}{\sqrt{x_0}}\sqrt{x_0 - x_0^2 + x^2} & x > x_0 \end{cases}$$

with $x_0 = \log(1.5)$. We allow for drift and leverage effects by setting $\mu_t = 0.0314$, $\rho_1 = -0.576$, and $\rho_2 = 0$ for GARCH(1,1) models, and $\mu_t = 0.030$ and $\rho_1 = \rho_2 = -0.30$ for the two-factor diffusion model.

We compare three methods to construct two-sided 95% confidence intervals: (i) the Wald-type interval (Wald), (ii) empirical likelihood (EL) and (iii) nonparametric likelihood (NL) with $\gamma = -1$ and $\phi = -1 + \frac{\sqrt{5}}{3}$.

Table 1 gives the actual coverage rates of all the intervals across 10,000 replications for five different sample sizes: n =1152, 288, 48, 24, and 12, corresponding to 1.25-minute, 5-minute, half-hour, 1-hour, and 2-hour returns. The Wald-type intervals tend to undercover for both models. The degree of undercoverage is especially large when sampling is not too frequent. The two-factor model implies overall larger coverage distortions than the GARCH(1,1) model. The nonparametric likelihood intervals (including EL intervals) outperform the Wald-type intervals in all cases.

Second, we consider two special cases to illustrate the second-order refinements proposed in the last section: (a) a benchmark model where volatility is constant, and (b) models where volatility is not constant (with no drift term and no leverage effect). Bartlett corrected nonparametric likelihood (BNL) with the Bartlett correction factor $1 + 3/n$ are compared with the above methods. Table 2 shows that the Bartlett corrected nonparametric likelihood intervals outperform all the other intervals even when there is stochastic volatility despite the fact that this correction does not theoretically provide an asymptotic refinement under the non-constant volatility case.

n	Wald	EL	NL	Wald	EL	NL
	GARCH(1,1) diffusion			Two-factor diffusion		
12	80.83	84.80	84.48	73.24	78.45	78.02
24	86.97	90.34	90.03	80.61	85.65	85.23
48	90.41	92.76	92.46	85.76	89.38	89.04
288	94.55	94.98	94.92	93.52	94.50	94.35
1152	94.72	94.83	94.79	94.91	95.31	95.22

TABLE 1. Coverage probabilities of nominal 95% confidence intervals for integrated volatility with leverage and drift

n	Wald	EL	NL	BNL
	Constant volatility			
12	81.20	85.18	84.77	87.46
24	87.63	90.66	90.35	92.00
48	91.04	93.54	93.14	94.08
288	94.24	94.85	94.78	94.89
1152	95.27	95.39	95.34	95.40
	GARCH(1,1) diffusion			
12	81.39	85.29	85.02	87.73
24	87.51	90.89	90.61	92.04
48	90.98	93.51	93.19	93.89
288	94.44	94.97	94.87	94.97
1152	95.07	95.18	95.14	95.18
	Two-factor diffusion			
12	73.74	77.97	77.63	80.87
24	80.90	85.72	85.33	87.06
48	86.05	86.69	89.45	90.32
288	92.83	94.08	93.95	94.08
1152	94.22	95.04	94.99	95.02

TABLE 2. Coverage probabilities of nominal 95% confidence intervals for integrated volatility with no drift and no leverage

Appendix A. Mathematical Appendix

A.1. **Proof of Theorem 1 .** We focus on the case of $\gamma, \phi \neq -1, 0$. Similar arguments apply to the cases of $\gamma, \phi = -1, 0$. For $q > 0$, define $\bar{\sigma}_q = \int_0^1 \sigma_u^q du$ and $\mu_q = E|Z|^q$ with $Z \sim N(0,1)$. From Barndorff-Nielsen *et al.* (2006, Theorem 1), Assumption X guarantees

$$R_q \xrightarrow{p} \mu_q \bar{\sigma}_q, \tag{15}$$

for any $q > 0$. This implies $R_2 \xrightarrow{p} \bar{\sigma}_2$, $R_4 \xrightarrow{p} \mu_4 \bar{\sigma}_4$, and

$$\frac{3}{2}\left(1 - \frac{R_2^2}{R_4}\right) \xrightarrow{p} \frac{3}{2}\left(\frac{\mu_4\bar{\sigma}_4 - \bar{\sigma}_2^2}{\mu_4\bar{\sigma}_4}\right). \tag{16}$$

Let $g_i = nr_i^2 - \theta$. By (2) and (15), we obtain

$$\left(\frac{2}{3}\mu_4\bar{\sigma}_4\right)^{-1/2} \frac{1}{\sqrt{n}} \sum_{i=1}^n g_i = \left(\frac{2}{3}\mu_4\bar{\sigma}_4\right)^{-1/2} \sqrt{n}(R_2 - \bar{\sigma}_2) \xrightarrow{d} N(0,1), \tag{17}$$

$$\frac{1}{n}\sum_{i=1}^n g_i^2 = R_4 - 2\bar{\sigma}_2 R_2 + \bar{\sigma}_2^2 \xrightarrow{p} \mu_4\bar{\sigma}_4 - \bar{\sigma}_2^2. \tag{18}$$

By these results combined with $E[g_i^2] < \infty$ for all $i = 1, \ldots, n$, we can apply the same argument to Owen (1998) to show $\max_{1 \leq i \leq n} |\eta + \lambda g_i| \xrightarrow{p} 0$. Thus, by expanding (6) around $(\eta, \lambda) = (0,0)$, we obtain

$$\begin{aligned}
\lambda &= -\phi\left(\frac{1}{n}\sum_{i=1}^n g_i^2\right)^{-1} \frac{1}{n}\sum_{i=1}^n g_i + O_p(n^{-1}), \\
\eta &= \frac{1}{2}\phi(\phi+1)\left(\frac{1}{n}\sum_{i=1}^n g_i^2\right)^{-1} \left(\frac{1}{n}\sum_{i=1}^n g_i\right)^2 + O_p(n^{-2}).
\end{aligned}$$

Based on these results, an expansion of $\ell_{\gamma,\phi}(\theta)$ around $(\eta, \lambda) = (0,0)$ yields

$$\ell_{\gamma,\phi}(\theta) = \frac{2}{\gamma(\gamma+1)}\sum_{i=1}^n \{(1+\eta+\lambda g_i)^{\frac{\gamma+1}{\phi}} - 1\} = \left(\frac{1}{n}\sum_{i=1}^n g_i^2\right)^{-1}\left(\frac{1}{\sqrt{n}}\sum_{i=1}^n g_i\right)^2 + O_p(n^{-1}).$$

Therefore, the conclusion follows by (16)-(18).

A.2. **Lemmas .** Here we present some approximation formulae for the moments of A_k. Lemma 1 is derived under Assumptions X and H, which allows non-constant volatility. Lemma 2 is derived for the constant volatility case. The proofs are available from the authors upon request.

Lemma 1. *Suppose Assumptions X and H hold true. Then*

$$E[A_1] = 0, \qquad E[A_1^2] = 2\bar{\sigma}_{4,n}V^{-1}n^{-1}, \qquad E[A_1^3] = 8\bar{\sigma}_{6,n}V^{-3/2}n^{-2},$$
$$E[A_1^4] = 12\bar{\sigma}_{4,n}^2V^{-2}n^{-2} + 48\bar{\sigma}_{8,n}V^{-2}n^{-3}, \qquad E[A_1^5] = 160\bar{\sigma}_{4,n}\bar{\sigma}_{6,n}V^{-5/2}n^{-3} + O(n^{-4}),$$
$$E[A_1^6] = 120\bar{\sigma}_{4,n}^3V^{-3}n^{-3} + O(n^{-4}), \qquad E[A_1A_2] = (12\bar{\sigma}_{6,n} - 4\theta\bar{\sigma}_{n,4})V^{-3/2}n^{-1}$$
$$E[A_1^2A_2] = (72\bar{\sigma}_{8,n} - 16\theta\bar{\sigma}_{6,n})V^{-2}n^{-2},$$
$$E[A_1^3A_2] = (72\bar{\sigma}_{4,n}\bar{\sigma}_{6,n} - 24\theta\bar{\sigma}_{4,n}^2)V^{-5/2}n^{-2} + O(n^{-3}),$$
$$E[A_1^4A_2] = (384\bar{\sigma}_{6,n}^2 + 864\bar{\sigma}_{4,n}\bar{\sigma}_{8,n} - 320\theta\bar{\sigma}_{4,n}\bar{\sigma}_{6,n})V^{-3}n^{-3} + O(n^{-4}),$$
$$E[A_1^5A_2] = (720\bar{\sigma}_{4,n}^2\bar{\sigma}_{6,n} - 240\theta\bar{\sigma}_{4,n}^3)V^{-7/2}n^{-3} + O(n^{-4}),$$
$$E[A_1A_3] = (90\bar{\sigma}_{8,n} - 36\theta\bar{\sigma}_{6,n} + 6\theta^2\bar{\sigma}_{4,n})V^{-2}n^{-1},$$
$$E[A_1^3A_3] = (540\bar{\sigma}_{4,n}\bar{\sigma}_{8,n} - 216\theta\bar{\sigma}_{4,n}\bar{\sigma}_{6,n} + 36\theta^2\bar{\sigma}_{4,n}^2)V^{-3}n^{-2} + O(n^{-3}),$$
$$E[A_1^5A_3] = (5400\bar{\sigma}_{4,n}^2\bar{\sigma}_{8,n} - 2160\theta\bar{\sigma}_{4,n}^2\bar{\sigma}_{6,n} + 360\theta^2\bar{\sigma}_{4,n}^3)V^{-4}n^{-3} + O(n^{-4}),$$
$$E[A_1^2A_2^2] = (192\bar{\sigma}_{4,n}\bar{\sigma}_{8,n} + 288\bar{\sigma}_{6,n}^2 - 288\theta\bar{\sigma}_{4,n}\bar{\sigma}_{6,n} + 48\theta^2\bar{\sigma}_{4,n}^2)V^{-3}n^{-2} + O(n^{-3}),$$
$$E[A_1^4A_2^2] = (1152\bar{\sigma}_{4,n}^2\bar{\sigma}_{8,n} + 3456\bar{\sigma}_{4,n}\bar{\sigma}_{6,n}^2 - 2880\theta\bar{\sigma}_{4,n}^2\bar{\sigma}_{6,n} + 480\theta^2\bar{\sigma}_{4,n}^3)V^{-4}n^{-3} + O(n^{-4}).$$

Lemma 2. *Suppose Assumptions X and H hold true. Furthermore, assume that $\sigma_t = \sigma$ over $t \in [0,1]$. Then*

$$E[A_1] = 0, \qquad E[A_1^2] = n^{-1}, \qquad E[A_1^3] = \alpha_3n^{-2}, \qquad E[A_1^4] = 3n^{-2} + (\alpha_4 - 3)n^{-3},$$
$$E[A_1^5] = 10\alpha_3n^{-3} + O(n^{-4}), \qquad E[A_1^6] = 15n^{-3} + O(n^{-4}),$$
$$E[A_1A_2] = \alpha_3n^{-1}, \qquad E[A_1^2A_2] = (\alpha_4 - 1)n^{-2},$$
$$E[A_1^3A_2] = 3\alpha_3n^{-2} + O(n^{-3}), \qquad E[A_1^4A_2] = (6\alpha_4 + 4\alpha_3^2 - 6)n^{-3} + O(n^{-4}),$$
$$E[A_1^5A_2] = 15\alpha_3n^{-3} + O(n^{-4}), \qquad E[A_1A_3] = \alpha_4n^{-1},$$
$$E[A_1^3A_3] = 3\alpha_4n^{-2} + O(n^{-3}), \qquad E[A_1^5A_3] = 15\alpha_4n^{-3} + O(n^{-4}),$$
$$E[A_1^2A_2^2] = (\alpha_4 + 2\alpha_3^2 - 1)n^{-2} + O(n^{-3}), \qquad E[A_1^4A_2^2] = (3\alpha_4 + 12\alpha_3^2 - 3)n^{-3} + O(n^{-4}).$$

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