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CRASHING OF EFFICIENT STOCHASTIC BUBBLES

By

Aloisio Araujo
(IMPA and FGV EPGE),

Juan Pablo Gama
(IMPA),

&

Mario R. Pascoa
(University of Surrey).

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School of Economics
University of Surrey
Guildford
Surrey GU2 7XH, UK
Telephone +44 (0)1483 689380
Facsimile +44 (0)1483 689548
Web https://www.surrey.ac.uk/school-economics
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Crashing of efficient stochastic bubbles

A. Araujo[1,2], J. Gama[*†1], and M. Pascoa[3]

[1]IMPA, Rio de Janeiro, Brazil
[2]FGV EPGE, Rio de Janeiro, Brazil
[3]University of Surrey, Guildford, U. K.

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Abstract

Efficiency is not commonly related to the crash of bubbles. However in the presence of wary agents, infinite-lived agents that are worried about distant losses, efficient bubbles may occur and, in a stochastic setting, these bubbles can crash. In this paper we characterize the Arrow-Debreu (AD) price and establish the relationship between the agents' concern about distant losses and the existence of pure charges in the AD price. We show that this pure charge induces efficient bubbles in the positive net-supply assets that complete the markets and that, as we enter some sub-tree, that pure charge may no longer present in the AD price for the sub-economy, implying the crash of the bubble. Finally, we give an example in which there is an efficient bubble with infinitely many crashes.

Keyword: crashing; efficient bubbles; complete markets; stochastic economies;

*jpgamat@impa.br

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1 Introduction

Rational bubbles have been extensively studied since the late 70's pioneering work by Blanchard (1979) (see (1)) and Blanchard and Watson (1982) (see (2)). Santos and Woodford (1997) (see (3)) made a theoretical and systematic study of rational bubbles in a general equilibrium model with finite and infinite-lived agents. However, efficient bubbles in positive net-supply assets seemed to be ruled out by the same portfolio constraints that avoided Ponzi schemes. Araujo, Novinski and Pascoa (2011) (see (4)) showed that this is no longer the case when standard impatience assumptions do not hold. More precisely, if agents are *wary*, that is, not willing to neglect losses at distant dates, the efficient allocations may be implemented by trading positive net supply assets with positive dividends and speculative prices. Araujo, Gama, Novinski and Pascoa (2019) (see (5)) extended this result to the case of fiat money, under appropriate taxes on money holdings that discourage inefficient savings. The drawback of this surprising set of results is that it was established in a deterministic setting, where the occurrence of a bubble implies that it will be present at all dates.

In a stochastic economy, bubbles can burst some time later, as we enter some sub-tree where the reason for the occurrence of the bubble does not prevail anymore. In this paper we address whether efficient bubbles generated by wariness may burst in a stochastic economy. Wariness is a lack of impatience that consists in neglecting distant gains but not distant losses. The lack of impatience can be interpreted in terms of ambiguous beliefs about the weight that different dates and states of nature should have in the utility function, in the sense of Gilboa and Schmeidler (1989) (see (6)) and Schmeidler (1989) (see (7)). A secondary goal of our paper is to study the *price volatility* caused by the crashing of bubbles in some states of nature.

We provide an example where a pure charge in the Arrow-Debreu price occurs but then disappears, as the infimum of consumption was a cluster point of the consumption process but is no longer in the relevant sub-tree. This induces the asset price bubble to appear and then burst. The bubble may even occur and then crash at infinitely many different dates depending on the path. In all cases, bubbles cannot reappear in the economy.

The article is organized as follows: In section 2, we introduce the notation, specify the preferences and define the AD equilibrium. In section 3, we characterize the superdiferential of the utility of a wary consumer. In section 4, we implement the AD allocation sequentially, and analyze the existence of efficient bubbles in this framework and how they affect the volatility of the asset prices. We conclude that section with an example of a bubble whose probability of crashing from a very distant date onwards is positive. In section 5, we provide some concluding remarks.

2 Model

We consider an economy with countably many dates. The event tree is defined as follows. The initial node is denoted by 1. Any node in the tree has a finite number N of immediate successors. A node occurring at date t is represented by $s^t = (s_1, s_2, \ldots, s_t) \in \{1\} \times N^{t-1}$. We denote by $s^{t,-}$ the predecessor of s^t and by $s^{t,-2}$ the predecessor of $s^{t,-}$.

An infinite path of the tree is represented by $\sigma := (s_1, \ldots, s_t, \ldots)$ and σ_t stands for its truncation up to date t. The set of all infinite paths is $\{1\} \times N^\infty$. Let $\mathcal{N}$ be the σ-algebra induced by $\{\sigma : \sigma_t = s^t\}$ for each $s^t \in \{1\} \times N^{t-1}$ and each $t \in \mathbb{N}$. Let $\mathbb{P}$ be a probability measure in $(\{1\} \times N^\infty, \mathcal{N})$. Therefore, $\mathbb{P}[\sigma_t = s^t] > 0$ is interpreted as the probability of node s^t occurring which is strictly positive for every node s^t and every $t \geq 1$.

2.1 Utility functions

The consumption space is $\mathcal{X} := \{X : \cup_{t \in \mathbb{N} \cup \{0\}} \{1\} \times N^t \to \mathbb{R}_+ : \exists K > 0 \text{ such that } X(s^t) \leq K \text{ for all } t\}$. Each element of $\mathcal{X}$ is a plan for consumption along each possible path of the event tree and can be seen as belonging to $L^\infty \left(\cup_{t \in \mathbb{N} \cup \{0\}} \{1\} \times N^t, \tilde{\mathcal{N}}, P \right)$ where $\tilde{\mathcal{N}}$ is the discrete σ-algebra in $\cup_{t \in \mathbb{N} \cup \{0\}} \{1\} \times N^t$ and P is the σ-additive measure induced by $P(\{s^t\}) = \mathbb{P}[\sigma_t = s^t]$[1].

In a stochastic infinite horizon economy, there are several ways to model a lack of impatience for losses. We consider a natural extension to the stochastic case of the one used in Araujo et al. (2011) (see (4)). It is generated by the $\varepsilon-$*Contamination*.

Recall that, given a probability measure μ and $\varepsilon \in [0, 1)$, the convex capacity ν_ϵ defined by $\nu_\varepsilon(A) = (1 - \varepsilon)\, \mu(A)$ for $A \subsetneqq \mathbb{N}$ and $\nu_\varepsilon(\mathbb{N}) = 1$ is

[1] The only purpose of the probability P is to define properly the consumption set that we are working on.

called the ε-*contamination capacity* of μ. The Choquet integral with respect of ν_ε can be written as $U(x) = (1-\varepsilon)\int_{\mathbb{N}} u \circ x \, d\mu + \varepsilon \inf u \circ x$.

This concept suggests the following preferences.

$$U(X) := \int_{N^\infty \times \mathbb{N}} u \circ X_{\sigma_t} d(\mathbb{P} \times \zeta)(\sigma, t) + \int_{N^\infty} \left(\beta(\sigma) \inf_t u \circ X_{\sigma_t}\right) d\mathbb{P}(\sigma) \qquad (1)$$

where $u : \mathbb{R}_+ \to \mathbb{R}$ is a differentiable, strictly increasing and strictly concave function, β is a $\mathcal{N}$-measurable non-negative function, and $\{\zeta(t)\}_t \in l^1_{++}$[2] is the temporal discount factor, therefore, $\{(\mathbb{P} \times \zeta)(\sigma, t)\}_t$ can be interpreted as the discount factor of the path σ.

The first term in the utility function is a standard discounted utility function for stochastic economies. The second term is the $\mathbb{P}$-average, among all possible paths, of the worst consumption. Agents with such preferences worry about consumption at each node if the discounted factors ζ are strictly positive and the probability of each node s^t, $\mathbb{P}[\sigma_t = s^t]$ is also positive. However, they are also concerned about the worst possible outcome of each path.

These preferences embody an ambiguity aversion in the sense of Schmeidler (1989) (see (7)). More precisely, in the Ascombe-Aumann framework, a preference defined over a set of acts is ambiguity averse if $1/2(f+g) \succsim f, g$ for all f, g acts. In the context of Schmeidler (1989) (see (7)), this is equivalent to the capacity being convex. In the deterministic economy in Araujo et al. (2019) (5) the ambiguity over discount factors translated into a weight given to the worst lifetime consumption. Now, in a stochastic economy, ambiguity translates into a weight given to the average, among all paths, of the worst consumption in each path.

A non-constant β allows for an agent to have different concerns about distant losses depending on the paths of the tree. More precisely, $\beta : N^\infty \to \mathbb{R}_+$ is a nonnegative, bounded and not necessarily constant function of the infinite paths.

2.2 Arrow-Debreu Equilibrium

Consider an economy with I consumers. Each consumer i is characterized by an endowment allocation $\{W^i_{s^t}\}_{s^t} \in L^\infty_+\left(\cup_{t \in \mathbb{N} \cup \{0\}} \{1\} \times N^t, \tilde{\mathcal{N}}, P\right)$ such that $\{W^i_{s^t}\}_{s^t} \ggg 0$[3], and a preference represented by a utility function U^i of the

[2] $l^1_{++} := \left\{\{\mu_t\}_t \in l^1 : \mu_t > 0 \ \forall t\right\}$

[3] $x \geq 0$ means $x_{s^t} \geq 0$ for all s^t, $x > 0$ means $x \geq 0$ and $x \neq 0$, $x \gg 0$ means $x_{s^t} > 0$ for all s^t, and $x \ggg 0$ means that there exists $\varepsilon > 0$ such that $x_{s^t} \geq \varepsilon$ for all s^t.

form 1 for some index u^i and some β^i.

Definition 1. An *Arrow-Debreu equilibrium* is a couple $\left(\pi, (x^i)_{i=1}^I\right)$ such that, for all i, x^i maximizes U^i on $B_{AD}(\pi, W^i) := \{x \geq 0 : \pi(x) \leq \pi(W^i)\}$, $\sum_{i=1}^I (x^i - W^i) = 0$ and $\pi \in L_+^{\infty *}\left(\cup_{t\in\mathbb{N}\cup\{0\}}\{1\} \times N^t, \tilde{\mathcal{N}}, P\right)$ with $\pi \neq 0$.

Prices are elements in the dual of the consumption space, not necessarily in the pre-dual. The dual space is $ba_+\left(\cup_{t\in\mathbb{N}\cup\{0\}}\{1\} \times N^t, \tilde{\mathcal{N}}, P\right)$ the set of the finitely additive measures in the set of nodes. Preferences given by 1 satisfy the hypotheses imposed by Bewley (see (8)) and Mas-Colell et al. (see (9)) to ensure existence of equilibrium, with consumption plans in $(L_+^\infty\left(\cup_{t\in\mathbb{N}\cup\{0\}}\{1\} \times N^t, \tilde{\mathcal{N}}, P\right)$ and prices in $ba_+\left(\cup_{t\in\mathbb{N}\cup\{0\}}\{1\} \times N^t, \tilde{\mathcal{N}}, P\right)$.

Theorem 1 (Bewley's Existence Theorems (8))**.**

(T1) Consider an economy with a finite number of agents where each agent has complete, convex, transitive, monotonous, norm continuous and Mackey usc preferences in l_+^∞, and $W^i \ggg 0$, then there exists an AD equilibrium with prices in ba_{++}.

(T2) Under the additional assumption of Mackey lsc, $\pi \in l_{++}^1$.

In the next section, we will characterize the supergradients of the utility function defined by 1 and relate such supergradients to the AD equilibrium prices.

3 Characterization of the Super-differential of the Utility Function

Recall that, for a concave function $U : B \to \mathbb{R}$ on a Banach space B, the super-differential of U at x is $\partial U(x) := \{\pi \in B^* : U(y) - U(x) \leq \pi(y - x) \forall y \in B\}$. Each $\pi \in \partial U(x)$ is a supergradient of U at x.

To fully characterize the super-differential of the utility function defined by (1) at any consumption plan $X \in \mathcal{X}$, we need to distinguish the following sets of paths in terms of what happens to the infimum of consumption. One is the set of paths where the infimum of consumption is never attained, $A_1(X) := [\sigma : \inf_t u \circ X_{\sigma_t} < u(X_{\sigma_t}) \forall t \in \mathbb{N}]$.

Another is the set of paths where it is at attained finitely many nodes and it is not a cluster point, $A_2(X) := \bigcup_K \left(\bigcup_{t_1 \leq \cdots \leq t_K} A^K_{2,t_1,\ldots,t_k}(X)\right)$, where $A^K_{2,t_1,\ldots,t_k}(X) := \Big\{\sigma : u\left(X_{\sigma_{t_1}}\right) = \cdots = u\left(X_{\sigma_{t_K}}\right) \leq u\left(X_{\sigma_t}\right) - \varepsilon$ for some $\varepsilon > 0, \forall t \neq t_1, \ldots, t_K\Big\}$.

Next, $A_3(X)$ consists of paths where it is attained finitely many times and is a cluster point, $A_3(X) := \bigcup_K \left(\bigcup_{t_1 \leq \cdots \leq t_K} A^K_{2,t_1,\ldots,t_k}(X)\right)$, where $A^K_{3,t_1,\ldots,t_k}(X) := \Big\{\sigma : \liminf_t u \circ X_{\sigma_t} = u\left(X_{\sigma_{t_1}}\right) = \cdots = u\left(X_{\sigma_{t_K}}\right) < u\left(X_{\sigma_t}\right) \forall t \neq t_1, \ldots, t_K\Big\}$.

Finally, the set of paths where it is attained at infinitely many nodes, $A_4(X) := \bigcup_{t_1 \leq t_2 \leq \ldots} A_{3,t_1,t_2\ldots}(X)$, where each $A_{4,t_1,t_2\ldots}(X)$ is the set $\Big\{\sigma : \forall t \neq t_1, t_2, \ldots, u\left(X_{\sigma_t}\right) > u\left(X_{\sigma_{t_1}}\right) = u\left(X_{\sigma_{t_2}}\right) = \ldots\Big\}$.

As we will see, for paths in $A_2(X) \cup A_3(X) \cup A_4(X)$ weights will be assigned, endogenously in each supergradient, to the several nodes where the infimum of consumption is attained. For each path in $A^K_{2,t_1,\ldots,t_k}(X)$ we consider weights $\alpha^\sigma_2(k) \geq 0$ on the K nodes where the infimum is attained, satisfying $\sum_{k=1}^K \alpha^\sigma_2(k) = 1$. Similarly, for each path in $A^K_{3,t_1,\ldots,t_k}(X)$ or in $A^K_{4,t_1,t_2\ldots}(X)$ we consider weights on the nodes where the infimum is attained and also a weight at infinity satisfying $\sum_{k=1}^K \alpha^\sigma_3(k) + \alpha^\sigma_3(\infty) = 1$ in $A_3(X)$, and $\sum_{k=1}^\infty \alpha^\sigma_4(k) + \alpha^\sigma_3(\infty) = 1$ in $A_4(X)$.

For paths in $A_1(X) \cup A_3(X) \cup A_4(X)$ the asymptotic behavior of consumption is relevant in terms of where the infimum of consumption is attained and supergradients will have a component capturing this through a generalized limit.

Recall that a generalized limit is a continuous linear functional LIM on l^∞ such that, for any $x \in l^\infty$, we have $\mathrm{LIM}(x) = \lim_t x_t$ when the limit exists, and $\mathrm{LIM}(x) \in [\liminf_t x_t, \limsup_t x_t]$ when it does not. We apply this concept in the following way. Given $X \in \mathcal{X}$ and $\sigma \in N^\infty$, let $\{t_k\}_k$ be the subsequence in which X_σ attains its infimum. We denote by LIM^X_σ a generalized limit such that, for any sequence $y \in l^\infty$, we have $\mathrm{LIM}^X_\sigma(y) = \lim_k y_{t_k}$ if y_{t_k} converges and $\mathrm{LIM}^X_\sigma(y) \in [\liminf_k y_{t_k}, \limsup_k y_{t_k}]$ otherwise.

The following result is an extension of the deterministic case analyzed in Araujo et al. (2011) (4).

Proposition 1. *Let $X \in \mathcal{X}$ be such that $X \ggg 0$ and $\inf_t X_{\sigma_t} = \underline{X}_\sigma$ for some $\underline{X} : N^\infty \to \mathbb{R}_{++}$. For U given by (1), with $u \in C^1(0,\infty)$, $\partial U(X)$ is the set of elements π in $ba_+\left(\cup_{t \in \mathbb{N}\cup\{0\}} \{1\} \times N^t, \tilde{\mathcal{N}}, P\right)$ such that πY is given by*

$$\sum_{s^t}\int_{[\sigma:\sigma_t=s^t]}u'\circ X_{\sigma_t}\zeta_t d\mathbb{P}(\sigma)Y_{s^t}+\int_{A_1}\beta(\sigma)u'\circ \underline{X}_\sigma \mathrm{LIM}_\sigma^X(Y_\sigma)d\mathbb{P}(\sigma)+$$
$$\sum_{t_1\leq\cdots\leq t_K}\int_{A^K_{2,t_1,\ldots,t_K}}\beta(\sigma)u'\circ \underline{X}_\sigma\left(\sum_{k=1}^K\alpha_2^\sigma(k)Y_{\sigma^{t_k}}\right)d\mathbb{P}(\sigma)+$$
$$\sum_{t_1\leq\cdots\leq t_K}\int_{A_{3,t_1,\ldots,t_K}}\beta(\sigma)\left(u'\circ \underline{X}_\sigma\right)\left(\sum_{k=1}^K\alpha_3^\sigma(k)Y_{\sigma^{t_k}}+\alpha_3^\sigma(\infty)\mathrm{LIM}_\sigma^X(Y_\sigma)\beta(\sigma)\right)d\mathbb{P}(\sigma)+$$
$$\sum_{t_1\leq t_2\leq\ldots}\int_{A_{4,t_1,t_2,\ldots}}\beta(\sigma)\left(u'\circ \underline{X}_\sigma\right)\left(\sum_{k=1}^\infty\alpha_4^\sigma(k)Y_{\sigma^{t_k}}+\alpha_4^\sigma(\infty)\mathrm{LIM}_\sigma^X(Y_\sigma)\beta(\sigma)\right)d\mathbb{P}(\sigma),$$

for any $Y\in L^\infty\left(\cup_{t\in\mathbb{N}\cup\{0\}}\{1\}\times N^t,\tilde{\mathcal{N}},P\right)$.

The proof can be found in A.

Under $\beta(\sigma)|_{A_1(X)}\gg 0$ $\mathbb{P}$-a.c.[4] in $A_1(X)$ and $\mathbb{P}[A_1(X)]>0$, the supergradient will have a positive pure charge component that is concerned about paths in $A_1(X)$ only. The supergradient has no pure charge component induced by what happens in $A_2(X)$ since for paths in this set the infimum of consumption is not a cluster point. While for $i=3,4$, a positive pure charge component will be induced by $A_i(X)$ if $\beta(\sigma)|_{A_i(X)}\gg 0$ $\mathbb{P}$-a.c. in $A_i(X)$, $\mathbb{P}[A_i(X)]>0$ and $\alpha_i^\sigma(\infty)>0$ in a positive measure subset of $A_i(X)$.

We see that a path σ which is β-positively-valued and such that the infimum of consumption is a cluster point but is also attained, will contribute towards the pure charge of the supergradient only if the weight $\alpha_i^\sigma(\infty)$ (for $i=3$ or $i=4$) happens to be positive. And such contributions will then be averaged over all such paths, according to the probability $\mathbb{P}$. There is, therefore, a multiplicity of supergradients arising from the various ways the weights $\alpha_i^\sigma(k)$ and $\alpha_i^\sigma(\infty)$ get to be distributed over the cluster point and the attainment nodes. Just like in the deterministic setting, the non-differentiability of the U results from two aspects: one is that indeterminacy in the assignment of weights, the other is the indeterminacy in the choice of the generalized limits.

The set $\partial U(x)$ is weak* compact. For any direction Y of changes in the consumption plan, the left (right) derivative of the utility function at X is the maximum (minimum, respectively) of the values that all elements of $\partial U(X)$ take at Y. The above multiplicity in the assignment of weights across all attainments of the infimum accommodates a non-differentiability that is intrinsic to the form of the utility function: if along some path, the infimum of consumption is attained at more than one node or only at one node but

[4] $x|_A$ means that x is restricted to the paths contained in A.

is also a cluster point, the canonical left derivative e_{s^t} at such a node has to greater than the right derivative. Decreasing consumption at that node lowers the infimum of consumption but increasing consumption does not, it just increases what consumption is at that node.

Remark 1. If $X \gg 0$ instead of $X \ggg 0$, there might be pure charges generated by the fact that $\{u'(X_{s^t})\}_{s^t}$ is not bounded. Moreover, if $\{\zeta_t \mathbb{P}[\sigma_t = s^t] u'(X_{s^t})\}_{s^t}$ is not in l^1, $\partial U(X) \nsubseteq l^1$.

The above characterization of the super-differential has an immediate implication in terms of how AD prices will look like. In fact, a necessary and sufficient (together with the budget constraint) optimality condition (see Zeidler (1984) (see (10)), p.391, Theorem 47.C) for $X^i \gg 0$ to maximize U^i subject to the AD budget constraint at prices π is that, for some $\rho^i > 0$, we have

$$\rho^i \pi \in \partial U^i(X^i) \tag{2}$$

4 Bubbles in sequential implementation of an AD equilibrium

Let us now define the sequential economy where AD allocations will be implemented. It has the event tree defined in section 2. Side by side with the single consumption good (the numeraire), there will be now $|N|$ **long-lived assets with nonnegative real payments** $\{R_{j,s^t}\}_{s^t} \in L^\infty_+ \left(\cup_{t \in \mathbb{N} \cup \{0\}} \{1\} \times N^t, \tilde{\mathcal{N}}, P\right)$ for each j. Each agent has initial nonnegative holdings of each asset $z^i_{j,0} \geq 0$, and sequential endowments $\{\omega^i_{s^t}\}_{s^t,i} \geq 0$. The budget constraint at the initial node is

$$x_1 - \omega_1 \leq q_1 \left(z^i_0 - z_1\right) \tag{3}$$

and at any other node s^t is

$$x_{s^t} - \omega_{s^t} \leq q_{s^t} \left(z_{s^t,-} - z_{s^t}\right) + R_{s^t} z_{s^t,-}. \tag{4}$$

Let us denote $x_{s^t}(z,i) := \omega_{s^t} + q_{s^t}\left(z_{s^t,-} - z_{s^t}\right) + R_{s^t} z_{s^t,-}$ for $s^t \neq 1$ and $x_1(z,i) := \omega_1 + q_1\left(z^i_0 - z_1\right)$.

The consumer's sequential problem SEQ^i consists in finding a portfolio $\{z^i_{s^t}\}_{s^t} \in \mathbb{R}^\infty$ that maximizes $U^i(\{x_{s^t}(z,i)\}_{s^t})$ subject to the non-negativity in consumption constraint $x_{s^t}(z,i) \geq 0$, at each node s^t, given prices $q_{1,s^t} \ldots, q_{J,s^t}$

for the J assets, the initial holdings $z^i_{j,0}$ and the sequential endowments $\{\omega^i_{s^t}\}_{s^t}$.

The following *Euler conditions* hold at a solution z^i to SEQ^i for which $x(z^i, i) \gg 0$:

$$\mu^i_{s^t} q_{s^t} \geq \sum_{s^{t+1,-}=s^t} \mu^i_{s^{t+1}} \left(q_{s^{t+1}} + R_{s^{t+1}}\right)$$

$$\tilde{\mu}^i_{s^t} q_{s^t} \leq \sum_{s^{t+1,-}=s^t} \tilde{\mu}^i_{s^{t+1}} \left(q_{s^{t+1}} + R_{s^{t+1}}\right)$$

where $\{\mu^i_{s^t}\}_{s^t}$ and $\{\tilde{\mu}^i_{s^t}\}_{s^t}$ are the l^1 components of two supergradients of U^i at $x(z^i, i)$[5].

Euler conditions hold with equality when all supergradients of U^i at $x(z^i, i)$ have the same l^1 components[6]. If that is the case, that common l^1 component is collinear with p, the l^1 component of the AD price (due to Equation (2). Moreover, in this case, efficient allocations can always be implemented sequentially.

Condition **U1**: an allocation x is said to satisfy Condition **U1** if, for any consumer i, all supergradients of U^i at x^i have the same l^1 component.

In other words, Condition **U1** requires, for every consumer i, the existence of the directional derivatives of U^i at x^i along every canonical direction e_{s^t}.

Remark 2. By Proposition 1, an efficient allocation $(x^i)_i \ggg 0$ satisfies Condition **U1** if, for all i,

1. $\inf_t x^i_{\sigma_t}$ is attained at most in one t and it is not a cluster point of $\left\{x^i_{\sigma_t}\right\}_t$, or

2. the infimum of $\left\{x^i_{\sigma_t}\right\}_t$ is never attained

for all path σ such that $\beta^i(\sigma) > 0$ $\mathbb{P}$-a.c..

An efficient allocation $(x^i)_i$ satisfying Condition **U1** is such that x^i can be attained as an optimal consumption in the sequential problem SEQ^i, for some initial holdings $z^i_{j,0}$ and sequential endowments $\{\omega^i_{s^t}\}_{s^t}$, only if the following Euler equations hold at every node s^t

[5]More precisely, the former is the supergradient that takes the highest value at the direction $v(s^t) := -q_{s^t} e_{s^t} + \sum_{s^{t+1,-}=s^t} \left(q_{s^{t+1}} + R_{s^{t+1}} e_{s^{t+1}}\right)$ and the latter is the one that takes the lowest value at $v(s^t)$. See Lemma 11 in Araujo et al. (4).

[6]Or under the weaker condition that the l^1 components of all supergradients take the same value at $v(s^t)$.

$$p_{s^t} q_{s^t} = \sum_{s^{t+1,-}=s^t} p_{s^{t+1}} \left(q_{s^{t+1}} + R_{s^{t+1}} \right) \tag{5}$$

Moreover, if $(x^i)_i \ggg 0$ satisfies Condition **U1**, the following *transversality condition* must also hold at a solution z^i to SEQ^i such that $x(z^i, i) = x^i$ (for some $z^i_{j,0}$ and $\{\omega^i_{s^t}\}_{s^t}$),

$$\nu^i \left(x - \omega \right) \geq \limsup_t \sum_{s^t} \mu^i_{s^t} q_{s^t} z_{s^t} \tag{6}$$

where μ^i and ν^i are the l^1 and the pure charge components, respectively, of some supergradient of U^i at $x(z^i, i)$

The necessity of this transversality conditions can be established as in the deterministic case (Proposition 4 in Araujo et al. (4)).

However, Euler and transversality conditions are not sufficient. Under impatience, a uniform upper bound on short sales together with a uniform impatience hypothesis[7] would ensure the sufficiency. For preferences are given by Equation (1), the problem is even harder, as consumers can find long-run improvement strategies for which the asymptotic gain of dishoarding exceeds the cost of investing on the assets along the lifetime (see Araujo et al. (4) and Araujo et al. (5), for the deterministic case). As usual in infinite horizon problems, the problem can be overcome by imposing additional portfolio constraints in the form of inequalities that are basically the converse of the transversality condition that the optimal plan must verify. Such constraints were proposed in Araujo et al. (4) (and replaced in Araujo et al. (5) by taxes play the same role) and when extended to the stochastic case take the following form:

Portfolio constraint (P1):

$$\limsup_t \sum_{s^t} p_{s^t} q_{s^t} z_{s^t} \geq \nu(x(z, i) - \omega^i), \tag{7}$$

where $\pi = p + \nu$ is the AD price. Assuming Condition **U1**, we can consider also the following alternative portfolio constraint (P2)

$$\limsup_t \sum_{s^t} p_{s^t} q_{s^t} z_{s^t} \geq \tilde{\nu}^i(x(z, i) - \omega^i), \tag{8}$$

where $\tilde{\nu}^i$ is collinear with the pure charge component $(\rho^i \tilde{\nu}^i)$ of the supergradient that takes the highest value in the AD net trade $(x^i - \omega^i)$ (where ρ^i is the multiplier given by Equation (2).

[7] This would hold for uniformly bounded sequential endowments and exponential discounting, but would fail under hyperbolic discounting, see Pascoa et al. (11)

We denote by SPj^i the optimization problem of agent i when constraints (Pj) are added to problem SEQ^i, for $j = 1, 2$.

An *equilibrium for the sequential economy* with constraints (Pj), given initial holdings $z^i_{j,0}$ and sequential endowments $\{\omega^i_{s^t}\}_{s^t,i}$, is a vector (q, z) such that z^i solves problem SPj^i at prices q, given $z^i_{j,0}$ and $\{\omega^i_{s^t}\}_{s^t,i}$.

We say that an AD equilibrium $\left(\pi, (X^i)_{i=1}^I\right)$ for the AD economy with endowments $W^i_{s^t}$ is implemented sequentially using constraints (Pj) if there exist initial holdings $z^i_{j,0}$ for the assets (i) satisfying $\omega^i_{s^t} \equiv W^i_{s^t} - \sum_j R_{j,s^t} z^i_{j,0} \geq 0$, for each s^t, and (ii) so that some sequential equilibrium (q, z) under (Pj), given $z^i_{j,0}$ and $\{\omega^i_{s^t}\}_{s^t,i}$, is such that $x(z^i, i) = X^i$.

Lemma 1. *Consider an AD equilibrium $((X^i)_i, \pi)$ with $(X^i)_i \gg 0$ and Condition (U1). Let (p, ν) be the decomposition of π in its l^1 and pure charge components. Then, $(X^i)_i$ can be implemented sequentially under (P1) if and only if initial holdings for the assets and asset prices satisfy the following condition*

$$0 = \sum_j \left(z^i_{j,0} \left(p_1 q_{j,1} - \sum_{s^t} p_{s^t} R_{j,s^t} - \nu\left(R_j\right) \right) \right) \tag{9}$$

Moreover, the implementing portfolio z^i is such that

$$\lim_t \sum_{s^t} p_{s^t} q_{s^t} z^i_{s^t} = \nu(x(z^i, i) - \omega^i) \tag{10}$$

Proof. Let us see that sequential budget set is contained in the AD budget set if and only if Equation (9) holds. In fact, let $\{x_{s^t}\}_{s^t,i}$ and $\{z_{s^t}\}_{s^t,i}$ satisfy Equations (3) and (4), given $\{q_{s^t}\}_{s^t,i}$ and z^i_0. By Equation (5) and (P1) we have $p.(x - \omega^i) \leq p_1 q_1 z^i_0 - \nu(x - \omega^i)$. Then, $\pi.(x - W^i) \leq p_1 q_1 z^i_0 - \pi.(W^i - \omega^i)$ where $p.(W^i - \omega^i) = \sum_j z^i_{j,0} \sum_{s^t} p_{s^t} R_{j,s^t}$ and $\nu.(W^i - \omega^i) = \sum_j \nu(R_j) z^i_{j,0}$. That is, $\pi.(x - W^i) \leq 0$ if and only if Equation (9) holds.

Moreover, for $X^i = x(z^i, i)$ satisfying Equations (3) and (4) as equalities we have $p.(X^i - \omega^i) = p_1 q_1 z^i_0 - \lim_t \sum_{s^t} p_{s^t} q_{s^t} z^i_{s^t}$ and, therefore, $p.(X^i - W^i) = p_1 q_1 z^i_0 - \sum_j z^i_{j,0} \sum_{s^t} p_{s^t} R_{j,s^t} - \lim_t \sum_{s^t} p_{s^t} q_{s^t} z^i_{s^t}$. Now, $\nu.(W^i - \omega^i) = \sum_j \nu(R_j) z^i_{j,0}$. Hence, $\pi.(X^i - W^i) = 0$ and Equation (9) imply Equation (10). This concludes the proof. □

When constraint (P2) is used instead we get the following implementation result,

Lemma 2. *Consider an AD equilibrium* $((X^i)_i, \pi)$ *with* $(X^i)_i \gg 0$ *and Condition (U1). Let* (p, ν) *be the decomposition of* π *in its* l^1 *and pure charge components. Then,* $(X^i)_i$ *can be implemented sequentially under (P2) if and only if initial holdings for the assets and asset prices satisfy the following condition*

$$\tilde{\nu}^i\left(X^i-\omega^i\right)-\nu\left(X^i-\omega^i\right)=\sum_j\left(z^i_{j,0}\left(p_1q_{j,1}-\sum_{s^t}p_{s^t}R_{j,s^t}-\nu\left(R_j\right)\right)\right). \tag{11}$$

Moreover, the implementing portfolio z^i *is such that*

$$\lim_t \sum_{s^t} p_{s^t}q_{s^t}z^i_{s^t} = \tilde{\nu}(x(z^i,i)-\omega^i). \tag{12}$$

Proof. As the implementing portfolio plan z^i must satisfy the transversality condition, Equation (6), and constraint (P2), we see that Equation (10) must hold. Now, $p.(X^i-\omega^i) = p_1q_1z^i_0 - \lim_t \sum_{s^t} p_{s^t}q_{s^t}z^i_{s^t}$ and, therefore, $\pi.(X^i - \omega^i) = p_1q_1z^i_0 - (\tilde{\nu}-\nu).(X^i-\omega^i)$. As $\pi.(W^i-\omega^i) = \sum_j z^i_{j,0}(\sum_{s^t} p_{s^t}R_{j,s^t} + \nu(R_j))$, we see that $\pi.(X^i - W^i) = 0$ if and only if Equation (11) holds.

The optimality of z^i for problem $SP2^i$ follows from Equations (5) and (10) by a supergradient estimation, as in Proposition 2 in Araujo et al. (5)). This concludes the proof. □

4.1 Characterization of Efficient Bubbles and their Crashing

As in Santos and Woodford (3), a rational bubble at a node s^t is defined as the difference between the price and the *fundamental value* of the asset at this node,

$$q_{s^t} - \frac{1}{a_{s^t}}\sum_{r=t+1}^{\infty}\sum_{s^{r,-(r-t)}=s^t} a_{s^r}R_{s^r} \geq 0,$$

where $\{a_{s^t}\}_{s^t}$ are state prices given by non-arbitrage conditions.

When implementing an efficient allocation $(X^i)_i \gg 0$, satisfying Condition (U1), it follows from Equation (5) **that the state price is the** l^1 component $\{p_{s^t}\}_{s^t}$ of the AD price and, we have that

$$p_{s^t}q_{s^t} - \sum_{r=t+1}^{\infty}\sum_{s^{r,-(r-t)}=s^t} p_{s^r}R_{s^r} = \lim_{r\to\infty}\sum_{s^{r,-(r-t)}=s^t} p_{s^r}q_{s^r}. \tag{13}$$

Therefore, the existence of efficient bubbles at a node s^t depends on the asymptotic behavior of the subtree that starts at this node. This implies that if a bubble crashes at some node s^t, a bubble can not reappear at any successor of s^t.

Proposition 2. *For any efficient allocation implemented in a sequential economy with a complete set of long-Lived assets. If there exists one node s^t and one asset j in which there is no bubble for the asset price $q^j_{s^t}$ then there is no bubble for the asset price $q^j_{s^r}$ for any state s^r sucessor of s^t.*

However, the crashing at s^t does not mean that there is no bubble in other subtrees that do not have s^t as a root.

Now, let us analyze conditions for the occurrence of bubbles. To do so, we will establish a relationship between bubbles and the existence of pure charges in the super-gradient of the utility function.

On one hand, pure charges are related to the agents' concerns about worst events and, on the other hand, the bubble captures the asymptotic behavior of the asset price. Hence, the occurrence of bubbles seems to be strongly related to the asymptotic behavior of the agents' consumption plans and, more precisely, the existence of the pure charges in the supporting prices of these plans. To be more precise, it follows from Lemma 1 that, $p_1 q_{j,1} = \sum_{s^t} p_{s^t} R_{j,s^t} + \nu(R_j)$ if constraints (P1) are imposed or also under constraints (P2) when $x^i_t - \omega^i_t$ converges for any i.

Moreover, under $SP2^i$, we have $p_1 q_{j,1} \geq \sum_{s^t} p_{s^t} R_{j,s^t} + \nu(R_j)$ for every j and, actually, $p_1 q_{j,1} > \sum_{s^t} p_{s^t} R_{j,s^t} + \nu(R_j)$ for at least one asset j, if $x^i_{s^t} - \omega^i_{s^t}$ doesn't converge for some i and $R \geq 0$.

These facts lead to the following results.

Proposition 3. *Given an efficient allocation such that $(X^i)_i \gg 0$ and Condition (U1) holds, if $\partial U^i(x^i) \subseteq \ell^1$ for all agent i, there is no bubble for any asset in positive net supply.*

Therefore, to find a bubble in assets in positive net supply, we must have that $\partial U^i(x) \nsubseteq \ell^1$. This result is consistent with what Santos and Woodford (1997) had shown (see (3)). Moreover, Equation (11) ensures that the converse is also true under some circumstances.

Proposition 4. *Let $(X^i)_i \ggg 0$ be an efficient allocation satisfying Condition (U1). If*

1. $x^i_{s^t} - \omega^i_{s^t}$ does not converge in a set of positive measure in $\mathbb{P}$ for some i, or

2. $\nu(R) > 0$,

then there is a bubble at the initial node for some asset j in positive net supply.

The following propositions study the condition for the occurrence or not of a bubble in the prices of assets at a node s^t. This conditions deal with what happens in paths in the sets A_1, A_2, A_3 and A_4 defined in Proposition 1.

Proposition 5. *Given an efficient allocation $(X^i)_i \ggg 0$ satisfying Condition (U1) and any node s^t, if there is one agent i such that in all paths σ that contain s^t, the infimum of X^i_σ is not a cluster point $\mathbb{P}$ almost certainly in the set $\{\sigma : \sigma_t = s^t \text{ and } \beta(\sigma) > 0\}$, then there is no bubble for any asset at the node s^t.*

Note that this proposition is driven by what happens in the set A_2 of paths, considered in Proposition 1 and the comments that followed that result.

This implies that, when a bubble occurred before a node s^t was reached, for it to burst at s^t a sufficient condition is that, almost surely, in the paths that follow on from node s^t, the infimum of consumption is attained in a finite number of dates but it is not a cluster point.

Let us present some conditions that ensure a positive bubble at a node s^t. These conditions are related to the infimum of consumption not being attained in finite time.

However, for a node s^t beyond the initial node 1, we can only establish conditions ensuring that there will be a bubble in the price of *some* asset.

Proposition 6. *Given an efficient allocation $(X^i)_i \ggg 0$ satisfying Condition (U1) and any state s^t, if there exists one agent i and a subset of paths A^i with the following properties*

1. A^i has positive $\mathbb{P}$-probability,

2. any σ in A^i satisfies that $\sigma_t = s^t$, and

3. for each σ in A^i, the infimum of $\left\{X^i_{\sigma_r}\right\}_{r\in\mathbb{N}}$, the sequence of consumption of agent i in the path σ, is a cluster point never attained.

Then there is a sequential implementation such that there is a bubble in the price of some asset at the node s^t if (1) there exists at least one asset in positive net supply such that $\nu\left(R^{j,s^t+}\right) > 0$ or (2) the net trade $X^i - \omega^i$ is not convergent in A^i.

From Proposition 5 and 6 and their proofs, the existence of efficient bubbles in this economy is related to the existence of positive pure charges in the super-gradients of the agents as mentioned for deterministic economies in Araujo et al (2011) (see (4)). In this case, Proposition 1 allows us to know the conditions that must be satisfied to ensure the existence of pure charges in the super-differential of each agent.

Note that, in Proposition 6, the set A^i is a subset of the A_1 defined in Proposition 1.

Similarly to the deterministic case, the desire of the wary agent of increasing consumption in the worst events of the subtree that contains the node s^t, produces a pure charge in the AD price that implies the occurrence of bubbles for the set of assets traded at that node.

4.2 Variation of prices in presence of Efficient Bubbles and Crashing

If, at some date, there is a bubble in some nodes, and there is no bubble in some other nodes, it means that, before that, the bubble has crashed in at least one node. Therefore, if we analyze the asset prices in these subtrees, we can analyze the changes in assets prices in the short and the long run.

Under the conditions described in Proposition 5 and 6, it is possible to know when and where, in the tree, there is a bubble and also when it will crash. Since these bubbles increase the price above the fundamental value of the assets, the crashing of them will, naturally, increase the dispersion of the asset prices across all successors s^t of a node s^r where the price of some asset had a bubble.

After the bubble crashes at the node s^t, the fundamental value of the asset and the market price are equal, this means that

$$\max_{\left\{s^r : s^{r,-(r-t)} = s^t\right\}} p_{s^r} q_{j,s^r} = \max_{\left\{s^r : s^{r,-(r-t)} = s^t\right\}} \left\{ \sum_{k>r} \sum_{s^{k,-(k-r)} = s^r} p_{s^k} R^j_{s^k} \right\} \to 0 \text{ when } r \to \infty.$$

However if there is a path σ such that there is always a bubble for the asset j we have that

$$p_{\sigma_t} q_{j,\sigma_t} = \sum_{r>t} \sum_{s^{r,-(r-t)}=\sigma_t} p_{s^r} R^j_{s^r} + \lim_r \sum_{s^{r,-(r-t)}=\sigma_t} \sum_j p_{s^r} q_{j,s^r}, \tag{14}$$

and, as we have already mentioned, there exists a relationship between the existence of bubbles and the pure charges in the super-gradient of the agents. Therefore, by proposition 1, we have:

Proposition 7. *Under the conditions of Proposition 1 and for any path σ, the component in Equation (14) that constitutes the bubble of any asset j at the node σ_t tends to zero as $t \to \infty$.*

This means that the bubble is being distributed among all nodes where the pure charge is positive, reducing its weight in the price of the asset in which the bubble is maintained.

Nevertheless it does not mean that the price will be bounded, in fact we have:

Proposition 8. *Under the conditions of Proposition 6 satisfying (1') $R^j|_{A_i} \ggg 0$ instead of (1). For any path $\sigma \in A^i$, we have that*

$$\limsup_{t\to\infty} \left\{ \frac{\lim_{r\to\infty} \sum_{s^{r,-(r-t)}=\sigma_t} \sum_j p_{s^r} q_{j,s^r}}{\mathbb{P}\left([\tilde{\sigma} : \tilde{\sigma}_t = \sigma_t]\right)} \right\} = \beta < \infty, \tag{15}$$

and there exists a path $\sigma \in A^i$ such that

$$\liminf_{t\to\infty} \left\{ \frac{\lim_{r\to\infty} \sum_{s^{r,-(r-t)}=\sigma_t} \sum_j p_{s^r} q_{j,s^r}}{\mathbb{P}\left([\tilde{\sigma} : \tilde{\sigma}_t = \sigma_t]\right)} \right\} = \alpha > 0. \tag{16}$$

And since there is a relationship between $\mathbb{P}\left([\tilde{\sigma} : \tilde{\sigma}_t = \sigma_t]\right)$ and μ_{s^t} due to the Euler equations (see Equation (5)), we will have:

Corollary 1. *Under the conditions exposed in Proposition 8, the bubble will tend to infinity when $t \to \infty$.*

This result is also consistent with what Santos and Woodford (1997) had shown (see (3)), and it implies in our framework that, when t is large, there are large variations of prices between states due to the existence of bubbles in some nodes s^t, which make the asset prices goes to infinity in several paths, but not in every path. Moreover, the set of paths in which the bubble crashes can have positive value.

Therefore, when there is a crashing of a bubble at the node s^t, the variations of the asset prices in a subtree that contains the node s^t and the predecessor of s^t tend to infinity in the long-run. Therefore, the existence of bubble is associated with high volatility of assets' prices if there is a chance of the bubble to crash.

4.3 Example

As can be noticed through this article, there is a variety of possibilities for the occurrence of bubbles, as the following example illustrates. It shows that bubbles can occur with positive probability even in the presence of an infinite number of paths in which the bubbles crash.

Example 1. Consider an economy with two agents and two states, $N = \{1,2\}$. Let $u^i(x) := \ln(x)$ be the utility index[8] and let $\xi^i_t := {}^1\!/_{2^t}$, $\beta^i = \beta > 0$ for each $i = 1,2$, $W : \cup_{t\in\mathbb{N}}\{1\} \times N^t \to \mathbb{R}_+$ be given by

$$W_{s^t} = \begin{cases} 8 + {}^1\!/_{2^{t-4}} & \text{if } s^{t,-(t-2)} = (1,2) \text{ and } (\exists k \in \mathbb{N} \text{ such that } 2k+1 \leq t \text{ and} \\ & s^{t,-(t-2k+1)}_{2k+1} = 2 \text{ or } s^{t,-k}_{t-k} = 1 \forall k = t-2), \\ 9 & \text{if } s^t = (1,2), (1,2,1,1) \text{ or } s^{t,-} = (1,2,2), \\ 10 & \text{if } t = 1 \text{ or } s^{t,-} = (1,2), \\ 11 & \text{otherwise if } t \text{ is even}, \\ 12 & \text{otherwise if } t \text{ is odd}, \end{cases}$$

$W^1_{s^t} = W_{s^t} + A_t$ and $W^2_{s^t} = W_{s^t} - A_t$ where A_t is 1 if t is even and $-{}^1\!/_2$ if t is odd. Within each date, all nodes are equally likely to occur, that is, $\mathbb{P}[\sigma : \sigma_t = s^t] = {}^1\!/_{2^{t-1}}$ for all $s^t \in \{1\} \times N^{t-1}$. As $W^i \ggg 0$, we can find an AD equilibrium. Given $a : \{1,2\} \to \mathbb{R}_{++}$, the consumption plan $x^i = a(i)\left(W^1 + W^2\right) = 2a(i)W$ is optimal under the budget constraint $\pi x \leq \pi W^i$ when the prices are given by

$$\pi x = \int \left(\sum_t \frac{x_{\sigma_t}}{2^t W_{\sigma_t}}\right) d\mathbb{P}(\sigma) + \beta \left({}^{x_{(1)}}\!/_{20} + {}^{x_{(1,2)}}\!/_{72} + \sum_{s\geq 2} \frac{x_{\tilde{\sigma}_{2s+1}}}{2^{2s+1}(8 + {}^1\!/_{2^{2t-3}})} + \int_{\bigcup_{t\in\mathbb{N}}[\sigma:\sigma_{2t+1}=(1,2,1,1,\ldots,1,2)]} \frac{\mathrm{LIM}_\sigma(x_\sigma)}{8} d\mathbb{P}(\sigma) \right) \tag{17}$$

where $\tilde{\sigma}_{2t+1} = (1,2,1,1,\ldots,1)$ and $a(i) = {}^{\pi(W^i)}\!/_{\pi(W)}$. One possible generalized limit is the Banach limit, B, which satisfies that $B\left((x_t)_{t\in\mathbb{N}}\right) =$

[8]In this case, each U^i is defined in all $x \gg 0$ uniformly bounded, but the preference induced by this utility function is well-defined for $x \geq 0$ uniformly bounded. Therefore, the utility function can be extended to the case in which $U^i(x) = -\infty$ for all $x = 0$ in at least one node.

$\lim_{n\to\infty} {}^1\!/\!_n \sum_{t=1}^{n} x_t$ when the limit of the sequence $\left\{{}^1\!/\!_n \sum_{t=1}^{n} x_t\right\}_n$ exists. Under this condition (π, x^1, x^2) is an AD equilibrium.

We consider two long-lived assets, one that pays 1 in each state 1 and 0 in each state 2, and the other one that pays 0 in each state 1 and 1 in each state 2. We can choose $q_1 \in \mathbb{R}^2_{++}$ and $z^i_0 > 0$ such that (x^i, z^i) is implemented sequentially, which implies that Equations (9) and (10) must be satisfied, that is, $0 = \sum_{j=1}^{2} \left(z^i_{j,0} \left(p_1 q_{j,1} - \sum_{s^t} p_{s^t} R_{j,s^t} - \nu\left(R_j\right)\right)\right)$ and $\lim_t \sum_{s^t} p_{s^t} q_{s^t} z^i_{s^t}(z^i_0) = \nu(x^i - \omega^i)$ where $\left\{z^i_{s^t}\left(z^i_0\right)\right\}_{s^t}$ is the portfolio that implements x^i with initial holdings z^i_0, ν is the pure charge of the AD price and $\{p_{s^t}\}_{s^t}$ is the l^1 component of the AD price (which is also the no-arbitrage state prices process) defined by

$$p_{s^t} = \begin{cases} \int_{[\sigma:\sigma_t = s_t]} \frac{1}{2^{t-1} W_{\sigma_t}} d\mathbb{P}(\sigma) & \text{if } s^{t,-(t-2)} = (1,2) \text{ and } \exists k \in \mathbb{N} \text{ such that} \\ & 2k+1 \leq t \text{ and } s^{t,-(t-2k+1)}_{2k+1} = 2, \\ \int_{[\sigma:\sigma_t = s_t]} \frac{1}{2^{t-1} W_{\sigma_t}} d\mathbb{P}(\sigma) + & \\ \sum_{s\geq 2} \frac{1}{2^{2s+1}(8 + 1/2^{2t-3})} & s^{2k+1} = (1,2,1,1,\ldots,1) = s^t \text{ for some } k \geq 1, \\ {}^1\!/\!_{20} + {}^\beta\!/\!_{20} & t = 1, \\ {}^1\!/\!_{72} + {}^\beta\!/\!_{72} & \text{if } s^t = (1,2), \end{cases}$$

$\{q_{s^t}\}_{s^t, t\geq 2}$ is defined based on what q_1 is[9] using the Euler equation $p_{s^t} q_{s^t} = p_{s^{t,1}}(q_{s^{t,1}} + R_{s^{t,1}}) + p_{s^{t,2}}(q_{s^{t,2}} + R_{s^{t,2}})$ for all s^t with $t \geq 1$. To ensure that the efficient allocation solves a sequential optimization problem of each agent, a (P1) constraint is imposed, $\lim_t \sum_{s^t} p_{s^t} q_{s^t} z_{s^t} \geq \nu(x(z,i) - \omega^i)$ where ν is the pure charge of the AD price.

Asset prices $(q^1_{s^t}, q^2_{s^t})_{s^t}$ satisfy condition (1) of Proposition 6: $\nu\left(R^{j,s^t+}\right) > 0$ for $j = 1,2$ and all nodes s^t such that the set of paths that contains s^t in which the infimum is not attained has positive $\mathbb{P}$-measure. The pattern for W implies hedging portfolios for which the infimum of consumption is a cluster point along many paths but will be no longer a cluster point when we enter some subtrees. This makes the bubble crash at the root of such subtrees, more precisely, at $s^2 = (1,1)$, $s^4 = (1,2,1,2)$, $s^6 = (1,2,1,1,1,2)$, $\ldots$, $s^{2t} = (1,2,1,\ldots,1,2)$, $\ldots$. In spite of the infinite number of paths where the bubbles crash, the proportion of paths with a positive bubble at any state is positive, in fact,

$$\begin{aligned} \mathbb{P}\left[\sigma : \inf_s X_{\sigma^s} < X_{\sigma_t}\, \forall t \in \mathbb{N}\right] &= \mathbb{P}[\sigma\colon \inf_s W_{\sigma^s} < W_{\sigma_t}\, \forall t \in \mathbb{N}] \\ &= \mathbb{P}[\sigma\colon \text{the bubble in the path } \sigma \text{ does not crash}] \\ &= {}^1\!/\!_3. \end{aligned}$$

[9]Note that $q_{j,1}$ also satisfies that $q_1 = \frac{1}{p_1}\left(\sum_{s^t} p_{s^t} R_{j,s^t} + \nu\left(R_j\right)\right)$.

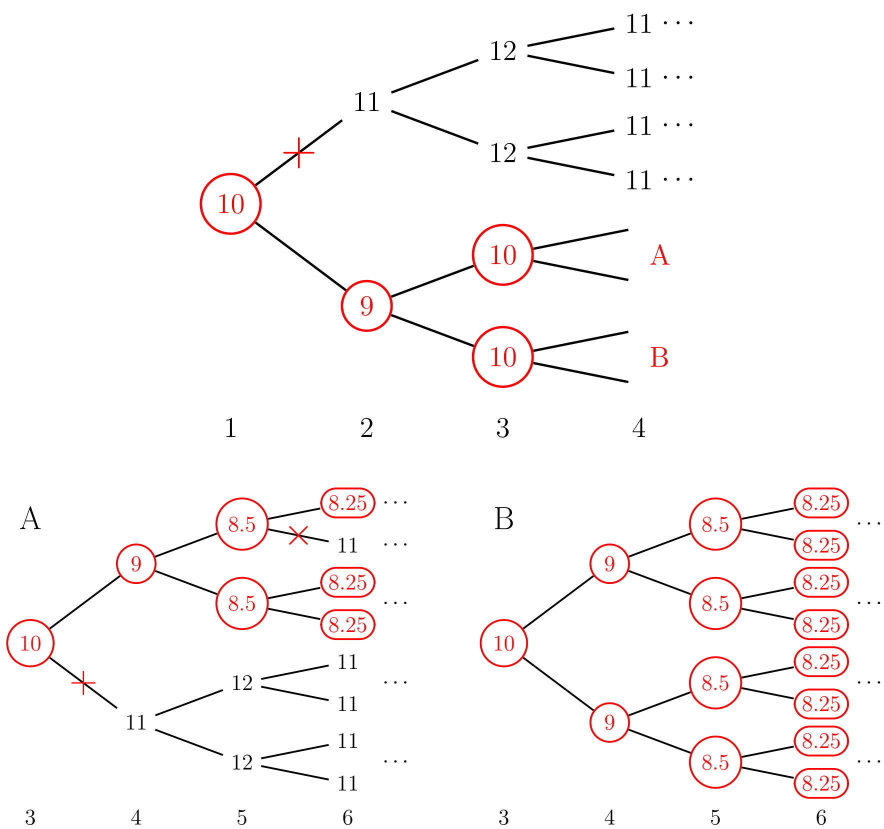

Figure 1: Distribution of W, existence of bubbles (with red circles) and crashing (with red $\times$'s)

This example points out that in stochastic economies, the existence of bubbles in a considerably large set of paths is consistent with the crashing of bubbles in a large set of nodes.

As in the deterministic case, the existence of rational efficient bubbles is related to the lack of impatience of the agents. However, the stochastic case is richer, as it allows for efficient bubbles to crash at the root of sub-trees for which the worst outcome is not a distant outcome. At the root of such a sub-tree, agents concern with worst outcomes shifts from the infinity to a finite future date and this makes the bubble burst: there is no longer a hedging reason to overvalue of the asset by attributing a value to it at infinity (the bubble) and, therefore, the asset price becomes equal to the fundamental value.

5 Concluding Remarks

Bubbles cannot burst in deterministic inter-temporal economies. This unappealing fact suggests looking at other settings that allow for a larger variety of bubble patterns. We propose as a natural setting a stochastic inter-temporal economy where infinite lived consumers have some lack of impatience. When agents are worried about distant losses, there will be room for pure charges in the AD prices, which will generate bubbles in the prices of assets that implement sequentially such efficient allocations. These results (Section 4.1) are independent on the specific form of caring for those distant losses.

We consider a general form for the utility function which is compatible with the model proposed by Schmeidler (see (7)) and Gilboa and Schmeidler (1989) (see (6)) and is a generalization of the $\varepsilon-$*contamination* utility function.

The way we model lack of impatience in a stochastic framework is compatible with the existence of AD equilibrium, as in the result by Bewley (see (8)). For such preferences, the existence of a positive pure charge in the AD price, pricing the asymptotic consumption of the agents, translates into the existence of efficient bubbles at the first date of the sequential economy. Moreover, such relation is still present at other nodes and, therefore, to analyze the occurrence of efficient bubbles at any node, it is enough to know how the efficient consumption plans behave in the sub-tree that rooted at that node. For the paths starting at that node, are the worst consumption outcomes attained only at some future successor node? or are they cluster points? If there was already a bubble before that node was reached, the former can make it burst, whereas the latter sustains the bubble. A bubble bursts when the value assigned to the asset's hedging at infinity can be dispensed with, as agents' concerns about worst outcomes shift from infinite to finite events at the root of that sub-tree.

We also analyzed the volatility in asset prices caused by bubbles and their crashes, showing that the crashing of bubbles will increment the variation in the asset price in the long-run,across nodes of a same distant date. For some paths the bubble has crashed, while for other paths it is still there, which makes the price volatility increase.

Finally we gave an example that illustrates how stochastic economies allow for bubble patterns that are much richer and more interesting than the one observed in deterministic economies. Crashes can occur at an infinite number of nodes and yet bubbles persist along infinitely many paths.

A Other proofs

Proof of Proposition 1. The idea of the proof is to separate each set A_1, A_2 and A_3 to analyze them separately, and then, apply the dominated convergence theorem to have the result. For each set A_1, $A^K_{2,t_1,\dots,t_K}$, $A_{3,t_1,t_2\dots}$ and σ belonging to any of the previous sets, we have that the analysis that can be done in the path σ is analogous to the deterministic case with a, ε-*contamination* utility function (see Araujo et al. (4)). Therefore the results will be true for each path σ that belongs to any of the sets described before.

Since the collection of all sets that have been described before is $2-2$ disjoint and non enumerable, there is an enumerable sub-collection with positive measure. Therefore we can rewrite the utility function in terms of these enumerable sub-collection only, and apply the deterministic case in each path that belongs to any of this subsets of the collection. Finally, if we apply the dominated convergence theorem (for a collection of generalized limits that are measurables in $(\{1\} \times N^\infty, \mathcal{N})$), we conclude one part of the proof.

To prove the other part, notice that what we have done is to prove that the integral in σ of elements of the super-differential of the utility function in each path σ, are in the super-differential of the utility function. And it can be easily observed that to prove second part is enough to prove that the super-differential is contained in the composition between the integral and the super-differential for each path.

Using some results of non differential analysis in Banach spaces, we have that, under the condition that we exposed before, the definition of super-gradient and Clarke super-gradient are equivalents (see Clarke (12), page 36 Proposition 2.2.7). We also have that, under our hypothesis, the Clarke super-differential of the utility function is contained in the integral in σ of elements in the Clarke super-gradients of $\int_{\mathbb{N}} u \circ X_{\sigma_t} d\zeta_i(t) + \beta(\sigma) \inf_t u \circ X_{\sigma_t}$ for each σ (see Clarke (12), page 76 Proposition 2.7.2). Which concludes the proof of the proposition. $\square$

Proof of Proposition 5 and Proposition 6. Let us suppose that we implement the efficient allocation by the portfolio constraint (P2). Note that, for each node s^t, the optimal allocation is also efficient in the subtree generated by s^t. This subtree will be denoted by s^t+. Therefore there is $\left\{W^{i,s^t}\right\}_i$ new "endowment allocation" such that:

1. $\sum_i W^{i,s^t}_{s^k} = \sum_i W^i_{s^k}\ \forall s^k$, $\pi W^{i,s^t} = \pi W^i\ \forall i$ where π is the AD price,

2. $W_{\sigma_r}^{i,s^t} = X_{\sigma_r}^i$ for all σ and r such that $r \leq t$ or σ^r is not in the subtree generated by s^t[10], and

3. $W_{s^{t+k}}^{i,s^t} = W_{s^{t+k}}^i$ $\forall k \geq 1$ for the rest of the endowment distributions in the subtree generated in s^t.

Note that for some nodes s^t the new "endowments allocation"might be negative in some states of the economy for some agents. However, it does not imply that we cannot analyze the economy that it defines[11] and its relationship with the original AD economy.

To establish this relationship, let us restrict the AD economy such that the agents maximize their consumption in the subtree generated by s^t only, that is, the agent consumption set will be:

1. $\left\{X_{\sigma_r}^i\right\}$ for all path σ and r such that $r \leq t$ or σ_r is not in the subtree generated by s^t, and

2. $\mathbb{R}_+$ for all s^r in the subtree generated by s^t.

Since the wealth of every agent and the efficient allocation are the same as their counterparts in the initial AD equilibrium, we have that the initial equilibrium price is in fact an equilibrium price for this restricted economy.

Let us analyze the stochastic sequential economy defined by the initial and the "new" endowment allocation for each s^t. To do so, let us define the new endowment distribution $\left\{\omega^{i,s^t}\right\}_i$ as $\omega^{i,s^t} = W^{i,s^t} - \sum_j R_j z_{j,s^t}^i$. Since the assets' prices are given by the Euler equations, and the FOC are the same in both cases, the prices are also the same in both economies. Using the same optimality conditions that were exposed in Araujo et al. (2019) (see (5)), we know that using the pure charge, $\nu_{s^t}^i$, the one that takes the highest value on the net trade $\left\{X^i - \omega^{i,s^t}\right\}$, we can implement the efficient allocation if we have

[10]This includes $\left\{s^{t,-(j)}\right\}_{j=1}^{t-1}$ all the predecessors of s^t.

[11]The economy defined by $\left\{W^{i,s^t}\right\}_i$ only differs with a standard economy on the fact that it may have a negative endowment in at most one date or event. In the case that you want to avoid the negative endowment in the node s^t, there is K^{s^t} such that for each $K \geq K^{s^t}$ such that you can choose an endowment allocation W^{i,s^t} such that $W_{s^{t+k}}^{i,s^t} > 0$ for $k = 0, \ldots, K$ and all s^{t+k} such that $s^{t+k,-k} = s^t$ and $W_{s^{t+K+r}}^{i,s^t} = W_{s^{t+K+r}}^i$ for $r \in \mathbb{N}$ and all s^{t+K+r} such that $s^{t+K+r,-(K+r)} = s^t$ which satisfies the other conditions mentioned above, conditions 1, 2 and, on the long run, 3.

$$\nu^i_{s^t}\left(X^i(z)-\omega^{i,s^t}\right)\leq \limsup_r \left(\sum_j \sum_{s^{r,-(r-t)}=s^t} p_{s^r} q_{j,s^r} z_{j,s^r}\right)$$

for every portfolio z, and with equality for X^i. Then, in order to implement sequentially this allocation, we have that

$$\lim_r \sum_{s^{r,-(r-t)}=s^t} \sum_j p_{s^r} q_{j,s^r} = \sum_j z^i_{j,s^t}\nu\left(R_{j,s^t+}\right)+\nu^i_{s^t}\left(X^i-\omega^{i,s^t}\right)-\nu\left(X^i-\omega^{i,s^t}\right), \tag{18}$$

and since we have that

$$p_{s^t} q_{j,s^t} = \sum_{r>t} \sum_{s^{r,-(r-t)}=s^t} p_{s^r} R^j_{s^r} + \lim_r \sum_{s^{r,-(r-t)}=s^t} \sum_j p_{s^r} q_{j,s^r}$$

for each node s^t, the existence of a bubble in the economy at the state s^t is given by the right part of Equation (18). Finally, since the subtree generated by s^t satisfies that $x^i|_{[\sigma:\sigma_t=s^t]}\ggg 0$, Proposition 1 can be used restricted to this set proving that, under the conditions of Proposition 5, the pure charges in the super-gradient are null in this subtree, and that under the conditions of Proposition 6, the pure charges in the subtree are such that the right part of Equation (18) is nonnegative. Moreover, if we sum Equation 18) over the agents we have that

$$\begin{aligned} I \lim_r \textstyle\sum_{s^{r,-(r-t)}=s^t}\sum_j p_{s^r} q_{j,s^r} &= \textstyle\sum_i\sum_j z^i_{j,s^t}\nu\left(R_{j,s^t+}\right)+\sum_i \nu^i_{s^t}\left(X^i-\omega^{i,s^t}\right) \\ &= \textstyle\sum_i\sum_j z^i_{j,0}\nu\left(R_{j,s^t+}\right)+\sum_i \nu^i_{s^t}\left(X^i-\omega^{i,s^t}\right) \\ &\geq 0, \end{aligned}$$

since $\nu^i_{s^t}\left(X^i-\omega^{i,s^t}\right)-\nu\left(X^i-\omega^{i,s^t}\right)\geq 0$ for each i. In the case in which $X^i-\omega^{i,s^t}$ are not convergent for some i, the right hand side is strictly positive.

The case in which we implement the efficient allocation with the portfolio constraint (P1), the right hand side of Equation 18 is equal to $\sum_j z^i_{j,s^t}\nu\left(R_{j,s^t+}\right)$ from which Proposition 5 follows, and Proposition 6 under *(1)*.

□

Proof of Proposition 7. Since:

- the bubble in the economy is characterized by the pure charges that exists in the super-gradient of the agent, more precisely given by 18,
- the pure charges that exist in the super-gradient of the agents are integral in σ of a collection of generalized limits, and

- the probability of each path, $\mathbb{P}(\{\sigma\})$, is zero;

we have that $\lim_r \sum_{s^{r,-(r-t)}=\sigma_t} \sum_j p_{s^r} q_{j,s^r} \to 0$ when $t \to \infty$. $\square$

Proof of Proposition 8. From the proof of Proposition 5 and 6, we know that the bubbles depends completely on the behavior of the pure charges in the super-differential of the optimal allocation in the subtree generated by the analyzed node σ_t. Moreover, Proposition 1 ensures that Equation 15 is satisfied.

Under condition (1') or (2), we have that the right part of Equation 18 is bounded from below by a positive constant multiplied by the capacity evaluated in the set $[\tilde{\sigma} \in A^i : \tilde{\sigma}_r = \sigma_r]$. Since $\mathbb{P}[A^i] > 0$, for each $r \geq t$, there is s^r such that $\mathbb{P}[\sigma \in A^i : \sigma_r = s^r] \geq \frac{\mathbb{P}[A^i]}{2|N|\mathbb{P}[\sigma:\sigma_r = s^r]}$ where $|N|$ is the number of elements in N, obtaining Equation 16. $\square$

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