Discussion Papers in Economics

NO-TRADE AND REFINED EQUILIBRIA FOR SECURED LOANS IN INFINITE HORIZON

By

Mario R. Pascoa
(University of Surrey),
&

Abdelkrim Seghir
(Ajman University).

DP 09/19

School of Economics
University of Surrey
Guildford
Surrey GU2 7XH, UK
Telephone +44 (0)1483 689380
Facsimile +44 (0)1483 689548
Web https://www.surrey.ac.uk/school-economics

ISSN: 1749-5075

No-trade and refined equilibria for secured loans in infinite horizon.

Mário R. Páscoa[a,1] **and Abdelkrim Seghir**[b]

.

Abstract

When loans are secured but subject to utility penalties on default, no-trade equilibria induced by unduly low repayment beliefs can be trivially found for finite horizon economies but not for infinite horizon ones. We illustrate this fact and propose also a refinement of equilibrium that gets rid of spurious no-trade outcomes when they do occur. Known existence results pass this refinement criterion.

Keywords: collateral, Ponzi schemes, incomplete markets.

M. Páscoa acknowledges support from FCT (Portugal) through project PTDC/IIM-ECO/5360/2012.

JEL Classification D52-D53-G33

[1]Corresponding author. E-Mail address: m.pascoa@surrey.ac.uk

a. University of Surrey, Guildford, GU27XH, UK.

b. Ajman University, UAE.

1 Introduction

Finite horizon economies with unsecured loans subject to utility penalties on default, have trivial no-trade equilibria induced by null repayment expectations, as pointed out by Dubey, Geanakoplos and Shubik (2005). These beliefs may be unduly low, in the sense of being inconsistent with agents' willingness to repay.

When loans are secured but subject to such penalties, finite horizon economies also have trivial no-trade equilibria, driven now by minimal repayment expectations (the minimum of the repayment due and the collateral value). These low expectations would always be correct if loans were non-recourse. However, utility penalties may induce agents to repay more and, therefore, mininal repayment expectations may inappropriate. Again, such pessimistic beliefs may be inconsistent with what agents were willing to repay, for example when strictly conscientious agents face marginal penalties above their marginal utilities of income.

However, infinite horizon economies do not always have such no-trade equilibria. No-trade outcomes induced by spuriously low repayment beliefs may leave room for Ponzi schemes or for extensions of such schemes taking also into account utility gains from the consumption of the collateral. When markets are complete all agents can be shown to be unable to do these long-run improvement stratagies, but, for incomplete markets, some agents may improve upon.

Some infinite horizon incomplete markets admit spurious no-trade outcomes, as in an example by Martins-da-Rocha and Vailakis (2012a). Others don't, as in an example provided in this paper, where one of the infinitely lived agents (the one discounting the future fastest) can construct a long-run strategy that improves upon the no-trade outcome.

Nevertheless, when infinite horizon economies admit such spurious no-trade outcomes it may be appropriate to rule them out by designing a refinement of equilibrium. In particular, it is interesting to check that for known existence results, the equilibrium set does not become empty once such a refinement is applied.

To be sure that promise prices are not too high to induce Ponzi schemes, the refinement that we propose has to be milder than a straightforward extension to secured loans of the refinement in Dubey, Geanakoplos and Shubik (2005). The no-trade outcome found by Martins-da-Rocha and Vailakis (2012a) was known to fail the latter and it also fails the former. Moreover, known results on existence of equilibrium pass this refinement criterion. More precisely, refined equilibria exist when penalties are moderate and the collateral does not yield any utility (as in the case of securities or productive assets) or when a real collateral secures nominal promises.

How does our refinement exactly differ from the one in Dubey, Geanakoplos and Shubik (2005)? To rule out trivial no-trade outcomes induced by unduly pessimistic beliefs, these authors used a refinement that checks whether the equilibrium is a limit of equilibria of economies where the government buys and sells an arbitrarily small amount of the promise, with full delivery. In this way, the authors got rid of all no-trade equilibrium with non-full delivery rates, duly or unduly.

However, when eliminating unduly pessimistic expectations, we should not use the same refinement, as it tends to eliminate too many no-trade finite horizon equilibria, also those with duly low delivery rates and duly low promise prices. That is, a straightforward extension of the Dubey-Geanakoplos-Shubik refinement to the collateral model is perfectly adequate in a finite horizon economy but seems to be too strong for its infinite horizon version, as it may not leave room for non-negative haircuts, as desired to avoid Ponzi schemes. For this reason, we look for a refinement that eliminates unduly expectations, whether these are optimistic or pessimistic.

Our refinement requires expectations about delivery rates to belong to the convex hull of the propensities to deliver of agents on the verge of selling. These are the agents that have their Kuhn-Tucker conditions with respect to selling the promise already holding as equalities and are already purchasing some amount of what can serve as collateral. Equivalently, a refined equilibrium can be approximated by equilibria of economies where the government buys a small amount of the promise and uses lump-sum taxes today and lump-sum subsidies tomorrow, to induce small sales.

The next section recalls the infinite horizon model with secured loans subject to utility penalties on default and the conditions for individual optimality. Section 3 discusses how spurious no-trade outcomes may fail arise. Section 4 addresses a refinement of equilibrium. Section 5 provides an example where no-trade equilibria do not exist. Some lengthy proofs are in the Appendix.

2 Secured loans and infinite horizon

2.1 The model

The event tree D has a countably infinite set of dates and there are finitely many branches at each node. Let $\mathbb{N}_0 = \{0, 1, \dots\}$ be the set of dates and ξ_0 be the root of the tree D. Given a node $\xi \in D$, let $t(\xi) \in \mathbb{N}_0$ be the date of node ξ. We denote by $D(\xi)$ the sub-tree that starts at ξ. We write $\xi' > \xi$ if $\xi' \in D(\xi)$ and $\xi' \neq \xi$. The immediate successors of node ξ constitute the set $\xi^+ \equiv \{\eta > \xi : t(\eta) = t(\xi) + 1\}$ while its immediate predecessor is denoted by ξ^-. We will also use the notation $D_T \equiv \{\xi \in D : t(\xi) = T\}$ and $D^T \equiv \bigcup\limits_{t=0}^{T} D_t$.

There are a finite number G of commodities and a finite number J of one-period promises. Let us start by assuming that promises have real returns. This assumption will be modified later, in one of our results.

Promises are secured by collateral, which is not necessarily a durable good, but may also be a productive asset or a security in positive net supply that pays real returns and cannot be short sold. Formally, there is a transformation matrix Y_ξ, of type $G.G$, indicating how commodities of the previous node convert into commodities of the node ξwe may have a non-diagonal. If g is a durable good, the only non-null element in column $(Y_\xi)^g$ is $(Y_\xi)_{gg}$, equal to the depreciation factor. If g is a security, $(Y_\xi)_{gg} = 1$ and, its non-negative dividends are given, for $g' \neq g$, by $(Y_\xi)_{g'g}$. If g is a productive asset the respective non-null column in Y_ξ represents its productive returns on other commodities.

As in Páscoa, M.R. and A. Seghir (2019), there is a finite set I of infinitely lived consumers satisfying the following assumptions

Assumption [E]. Endowments of consumer i of commodity g at node ξ, denoted by $\omega^i_{g\xi}$, satisfy

(i) $\exists W \in \mathbb{R}_{++} : \forall i \in I,\ \forall \xi \in D,\ \sum\limits_{g \in G} \omega^i_{g\xi} \leq W.$

(ii) $\omega_{\xi_0} \gg 0$ and, for $\xi > \xi_0$ and any g, $\omega_{g\xi} > 0$ whenever the $g-$th row of Y_ξ is null.

Let $Y_{\xi_0,\xi_n} = Y(\xi_n)Y(\xi_{n-1})\dots Y(\xi_1)$ for $\xi_{k+1} \in {\xi_k}^+$. The aggregate physical resources available at node ξ are given by $\Omega_\xi = \sum\limits_i W^i_\xi$, where $W^i_\xi = \sum\limits_{\eta \in \{\xi_0,\dots,\xi^-,\xi\}} Y_{\eta,\xi}\, \omega^i_\eta$.

We say that good g is perishable at node ξ if the g-th column of Y_η is null for any $\eta \in \xi^+$.

Assumption [U]. $\forall i \in I$, preferences over consumption are described by a time and state separable utility U^i with instantaneous utility $v^i_\xi : \mathbb{R}^G_+ \longrightarrow \mathbb{R}_+$ such that

(i) v^i_ξ is monotone and concave,

(ii) v^i_ξ is differentiable on $\mathbb{R}^G_{++}$,

(iii) $\forall \alpha \in \mathbb{R}^G_+$ we have $\sum_{\xi \in D} v^i_\xi(\alpha) < \infty$ and

(iv) $\sum_{\xi \in D} v^i_\xi(\Omega_\xi) < \infty$[2].

Promises and collateral satisfy the following

Assumption [R].

(i) Promised returns are real and given by $A_{j\xi} \in \mathbb{R}^G_+$, $\forall j \in J, \xi > \xi_0$.

(ii) At each node ξ, collateral must be posted in at least one $g \in G$ for which the column Y^g_η is non-null at every node $\eta \in \xi^+$. Collateral requirements are given by a $G \times J$ matrix C_ξ .

Consumers take as given prices p for goods, prices q for promises and delivery rates K on the promises. A choice variable is a *non-negative* plan $(x, \theta, \varphi, \psi)$ consisting of purchases of goods not for collateral purposes, promises purchases, promises sales and defaults, respectively. We denote $\tilde{x}^i_\xi = x^i_\xi + C_\xi \varphi^i_\xi$. *Budget constraints* at the initial node or at subsequent nodes $\xi \in D \setminus \{\xi_0\}$, are given, respectively, by:

$$p_{\xi_0}(\tilde{x}^i_{\xi_0} - \omega^i_{\xi_0}) + q_{\xi_0}(\theta^i_{\xi_0} - \varphi^i_{\xi_0}) \leq 0, \tag{1}$$

$$p_\xi\big(\tilde{x}_\xi - \omega^i_\xi - Y_\xi \tilde{x}^i_{\xi^-} - \sum_{j \in J(\xi^-)} A_{j\xi}(K_{j\xi}\theta^i_{j\xi^-} - \varphi^i_{j\xi^-})\big) + q_\xi(\theta^i_\xi - \varphi^i_\xi) \leq \sum_{j \in J(\xi^-)} \psi^i_{j\xi}, \tag{2}$$

[2]When Y is diagonal with elements uniformly bounded away from one, the assumptions that endowments are uniformly bounded and that the utility of a bounded plan is finite are sufficient to ensure $\sum_{\xi \in D} v^i_\xi(\Omega_\xi) < \infty$. (see Páscoa and Seghir (2009)).

To shorten the notations, we define $M_{j\xi} = \min\{p_\xi A_{j\xi}, p_\xi Y_\xi C^j_{\xi^-}\}$, for each node ξ and for each promise $j \in J_{\xi^-}$. The *minimal repayment constraint* requires consumers to repay at least $M_{j\xi}\varphi^i_{j\xi^-}$, that is,

$$\psi^i_{j\xi} \leq (p_\xi A_{j\xi} - M_{j\xi})\varphi^i_{j\xi^-} \tag{3}$$

Utility penalties may encourage consumers to repay above that minimum value. For this reason non-recourse might not prevail. The utility penalty is given by $\tilde{\lambda}^i_{j\xi} = \frac{\lambda^i_{j\xi}}{p_\xi\, b_\xi}$, where $b_\xi \in \mathbb{R}^G_{++}$ is a *reference bundle*. The entire payoff of consumer i is

$$\Pi^i(x^i, \theta^i, \varphi^i, \psi^i; p, q, K) := \sum_{\xi \in D} v^i_\xi(\tilde{x}^i_\xi) - \sum_{\xi \in D\setminus\{\xi_0\}} \sum_{j \in J(\xi^-)} \tilde{\lambda}^i_{j\xi}[\psi^i_{j\xi}]^+$$

where $[a]^+ = \max\{a, 0\}$, for any $a \in I\!R$. Consumer i problem consists in maximizing Π^i subject to (1), (2) and (3) and the following non-negativity constraint

$$x^i, \theta^i, \varphi^i \geq 0 \tag{4}$$

Definition 1. *An equilibrium is a process $(p, q, K, (x^i, \theta^i, \varphi^i, \psi^i)_{i \in I})$ such that $p_\xi > 0$* [3] *at any node $\xi \in D$ and verifying:*

(i) $\forall i \in I,\ (x^i, \theta^i, \varphi^i, \psi^i) \in \text{argmax } \Pi^i(x, \theta, \varphi, \psi; p, q, K)$ *subject to (1), (2), (3) and (4).*

(ii) $\sum_{i \in I}[x^i(\xi_0) + C(\xi_0)\varphi^i(\xi_0)] = \sum_{i \in I} \omega^i(\xi_0)$,

(iii) $\sum_{i \in I}[x^i_\xi + C_\xi \varphi^i_\xi] = \sum_{i \in I}[\omega^i_\xi + Y_\xi x^i(\xi^-) + Y_\xi C(\xi^-)\varphi^i(\xi^-)],\ \forall \xi \in D \setminus \{\xi_0\}$,

(iv) $\sum_{i \in I} \theta^i = \sum_{i \in I} \varphi^i$,

(v) $\forall j, \xi \in D \setminus \{\xi_0\},\ p_\xi A_{j\xi}(1 - K^j_\xi) \sum_{i \in I} \theta^i_j(\xi^-) = \sum_{i \in I} \psi^i_{j\xi}$.

[3]The reason why we require $p_\xi > 0$ to be an equilibrium condition has to do with the fact that the default penalty coefficient $\tilde{\lambda}^i_{j\xi} \equiv \frac{\lambda^i_{j\xi}}{p_\xi\, b_\xi}$ is only well defined in this case.

2.2 Individual optimality conditions

Definition 2. *Given prices $(p, q.K)$ and a plan $Z^i := (x^i, \theta^i, \phi^i, \psi^i)$ that verifies at these prices the constraints (1), (2), (3) and (4),* we say that Z^i satisfies the Euler conditions at (p, q, K) *if there exist supergradients $(d^i_j)_{j\in J}$ of the function* $\max\{0, \cdot\}$ *evaluated at $\psi^i_{j\xi}$ and a non-negative process $(\gamma^i, (\rho^i_j,)_{j\in J})$ of multipliers such that, for any promise $j \in J$ and any node ξ, the following hold*

(i)

$$\tilde{\lambda}^i_{j\xi} d^i_{j\xi} + \rho^i_{j\xi} = \gamma^i_\xi \tag{5}$$

$$\begin{aligned}\gamma^i_\xi\Big(p_\xi C_{j\xi} - q^j_\xi\Big) - v^{i\,\prime}_\xi(\tilde{x}_\xi) C_{j\xi} \quad \geq \quad & \sum_{\eta\in\xi^+}\Big[\gamma^i_\eta\Big(p_\eta Y_\eta C_{j\xi} - M_{j\eta}\Big) \\ & -\tilde{\lambda}^i_{j\eta} d^i_{j\eta}\Big(p_\eta A_{j\eta} - M_{j\eta}\Big)\Big]\end{aligned} \tag{6}$$

$$\gamma^i_\xi q^j_\xi \geq \sum_{\eta\in\xi^+} \gamma^i_\eta K_{j\eta} p_\eta A_{j\eta} \tag{7}$$

$$\forall g \in G, \quad \gamma^i_\xi p_{g\xi} \geq v^{i\prime}_\xi(\tilde{x}_\xi, g) + \sum_{\eta\in\xi^+} \gamma^i_\eta p_\eta (Y_\eta)^g, \tag{8}$$

(ii) equalities in (6), (7) or (8) hold when $\varphi_{j\xi} > 0$, $\theta_{j\xi} > 0$ or $x_{g\xi} > 0$, respectively.
(iii) $\rho^i_{j\xi}[\psi^i_{j\xi} - (p_\xi A_{j\xi} - M_{j\xi})\varphi^i_{j\xi^-}] = 0$

Lemma 1. *Under assumption [U], if the plan $(x^i, \theta^i, \phi^i, \psi^i)$ is a maximizer of $\Pi^i(x, \theta, \varphi, \psi)$ subject to (1), (2), (3) and (4) at prices (p, q, K), then this plan satisfies the Euler conditions and following transversality condition at (p, q, K),* $\limsup_T \sum_{\xi:\, t_\xi = T} \left(\gamma^i_\xi \left[p_\xi \tilde{x}^i_\xi - q_\xi(\theta^i_\xi - \varphi^i_\xi)\right] - v^{i\,\prime}_\xi \tilde{x}^i_\xi\right) \leq 0$

Actually, the above transversality condition implies the following

$$\limsup_T \sum_{\xi:\, t_\xi = T} [\gamma^i_\xi\,(p_\xi C_\xi - q_\xi) - v^{i\,\prime}_\xi C_\xi]\varphi^i_\xi \leq 0 \tag{9}$$

Lemma 2. *Let $\overline{Z}^i$ be a plan for consumer i that satisfies at prices (p, q, K) constraints (1), (2), (3) and (4), together with Euler conditions and the transversality condition. Suppose that any promises sales trajectory $\hat{\varphi}$ which is part of a plan $\hat{Z} = (\hat{x}, \hat{\theta}, \hat{\varphi}, \hat{\psi})$ satisfying constraints (1), (2) and (3) at prices and delivery rates (p, q, K) is such that for $v_\xi^{i\prime}$ evaluated at $\tilde{x}_\xi^i$ we have*

$$\limsup_T \sum_{\xi:\, t_\xi = T} [v_\xi^{i\prime} C_\xi - \gamma_\xi^i \, (p_\xi C_\xi - q_\xi)] \hat{\varphi}_\xi \leq 0, \tag{10}$$

then, under assumptions [U], [E] and [R], the plan $\overline{Z}^i$ is optimal for i at (p, q, K).

3 On no-trade equilibria

When promises are not collateralized, a trivial no-trade equilibrium can be found by setting $K_{j\xi} = 0$ and $q_{j\xi} = 0$, as already remarked by Dubey, Geanakoplos and Shubik (2005). However, when promises are collateralized, $K_{j\eta}$ is bounded from below by $\frac{M_{j\eta}}{p_\eta A_{j\eta}}$ when $p_\eta A_{j\eta} > 0$. In finite horizon economies, no-trade equilibrium can be trivially found by setting $K_{j\eta}$ equal to this lower bound.

In fact, we know that a finite horizon equilibrium $(p, q, \tilde{K}, x, \varphi, \theta, \psi)$ always exists, with $p_\eta A_{j\eta} > 0$ at every node η. If promise trades are chosen to be zero, replacing $\tilde{K}_{j\eta}$ by $K_{j\eta} = \frac{M_{j\eta}}{p_\eta A_{j\eta}}$, trivially satisfies the equilibrium condition on repayment rates (item (v) of definition 1) and the Kuhn-Tucker conditions still hold (as there is no increase in the willingness to pay that enters in the condition (7) on promise purchases).

Such no-trade outcomes are not trivially found in infinite horizon. The cluster points of trivial no-trade finite horizon equilibria might not satisfy condition (10). Strict sense Ponzi schemes or generalized Ponzi schemes may be doable. Under complete markets, more precisely, if all agents had the same Kuhn-Tucker deflator processes $\gamma_\xi^i / \gamma_{\xi^-}^i$, finding such no-trade infinite horizon equilibria would be trivial: setting the promise price equal to the common willingness to pay, we get $\gamma_\xi^i (p_\xi C_{j\xi} - q_{j\xi}) - v_\xi^{i\prime} C_{j\xi} \geq \sum_{\eta \in \xi^+} \gamma_\eta^i (p_\eta Y_\eta C_{j\xi} - M_{j\eta}) \geq 0$ and, therefore, (10) holds. In incomplete markets, such argument can be done for just one consumer, the one with the highest willingness to pay for the promise. For the other consumers, (10) might not hold.

There are infinite horizon economies where such no-trade equilibrium can be constructed and others where it cannot. Martins-da-Rocha and Vailakis (2012a) illustrated the former and we will illustrate the latter (in section 5).

Nevertheless, we may still want to refine the equilibrium concept, to be sure that the equilibria found in known existence results are not just no-trade outcomes driven by spurious delivery expectations.

When no-trade equilibria exist it becomes important to check if that minimal repayment rate actually captures the propensity to deliver of agents or if it such rate is just being chosen unduly low. As Martins-da-Rocha and Vailakis (2012a) acknowledge, in their no-trade example, the repayment rates are unduly low, as all agents have default penalty coefficients exceeding their marginal utilities of income and would, therefore, be prone to fully honor their debts. The authors also suggest that equilibria should be refined in order to rule out spurious outcomes that overestimate the propensity to default of the consumers. In the next section we address the refinement of equilibria.

Once such equilibria with pathologically low delivery rates are eliminated, we may end up with no equilibria for such economies. This should make it clear why, in such economies, the presence of penalties opens up the opportunity for a Ponzi scheme (or generalized versions of it). In the absence of that refinement, a spurious equilibrium may prevail and such understanding gets all distorted. In the next section, we address how equilibria should be refined and, to illustrate, show that the equilibrium found by Martins-da-Rocha and Vailakis (2012a) is not a refined equilibrium.

4 Removing unduly expectations.

In the absence of collateral, the refinement proposed by Dubey, Geanakoplos and Shubik (2005) consisted in picking the limits (as $\varepsilon \longrightarrow 0$) of equilibria of economies where an artificial agent was buying and selling ε units of each asset, delivering fully but receiving $K_{j\xi} p_\xi A_{j\xi}$. Commodity market clearing was adjusted to accommodate the fact that this agent was injecting $\sum_j \varepsilon(1-K_{j\xi})A_{j\xi}$ goods in the economy. The delivery rate $K_{j\xi}$ was such that $(1-K_{j\xi})p_\xi A_{j\xi}\left(\varepsilon + \sum_i \theta^i_{j\xi}\right) = \sum_i \psi^i_{j\xi}$. Such refinement ruled out the pathological no-trade outcome resulting from setting $K_{j\xi} = 0$

even though all agents were "strictly conscientious" (that is, $\tilde{\lambda}^i_{j\xi} > \gamma^i_\xi$).

If the same refinement were to be applied in the collateral model, it would eliminate pathological equilibria where $K_{j\xi} = \frac{M_{j\xi}}{p_\xi A_{j\xi}}$, even though $\tilde{\lambda}^i_{j\xi} > \gamma^i_\xi$, $\forall i$. However, this refinement seems to be too strong, as it would also eliminate no-trade outcomes where $\tilde{\lambda}^i_{j\xi} < \gamma^i_\xi$ for some agents and $K_{j\xi}$ is duly equal to $\frac{M_{j\xi}}{p_\xi A_{j\xi}}$.

This would be particularly awkward when along the sequence of equilibria for the $\varepsilon-$economy the true agents are not trading promise j but some of them are on the verge of selling it (with (6) holding with equality). Such shortcoming could cause a serious problem. In fact, unduly high expectations about $K_{j\eta}$ make prices of non-traded promises become over estimated and Ponzi schemes may occur spuriously (as $p_\xi C_{j\xi} - q^j_\xi$ may be fictitiously negative).

Actually, when avoiding unduly (pessimistic and optimistic) expectations, what is important is to eliminate delivery beliefs that are inconsistent with the penalty functions of agents who are on the margin of selling the promise. As Dubey, Geanakoplos and Shubik (2005) stressed, these on-the-verge agents should pin down what the delivery rates are.

We propose a different refinement. Let $E = \left(\overline{p}, \overline{q}, \overline{K}, \overline{x}, \overline{\theta}, \overline{\varphi}, \overline{\psi}\right)$ be an equilibrium and $\tilde{J}_\xi(E)$ be the set of promises that are not traded at node ξ. The delivery rate of promise j at each node $\eta \in \xi^+$ should be consistent with the penalty coefficients and marginal utilities of income of agents that might be already purchasing what can serve as collateral for j and are also on-the-verge of selling j.

Let us be more precise. An agent i is *on-the-verge of selling* promise j at node ξ if (6) holds with equality *for some vector* $(\rho^i_{j\eta}, d^i_{j\eta})$ (of multipliers of the minimal repayment constraint and supergradients for the penalty function) satisfying (5) and agent i is already purchasing the commodities that serve as collateral, which implies that (8) should hold with equality.

Let us see what is the desiderata on $(\rho^i_{j\eta}, d^i_{j\eta})$. If $j \in \tilde{J}_\xi(E)$ and $\tilde{\lambda}^i_{j\eta} < \gamma^i_\eta$ at any $\eta \in \xi^+$, then agent i is on-the-verge of selling if Euler conditions are ready for him to become a seller and, therefore, deliver with maximal default, so with $d^i_{j\eta} = 1$ for $\eta : p_\eta A_{j\eta} > M_{j\eta}$. For $\tilde{\lambda}^i_{j\eta} > \gamma^i_\eta$, we require instead $\rho^i_{j\eta} = 0$ at any $\eta \in \xi^+$ where $p_\eta A_{j\eta} > M_{j\eta}$, so that Euler conditions are ready for him to sell at ξ and not default at η. For $\tilde{\lambda}^i_{j\eta} = \gamma^i_\eta$, there are no restrictions on $d^i_{j\eta}$ or $\rho^i_{j\eta}$.

That is, an agent on-the-verge of selling promise j at node ξ should be ready to deliver at the next nodes $\eta \in \xi^+$ according to the optimality criterion on deliveries

(5) (for the right supergradient $d^i_{j\eta}$), given his penalty coefficients $\tilde{\lambda}^i_\eta$ and the marginal utilities of income γ^i_η.

Let us assume that different promises use different collateral instruments and that each promise uses just one collateral instrument.

Assumption [C]. The mapping $j \longmapsto \{g \in G : \ C^j_{g\xi} > 0\}$ is an injective function that does not change from node to node. Denote by $g(j)$ the element $g \in G$ such that $C^j_{g\xi} > 0$.

The node-invariance was assumed to simplify the notations. Now, for each node ξ and each promise j, let $N^j_\xi(E) = \{i : \ \tilde{x}^i_{g\xi} > 0 \text{ at E}\}$ and let $V^j_\xi(E) = \Big\{ i \in N^j_\xi(E) :$ (6) holds with equality for i at E with $d^i_{j\eta} \in [0,1]$ and $\rho^i_{j\eta}$ satisfying (5) and such that $d^i_{j\eta} = 1$ if $\tilde{\lambda}^i_{j\eta} < \gamma^i_\eta$, while $\rho^i_{j\eta} = 0$ if $\tilde{\lambda}^i_{j\eta} > \gamma^i_\eta, \eta \in \xi^+ \Big\}$.

Definition 3. *Given an equilibrium $E = \left(\overline{p}, \overline{q}, \overline{K}, \overline{x}, \overline{\theta}, \overline{\varphi}, \overline{\psi}\right)$, for which i is in the set $V^j_\xi(E)$ of agents on the verge of selling an un-traded promise j at node ξ, we define $\zeta^i_{j\eta} \in [0,1]$, the propensity to deliver at node $\eta \in \xi^+$, by*

$$\zeta^i_{j\eta} p_\eta A^j_\eta = \upsilon^i_{j\eta} p_\eta A_{j\eta} + (1 - \upsilon^i_{j\eta}) M_{j\eta} \tag{11}$$

where $\upsilon^i_{j\eta} \in [0,1]$ is such that $\upsilon^i_{j\eta} = 1$ if $\tilde{\lambda}^i_{j\eta} > \gamma^i_\eta$ and $\upsilon^i_{j\eta} = 0$ if $\tilde{\lambda}^i_{j\eta} < \gamma^i_\eta$.

Definition 4. *Given an equilibrium $E = \left(\overline{p}, \overline{q}, \overline{K}, \overline{x}, \overline{\theta}, \overline{\varphi}, \overline{\psi}\right)$, we say that E is a refined equilibrium if, for any non-traded promise j at some node ξ, we have at $\eta \in \xi^+$ that $K_{j\eta}$ is a convex combination of the propensities to deliver $\zeta^i_{j\eta}$ of the agents i in the set $V^j_\xi(E)$.*

In finite horizon, under Assumptions [E],[U] and [C], there exists always a refined equilibrium. This result is shown in Lemma A.1 of Appendix 6.1, by adapting the proof of Theorem 1 in Dubey, Geanakoplos and Shubik (2005).

For the sake of comparison with the work by Dubey, Geanakoplos and Shubik (2005), the refined equilibria we propose now can be shown to be limits of equilibria of hypothetical economies where an artificial agent (the government) induces agents in V^j_ξ to do small sales ε of what was a non-traded promise (rather than being himself a seller). This is done by collecting lump-sum taxes $t^i_{j\xi}$ from consumers in V^j_ξ (with $\sum_{i \in V^j_\xi(E)} t^i_{j\xi} = \overline{q}^j_\xi \, \varepsilon$), spending the tax revenue purchasing $\theta^G_{j\xi}$

units of promise j at ξ, and, then, at $\xi \in \xi^+$, giving *lump-sum subsidies* $s^i_{j\eta}$ to consumers in $V^j_\xi(E)$ using returns from the purchase done at ξ. Details can be found in Lemma A.2 of Appendix 6.2.

Are there refined equilibria under the assumptions of Theorems 2 or 3 in Páscoa and Seghir (2019)? The former assumed that the collateral does not provide any utility and that penalties are moderate, in the sense that $\lambda^i_{j\xi}$ is always below the minimum of the derivative of v^i_ξ along the direction b_ξ, taken over all feasible bundles z.

Proposition 1 *Under the assumptions of Theorem 2 in Páscoa and Seghir (2019) and Assumption [C], there exists a refined equilibrium for the infinite horizon economy, where* $K_{j\xi} p_\xi A_{j\xi} = M_{j\xi},\ \forall j,\ \forall \xi.$

Proof. In fact, at the equilibrium E found in that theorem, we had $K_\xi p_\xi A_{j\xi} = M_{j\xi},\ \forall j,\ \xi$. If $V^j_\xi(E) \neq \emptyset$, for $j \in \tilde{J}_\xi(E)$, then at the E_ε equilibrium for each $\varepsilon-$economy, we have $p_\eta A_{j\eta} \hat{K}^j_{\varepsilon\eta} = \frac{\sum_{i \in V^j_\xi(E)} (1-v^i_{j\eta}) M_{j\eta}}{\# V^j_\xi(E)}$, $\forall \eta \in \xi^+$, where $v^i_{j\eta} = 0,\ \forall i \in V^j_\xi(E)$. So, $K_{j\eta} = \hat{K}^j_{\varepsilon\eta},\ \forall \eta \in \xi^+$. If $V^j_\xi(E) = \emptyset$, the result is immediate. □

The latter replaced real promises of assumption [R] by nominal promises with returns given by $b^j_\eta \in \mathbb{R}_+$ at $\eta \in \xi^+$, for $j \in J$. The collateral is still real as in [R (ii)]. As usual, for $A_{j\xi} = \frac{b^j_\xi}{S_\xi} \mathbb{1}$ where S_ξ stands for $\|p_\xi\|_1$ and $\mathbb{1} = (1, ..., 1)$, equilibrium is still given by Definition 1. Notice that (10) holds for any budget feasible plan if all ξ and all j, we have

$$\sum_{\eta \in \xi^+} \gamma^i_\eta (p_\eta Y_\eta C_{j\xi} - M_{j\eta}) - \sum_{\eta \in \xi^+} \tilde{\lambda}^i_{j\eta} d^i_{j\eta} (p_\eta A_{j\eta} - M_{j\eta}) \geq 0,\ \forall i,\ \forall \xi, \tag{12}$$

where $d^i_{j\eta}$ satisfies (5).

In the case of nominal promises secured by real collateral, 12 holds if:

$$\sum_{\eta \in \xi^+} S^{-1}_\eta \max\{\lambda^i_{j\eta}, \gamma^i_\eta\}\, b^j_\eta \leq \sum_{\eta \in \xi^+} \min\{\lambda^i_{j\eta}, \gamma^i_\eta\}\, p_\eta Y_\eta C^j_\eta,\ \forall i,\ \forall \xi \tag{13}$$

Proposition 2 *Under the assumptions of Theorem 3 in Páscoa and Seghir (2019) and Assumption [C], there exists a refined equilibrium for the infinite horizon economy.*

Proof. Equilibria of economies with finite horizon H satisfy (12)or equivalently (13), by an appropriate choice of inter-nodes inflation rates (as in the proof of Theorem 3 in Páscoa and Seghir (2019)). As $H \to \infty$, their cluster point satisfies (10) and is therefore an infinite horizon equilibrium. Moreover, it is refined since $K_{j\eta} = \sum_{i \in V_\xi^j(E)} \beta_{j\xi}^i \zeta_{j\eta}^i, \ \forall \eta \in \xi^+, \ \left(\beta_{j\xi}^i\right)_i \in \Delta^{\# V_\xi^j(E)-1}$, whenever $V_\xi^j(E) \neq \emptyset$, for $j \in \tilde{J}_\xi(E)$. □

What does the above refinement do when agents are strictly conscientious?

Proposition 3 *If E is such that $V_\xi^j(E) \neq \emptyset$ and $\tilde{\lambda}_{j\eta}^i > \gamma_\eta^i, \ \eta \in \xi^+, \ \forall i \in V_\xi^j(E)$, then E is a refined equilibrium if and only if $K_{j\eta} = 1$ for $\eta : \ p_\eta A_{j\eta} > 0$.*

Proof. In fact, $K_{\varepsilon\eta}^j = \dfrac{\sum_{i \in V_\xi^j(E)} \upsilon_{j\eta}^i}{\# V_\xi^j(E)} = 1$ (as $\upsilon_{j\eta}^i = 1, \ \forall i \in V_\xi^j(E)$), for $\eta : \ p_\eta A_{j\eta} > 0$. □

The next example illustrates Proposition 3. We take the example in Martins-da-Rocha and Vailakis (2012a). Apparently, in the no-trade equilibrium found by these authors no agent has a zero slack in the Euler condition (6) on promise sales. This would imply that such equilibrium is a refined equilibrium. A more careful analysis shows that this is not true.

Example 1: the example by Martins-da-Rocha and Vailakis (2012a)

The economy is deterministic, there is a single good and agents utilities are given by $U^i(x) = \sum_{t=1}^{\infty} \beta_i^t x_t$. The good is durable, depreciates at a constant rate $Y < 1$ and serves as numeraire. There is a promise whose constant returns A and constant collateral coefficients C are such that $A > YC$. Penalty coefficients are given by $\lambda_t^i = \beta_i^t \sigma^i$, where $\sigma^i \geq \frac{1}{1-\beta_i Y}$. Let us focus on the case where $\sigma^i = \frac{1}{1-\beta_i Y}$

No-trade equilibrium was shown by these authors to exist and is such that $\max_i\{\beta_i\} YC < q < \min\{C, \min_i\{\beta_i\} A\}$ and $K = YC/A$.

Euler conditions were shown, with equality in (8), for multipliers $\hat{\gamma}$ and ρ^i together with supergradients $\hat{d}_t^i$ given by $\rho_t^i = 0$, $\hat{\gamma}_t^i = \lambda_t^i \hat{d}_t^i = \beta_i^t \sigma^i \hat{d}^i = \beta_i^t \frac{1}{1-\beta_i Y}$. The slack in (6) is $\beta_i^t \sigma^i \hat{d}^i (\beta_i A - q)$ and the slack in (7) is $\beta_i^t \sigma^i \hat{d}^i (q - \beta_i YC)$. To see that this is indeed an infinite horizon equilibrium, it suffices to check that (10) is satisfied. Notice that $\hat{\gamma}_t^i(C-q) - v_t^{i'}C = \beta_i^t[\frac{C-q}{1-\beta_i Y} - C]$, which is a negative sequence (as $q > \beta_i YC$) tending to zero, while any budget feasible promise sales plan is bounded (as $C-q$ is a positive constant). This makes (10) hold.

Claim 1(i): we can choose a lower supergradient d_t^i for the function $\psi \mapsto \psi^+$ so that the slack in (6) vanishes (and this can be done actually for every agent and every date in this example as (8) holds with equality for every agent and date).

To see this, we keep $\rho_t^i = 0$ and let $\gamma_t^i = \beta_i^t \tilde{\gamma}_t^i$, where $\tilde{\gamma}_t^i$ is to be determined. The equality in (8) implies that (6) can be written as follows:

$$\tilde{\lambda}_{t+1}^i d_{t+1}^i (A - M) \geq \gamma_t^i q - \gamma_{t+1}^i M, \tag{14}$$

We want (14) to hold with equality. Now, (5) implies that $\tilde{\lambda}_{t+1}^i d_{t+1}^i = \gamma_{t+1}^i$ and we know that $M = YC$. Hence, the slack in (14) vanishes if $q = \beta_i \frac{\tilde{\gamma}_{t+1}^i}{\tilde{\gamma}_t^i} A$. As $q < \beta_i A$, it follows that we must have

$$\frac{\tilde{\gamma}_{t+1}^i}{\tilde{\gamma}_t^i} < 1. \tag{15}$$

Now, the equality in (8) implies that $\tilde{\gamma}_t^i = \tilde{\gamma}_{t+1}^i \beta_i Y + 1$. Hence, (15) holds if and only if $\beta_i Y + 1/\tilde{\gamma}_{t+1}^i > 1$, or equivalently, $\tilde{\gamma}_{t+1}^i < \frac{1}{1-\beta_i Y}$. Let $\tilde{\gamma}_t^i = \frac{1}{1-\beta_i Y(1-\epsilon_t)}$ where (ϵ_t) is an increasing sequence in $[0,1]$ converging to some $\epsilon \in (0,1)$. Notice that the supergradient is given by $d_t^i = \frac{\tilde{\gamma}_t^i}{\sigma^i} = \frac{1-\beta_i Y}{1-\beta_i Y(1-\epsilon_t)} < 1$.

Let us check that (7) holds. Denote its slack by $S(\theta_t^i)$ and make $S(\theta_t^i) = \beta_i^t \tilde{S}(\theta_t^i)$, where $\tilde{S}(\theta_t^i) = \tilde{\gamma}_t^i q - \beta_i \tilde{\gamma}_{t+1}^i YC$. Now, (7) holds if and only if $\tilde{S}(\theta_t^i) \geq 0$. We know that $q > \beta_i YC$, then $\tilde{S}(\theta_t^i) = (\tilde{\gamma}_t^i - \tilde{\gamma}_{t+1}^i)q + \tilde{\gamma}_{t+1}^i(q - \beta_i YC) > 0$[4]. $\square$

That is, all agents are on-the-verge of selling the promise. The promise price is equal to their reservation prices as sellers of the promise. Martins-da-Rocha and Vailakis (2012a) suggested that no trade prevailed because the promise was too expensive to buy and too cheap to sell. We have shown that the promise is not too cheap to sell, it is just too expensive to buy, due to the spuriously low delivery rates K, and this is the only reason why it is not traded.

[4]For this alternative way of defining γ_t^i, we still have $\gamma_t^i(p_t C_t - q_t) - v_t^{i'} C_t < 0$ and tending to zero. Hence, (10) is satisfied as before (as all budget feasible sales plans happen to be bounded).

Claim 1(ii): the equilibrium is not a refined equilibrium.

In fact, all agents are on-the-verge of selling and all have a propensity to deliver ζ_t^i equal to 1, but the equilibrium delivery rate K is $YC/A < 1$. □

5 On non-existence of no-trade equilibria

We present next a variant of Example 1 which illustrates why no-trade equilibria might not exist in infinite horizon economies.

Example 2: non-existence of no-trade equilibria
Let us modify Example 1, replacing the constant collateral and returns coefficients by unbounded sequences. Take the multipliers choice we proposed in Example 1: $\gamma_t^i = \beta_i^t \frac{1}{1-\beta_i Y(1-\epsilon_t)}$ and let $\epsilon_t = \epsilon(1 - 1/2^t)$, where $\epsilon = 1/2$. We assume that $C_t = (1-\epsilon_t)^{-1} C_{t-1}$, where we set $C_0 = 1$, and $A_t = YC_{t-1}(1-\epsilon_t)^{-1}$. Finite horizon no-trade equilibria are such that $\max_i\{\beta_i\} YC_{t-1} < q_t < \min\{C_t, \min_i\{\beta_i\} A_t\}$ and $K_t = YC_{t-1}/A_t = 1-\epsilon_t$. Actually, if an infinite horizon no-trade equilibrium would exist, Euler conditions should hold and these same inequalities would prevail.

Suppose $\min_i\{\beta_i\} = 1 - \epsilon < \max_i\{\beta_i\}$ and let m be an agent for whom $\beta_m = \min_i\{\beta_i\}$.

Claim 2(i): $\gamma_t^m(C_t - q_t) - v_t^{m'} C_t$ has a negative uniform upper bound.

In fact, these front leg gains are $\beta_m^t [\frac{C_t - q_t}{1-\beta_m Y(1-\epsilon_t)} - C_t] = \beta_m^t C_{t-1} \frac{\beta_m Y - q_t/C_{t-1}}{1-\beta_m Y(1-\epsilon_t)}$, where $\beta_m Y$ is below the infimum of q_t/C_{t-1}.

Let us show that $\beta_m^t C_{t-1}$ is bounded away from zero. Notice that $\beta_m^t C_{t-1} = \beta_m \prod_{\tau=1}^{t-1} (1-\epsilon_\tau)^{-1}(1-\epsilon)$. Let $Z_t \equiv 1-(1-\epsilon_\tau)^{-1}(1-\epsilon) = (1+2^t)^{-1}$. The infinite product $\prod_{t=1}^{\infty}(1 - Z_t)$ converges to a non-zero real number if the series $\sum_{t=1}^{\infty} Z_t$ converges. The latter follows by the D'Alembert criterion since $Z_{t+1}/Z_t = (2^t + 1)/(2^{t+1} + 1)$, which tends to 1/2. □

Claim 2(ii): $(C_t - q_t)_t \ggg 0$. For bounded endowments, this implies that *any budget feasible deviation from no-trade, consisting of a process of promise sales φ with maximal default accommodated by a sacrifice in consumption, must be bounded.*

In fact, $C_t - q_t > C_t - \beta_m A_t = C_t - \beta_m Y C_{t-1}(1-\epsilon_t)^{-1} = C_{t-1}(1-\epsilon_t)^{-1}(1-\beta_m Y)$, where C_{t-1} increases unboundedly while ϵ_t tends to $1/2$. Budget feasibility requires $x_t + (C_t - q_t)\varphi_t \leq \omega_t^i + Y x_{t-1}$, as default is maximal and the promise is not purchased, where $x_t = x_t^i - \hat{x}_t^i$ for a sacrifice $\hat{x}_t^i \in (0, x_t^i)$. Hence, $\varphi_t \leq \frac{\omega_t^i + Y(x_{t-1}^i - \hat{x}_{t-1}^i)}{C_t - q_t}$, where the numerator is bounded from above by $W_t^i \leq \sup_t \omega_t^i/(1-Y)$. $\square$

Claim 2(iii): If $\omega^{(m)} \ggg 0$, then the deviation contemplated in Claim 5(ii) makes consumer m improve upon the no-trade plan.

In fact, in this simple economy with just one good and one promise, when the promise is not traded, agents end up consuming the respective resources $\omega_t^{(i)} + Y x_{t-1}^{(i)}$. So $x^{(m)} \ggg 0$ and a generalized Ponzi scheme can be constructed as in Example 1 in in Páscoa and Seghir (2019), by sacrificing consumption in an amount $s x_t^{(m)}$ at every date. By Claims 2(i) and 2(ii), the resulting process of promise sales $\varphi_t = s x_t^{(m)}/(C_t - q_t)$ by agent m is such that the strict inequality opposite to (10) holds instead[5], which says that *this* consumer can improve upon the no-trade outcome. $\square$

No-trade is not an equilibrium for the economy with strictly consciencious agents of Example 2, in spite of the fact that the haircut $C_t - q_t$ on the secured loan has a positive uniform lower bound. When front leg gains $v_t^{m'} C_t - \gamma_t^m (C_t - q_t)$, per unit of the promise, are positive and bounded away from zero, having a positive lower bound for the haircut is actually not enough to prevent generalized Ponzi schemes.

[5]More precisely, the right-hand-side derivative of utility along the direction $(C_t\varphi_t - \epsilon x_t^{(m)})_t$ is well defined (since φ is bounded) and positive.

6 Appendix

6.1 Finite horizon refined equilibria

Lemma A.1 *Under Assumptions [E],[U] and [C], in a finite horizon economy, there exists a refined equilibrium.*

Proof. We adapt the proof of Theorem 1 in Dubey, Geanakoplos and Shubik (2005). As in that proof, for each s, the price set is such that $p(\xi, g) \geq s$, $\sum_g p(\xi, g) = 1$ and $q_\xi^j \leq \frac{1}{s}$. We define the function F mapping each vector $(\tilde{x}^i(\xi, g(j)))_i$ into the value $F(\xi, g(j))$ equal to $\min_i \{\tilde{x}^i(\xi, g(j))) : \tilde{x}^i(\xi, g(j)) > 0\}$ if $(\tilde{x}^i(\xi, g(j)))_i \neq 0$ and equal to zero otherwise. It is a continuous function. Now, for each i, let $\overline{\beta}^i_{j\xi} = \frac{\tilde{x}^i(\xi, g(j)))}{s + F(\xi, g(j))}$.

Define also the following correspondences: $G^i_{j\xi} = \operatorname{argmax}\{v^i_{j\xi}(\tilde{\lambda}^i_{j\xi} - \gamma^i_\xi) : v^i_{j\xi} \in [0,1]\}$ and let $\tilde{G}^i_{j\xi} = \operatorname{argmin}\{[(\zeta^i_{j\xi} p_\xi A_{j\xi} - v^i_{j\xi} p_\xi A_{j\xi} - (1 - v^i_{j\xi}) M_{j\xi}]^2 : \zeta^i_{j\xi} \in [0,1]\}$.

Delivery rates $K_{j\eta}$ are defined jointly with weights $\beta^i_{j\xi}$ by the next correspondence. The conditions this correspondence imposes on the weights $\beta^i_{j\xi}$ become redundant when the promise is traded. For each (ξ, j) : $\Big((K_{j\eta})_{\eta \in \xi^+}, (\beta^i_j)_i\Big) \in \mathbb{H}^j_\xi$, where

$$
\begin{aligned}
\mathbb{H}^j_\xi \quad &\equiv \quad \operatorname{argmin}\Big\{ \sum_{\eta \in \xi^+} \Big[((\sum_i \theta^i_{j\xi})(1 - K_{j\eta}) p_\eta A_{j\eta} - \sum_i \psi^i_{j\eta})^2 + (K_{j\eta} - \sum_i \beta^i_{j\xi} \zeta^i_{j\eta})^2 \\
&+ \quad \sum_i \Big(\alpha^i_{j\xi} \beta^i_{j\xi} + \beta^i_{j\xi} [\gamma^i_\eta - \tilde{\lambda}^i_{j\eta}]^+ (1 - d^i_{j\eta}) + \beta^i_{j\xi} [\tilde{\lambda}^i_{j\eta} - \gamma^i_\eta]^+ \rho^i_{j\eta} [p_\eta A_{j\eta} - M_{j\eta}]^+ \\
&+ \quad (\beta^i_{j\xi} \sum_k \varphi^k_{j\xi} - \varphi^i_{j\xi})^2 \Big) : \; K_{j\eta} \in [0,1], \sum_i \beta^i_{j\xi} \leq \frac{1}{1+s}, \; 0 \leq \beta^i_{j\xi} \leq \overline{\beta}^i_{j\xi} \Big\},
\end{aligned}
$$

where $\alpha^i_{j\xi}$ is the slack in the Kuhn-Tucker condition on promise sales.

Lagrange multipliers $(\gamma^i_\xi, \rho^i_\xi)$ are given by $\mathbb{L}^i_\xi \equiv \operatorname{argmin}\Big\{ L^i_\xi(x^i, \varphi^i, \theta^i, \psi^i, p, q, K) : \gamma^i_\xi \in [0, \overline{\gamma}^i_\xi], \; \rho^i_\xi \in [0, \overline{\gamma}^i_\xi] \Big\}$.

The supergradients $d^i_{j\eta}$ are given by $\mathbb{D}^i_{j\eta} \equiv \operatorname{argmin}\Big\{ (\tilde{\lambda}^i_{j\eta} d^i_{j\eta} - \gamma^i_\eta)^2 : d^i_j \in \partial(+)(\psi^i_{j\eta}) \Big\}$, where $+$ stands for the function $z \longmapsto z^+$.

For each s, a fixed point of $\tau^0_s \times F \times \overline{\beta} \times \prod_{(j,\xi)} \mathbb{H}^j_\xi \times \prod_i \tau^i_s \times \prod_{i,\xi} \mathbb{L}^i_\xi \times \prod_{(i,\xi,j)} \mathbb{D}^i_{j\xi} \times \prod_{(i,\xi,j)} G^i_{j\xi} \times \prod_{(i,\xi,j)} \tilde{G}^i_{j\xi}$ exists.

As $s \longrightarrow 0$, the sequence of fixed points has a cluster point, at which market clearing holds, $p(\xi, g)$ does not converge to zero and q^j_ξ stays bounded. This cluster point is an equilibrium E for the finite horizon economy such that if $j \in \tilde{J}_\xi(E)$, $K_{j\eta} = \sum_i \beta^i_{j\xi}\zeta^i_{j\eta}$ where $\beta^i_{j\xi} = 0$ for i with $x^i(\xi, g(j)) = 0$ or $\alpha^i_{j\xi} = 0$. Moreover, if $V^j_\xi(E) \neq \emptyset$, then $\beta^i_{j\xi} > 0$ only for $i \in V^j_\xi(E)$. □

6.2 Refined equilibria and perturbed economies

For the sake of comparison with the work by Dubey, Geanakoplos and Shubik (2005), we can characterize refined equilibria in terms of asymptotic equilibria of hypothetical economies where an artificial agent ("'the government"') would induce small trades of the non-traded promise, rather than trading himself as was the case in the Dubey, Geanakoplos and Shubik (2005) refinement.

We consider an *auxiliary* $\epsilon-$*economy* that differs from the original economy by adding another agent, called "the government", that collects at node ξ *lump-sum taxes* $t^i_{j\xi}$ from consumers in $V^j_\xi(E)$, $j \in \tilde{J}_\xi(E)$, with $\sum_{i \in V^j_\xi(E)} t^i_{j\xi} = \overline{q}^j_\xi \varepsilon$, spends the tax revenue purchasing $\theta^G_{j\xi}$ units of promise j at ξ, and, then, at $\xi \in \xi^+$, gives *lump-sum subsidies* $s^i_{j\eta}$ to consumers in $V^j_\xi(E)$ using returns from the purchase done at ξ. In an equilibrium for the $\varepsilon-$economy the government choice variables should satisfy $K_{j\eta}p_\eta A_{j\eta}\theta^G_{j\xi} = \sum_{i \in V^j_\xi(E)} s^i_{j\eta}$. At the same time, the delivery rate should be such that $(1 - K_{j\eta})p_\eta A_{j\eta}(\theta^G_{j\xi} + \sum_i \theta^i_{j\xi}) = \sum_i \psi^i_{j\eta}$. Market clearing for j requires now $\sum_i \varphi^i_{j\xi} = \theta^G_{j\xi} + \sum_i \theta^i_{j\xi}$.

We focus on a special class of equilibria for the $\varepsilon-$economy, called E_ε *equilibria*, where only the government purchases promise $j \in \tilde{J}_\xi(E)$, only consumers in $V^j_\xi(E)$ sell it and marginal utilities of income γ^i_η, $\eta \in \xi^+$, for $i \in V^j_\xi(E)$, are as in the original equilibrium E, so that agents in $V^j_\xi(E)$ are just as willing to default as they were at the original equilibrium E. Let $\upsilon^i_{j\eta} \in [0,1]$ be such that $\upsilon^i_{j\eta} = 1$ if $\tilde{\lambda}^i_{j\eta} > \gamma^i_\eta$ and $\upsilon^i_{j\eta} = 0$ if $\tilde{\lambda}^i_{j\eta} < \gamma^i_\eta$. In such equilibria, $t^i_{j\xi} = \frac{\overline{q}^j_\xi \varepsilon}{\# V^j_\xi(E)}$, $s^i_{j\eta} = \left[\upsilon^i_{j\eta}p_\eta A_{j\eta} + (1 - \upsilon^i_{j\eta})M_{j\eta}\right] \frac{\varepsilon}{\# V^j_\xi(E)}$ and $K_{j\eta}p_\eta A_{j\eta} = \frac{1}{\varepsilon} \sum_{i \in V^j_\xi(E)} s^i_{j\xi}$, whenever $\overline{q}^j_\xi > 0$.

Lemma A.2 An equilibrium $E = \left(\overline{p}, \overline{q}, \overline{K}, \overline{x}, \overline{\theta}, \overline{\varphi}, \overline{\psi}\right)$ is a *refined equilibrium* if, whenever $V_\xi^j(E) \neq \emptyset$, for some $j \in \tilde{J}_\xi(E)$ and some ξ, E is a limit (in the product topology of the countable tree) of a sequence of equilibria E_ε for ε_n-economy (as $\varepsilon_n \longrightarrow 0$).

In fact, if at the original equilibrium E, $\overline{q}_\xi^j > 0$, and, for every $\eta \in \xi^+$ such that $p_\eta A_{j\eta} > 0$, $\overline{K}_\eta^j$ is a weighted average with weights $\beta_{j\xi}^i$, of the individual delivery rates $\zeta_{j\eta}^i$ of agents in $V_\xi^j(E)$ (possibly with some null weights) for every j in $\tilde{J}_\xi(E)$, (just as it was the case for traded promises) then for each ε, we have an equilibrium E_ε that differs from E only by making, for $i \in V_\xi^j(E)$, $\varphi_{j\xi}^i = \beta_{j\xi}^i \varepsilon$, $t_{j\xi}^i = \beta_{j\xi}^i \varepsilon \overline{q}_\xi^j$ and $x_{g(j)\xi}^i = \overline{x}_{g(j)\xi}^i - C_{g(j)}^j \beta_{j\xi}^i \varepsilon$. Then, E is a refined equilibrium.

References.

Araujo, A. and M.R. Páscoa, 2002. Bankruptcy in a Model of Unsecured Claims. *Economic Theory*, 20, 455-481.

Araujo, A., R. Novinski and M.R.Páscoa, 2011. General equilibrium, wariness and efficient bubbles. *Journal of Economic Theory*, 146, 785-811.

Araujo, A., M.R. Páscoa and J.P. Torres-Martínez, 2002. Collateral avoids Ponzi schemes in incomplete markets. *Econometrica*, 70, 1613-1638.

Araujo, A., M.R. Páscoa and J.P. Torres-Martínez, 2011. Long Lived Collateralized Assets and Bubbles, *Journal of Mathematical Economics*, 47, 260-271.

Dubey, P., J. Geanakoplos and M. Shubik, 2005. Default and Punishment in General Equilibrium. *Econometrica*, 73, 1-37.

Ferreira , R. and J.P. Torres-Martinez, 2010. The Impossibility of Effective Enforcement Mechanisms in Collateralized Credit Markets. *Journal of Mathematical Economics*, 46, 332-342.

Fostel, A. and J. Geanakoplos, 2008. Leverage Cycles and the Anxious Economy. *American Economic Review*, 98(4): 1211-1244.

Fostel, A. and J. Geanakoplos, 2015. Leverage and Default in Binomial Economies: a Complete Characterization. *Econometrica,* 83(6), 2191-2229.

Geanakoplos, J., 1997 . Promises, Promises. *The Economy as an Evolving*

Complex System, W.B. Arthur, S. Durlauf and D. Lane, eds., Addison-Wesley, Reading MA, 285-320.

Geanakoplos, J. and W. Zame, 1997. Collateral, Default and Market Crashes. Cowles Foundation discussion paper.

Geanakoplos, J. and W. Zame, 2014. Collateral Equilibrium I: a Basic Framework, *Economic Theory*, 56(3), 443-492.

Kubler, F. and K. Schmedders, 2003. Stationary Equilibria in Asset-Pricing Models with Incomplete Markets and Collateral. *Econometrica* 71, 1767-1793.

Martins-da-Rocha, V.F. and Y. Vailakis, 2012a, Harsh Default Penalties Lead to Ponzi Schemes: a Counter-Example. *Games and Economic Behavior*, 75, 277-282.

Martins-da-Rocha, V.F. and Y. Vailakis, 2012b. On Ponzi Schemes in Infinite Horizon Collateralized Economies with Default Penalties. *Annals of Finance*, 8, 455-488.

Páscoa, M.R. and A. Seghir, 2009. Harsh Default Penalties Lead to Ponzi Schemes. *Games and Economic Behavior*, 65, 270-286.

Páscoa, M.R. and A. Seghir, 2019. Recourse Loans and Ponzi Schemes. University of Surrey Discussion Papers in Economics DP07-2019, https://www.surrey.ac.uk/school-economics ISSN: 1749-5075.

Poblete-Cazenave, R. and J.P. Torres-Martinez, 2013. Equilibrium with Limited-Recourse Collateralized Loans. *Economic Theory*, 53, 181-211.

Shubik, M. and C. Wilson, 1977. The Optimal Bankruptcy Rule in a Trading Economy Using Fiat Money. *Zeitschrift fur Nationalokonomie*, 37, 337-354.