1.1 Introduction

The aim of this unit is to assist you in consolidating and developing some fundamental numerical processing skills.

Some of the problems relate to concepts covered in later core units. You are encouraged to revisit some of these problems after studying the subsequent units as this will further help to reinforce your knowledge and skills.

While studying these slides you should attempt the 'Your Turn' questions and the 'Quick Questions' in the separate document.

After studying the slides and completing the short questions, you should attempt the Consolidation Questions.

The 'Surds Extra' document covers further useful and interesting information and problems involving surds.

1.1 Learning checklist

Learning resource	Notes	Tick when complete
Slides		
Your turn questions		
Quick Questions		
Consolidation questions		
Surds Extra		

1.1 Learning objectives

At the end of the following lecture and seminar, you should be able to

1.1.1   Identify different types of Numbers

1.1.2.   Round a number to a given number of decimal places

1.1.3.   Round a number to a given number of significant figures

1.1.4.   Convert a number to standard index form and vice-versa

1.1.5.   State the base dimensions and base units in mechanics

1.1.6.   Standardise units in calculations

1.1.7.   Carry out calculations within limits of accuracy

1.1.8.   State the difference between rational and irrational numbers

1.1.9.   Carry out operations of ASMD with simple surds

1.1.10.   Rationalise the denominator of fractions containing surds

1.1.1 Numbers

Numbers are part of a wider symbolic language used for describing mathematical quantities, operations, theorems or objects.

Their representation has been standardised over thousands of years and they have been classified into different types

Natural numbers

Natural numbers are positive whole numbers (positive integers).

$\mathbb{N} = \{0, 1, 2, 3, \ldots\}$. It is an infinite but **countable** set

With natural numbers we can solve equations such as, $x - 2 = 1$ but NOT $x + 2 = 1$

Positive and negative integers

To solve the equation $x + 2 = 1$, we have to invent a new set of numbers!

We define new "negative" whole numbers (negative integers) to have certain properties and they must obey rules that we specify.

Together, these positive (Natural) and negative whole numbers form the set of integers, $\mathbb{Z} = \{\ldots -3, -2, -1, 0, 1, 2, 3, \ldots\}$. Again, it is infinite, but countable.

These properties will ensure that we can solve equations such as

$x + 2 = 1$ AND $x - 2 = 1$

Rational numbers

So far, we can solve $x + 2 = 1$ and $x - 2 = 1$ what about $2x + 2 = 1$?

$2x + 2 = 1$ has no solution in the set of integers, so again, we have to invent another new set of numbers!

The rational numbers $\mathbb{Q}$ is the set of numbers of the form a/b, where a and b are integers.

Irrational numbers and real numbers

The non-linear equations, $x^2 - 4 = 0$ and $x^2 - 16/9 = 0$, have solutions in the integer and rational systems respectively.

However, to solve $x^2 - 2 = 0$, we need yet another new set of numbers called irrationals, I. The set *I is* infinite and **uncountable** (because each interval, no matter how small, between two different numbers is infinitely divisible). It contains all real numbers that cannot be expressed in the form a/b, where a and b are integers.

Together, the sets, $\mathbb{N}, \mathbb{Z}, \mathbb{Q}$ and *I* form the set of real numbers, $\mathbb{R}$. It is also an uncountable set.

Relationship between number sets

1.1.2/3 Significant figures and decimal places

When working to a specified number of decimal places, you should only round the *final answer* to the specified number of decimal places.

Any calculation steps done prior to the final answer should use *all of the digits* in your calculator display or stored values in the calculator. For example, you should use the π button rather than entering say 3.14195.

Decimal places

The number of decimal places (dp) is just the number of digits after the decimal point.

A number with zero decimal places is an integer.

Place value is the value of the digit at a given decimal place.

Rounding to a given number of dp

To round a number to n decimal places.

1. Identify the place value of the digit in the $(n+1)^{th}$ decimal place.

2. If the place value is less than 5, round down by truncating the number at the nth decimal place. If the place value is 5 or greater, round up by incrementing the nth place value by 1 before truncating.

3. With negative numbers, the procedure is exactly the same, you can just imagine that the negative sign isn't there.

Examples

1. Round 0.0355091 to 3 dp

   4th place value : $5 \geqslant 5$

   $\therefore$ round up

   $\underline{0.036 \text{ to } 3 \text{ dp}}$

2. Round 5.096 to 2 dp

   3rd place value $6 \geqslant 5$

   $\therefore$ round up

   $\underline{5.10 \text{ to } 2 \text{dp}}$ (5.1 is not correct !)

3. Round 2.004 to 2dp

   3rd place value $4 < 5 \therefore$ round down

   $\underline{2.00 \text{ to } 2 \text{dp}}$

1.1.2. Round a number to a given number of significant figures (digits)

When we round to a given number of decimal places we are only concerned with digits after the decimal point.

With significant figures (digits) this is no longer the case. There are a number of conditions that we need to be familiar with before we start to do some rounding.

Your turn!

Consider these numbers, how many significant figures are there in each number?

20000

20001

20600

20000.00

0.0001700

20000 – Has **one** significant figure. The leading (first) figure is non-zero. There is no decimal point. The remaining figures are trailing zero's.

20001 – Has **five** significant figures. The leading (first figure) is non-zero. There is no decimal point. The last digit is non-zero. We have to count the zero's between the first and last significant figures.

20600 - Has **three** significant figures. The leading (first figure) is non-zero. There is no decimal point. The third figure is non-zero and we have to count the zero in between the first and third significant figures. The two trailing zero's are not significant.

20000.00 - Has **seven** significant figures. The leading (first) figure is non-zero. There are two decimal places after the point. We have to count all of the zero's.

0.0001700 - Has **four** significant figures. The leading zeros up to the first significant figure (the "1") are not significant and we do not count them. We must count any trailing zero's after a decimal point though.

Examples

1. Round   17003.05   to 1 sf
   $\uparrow$ 2nd sf : $7 \geqslant 5$ $\therefore$ round up

   20 000   to 1 sf

2. Round   2.002   to   2 sf
   3rd sf : $0 < 5$ $\therefore$ round down

   2.0   to   2 sf

3. Round   0.00359   to 2 sf
   3rd sf : $9 > 5$ $\therefore$ round up

   0.0036 to 2 sf

4. Round   1.00701   to 3 sf
   4th sf : $7 \geqslant 5$ $\therefore$ round up

   1.01   to 3 sf

Your turn!

Round each of the numbers below

a) 101. 109 to 2dp and to 5 sf

b) 0.000098714 to 6 dp and to 3 sf

Answers

a) 101. 109 to 2dp and to 5 sf

   101.11   2dp
   101.11   5sf

b) 0.000098714 to 6 dp and to 3 sf

   0.000099   6dp
   0.0000987   3sf

1.1.4. Convert a number to standard index form and vice-versa

In standard index form, also called standard form or scientific notation, a number is written as follows;

$$A \times 10^n$$

Where, $1 \leq A < 10$ and n is an integer, $n \neq 0$.

You may come across various words to describe an "ordinary number" (a number that is not written in standard form). We need to be able to convert a number in standard form to an ordinary number and vice versa, and evaluate numerical operations on numbers written in standard form. You can use any scientific calculator to do this.

Large and small numbers

23 000 000 (23 million) can be written as 2.3×10^7.

It would be incorrect to write 23×10^6 as a number in standard form because A must satisfy $1 \le A < 10$. Large numbers have positive powers of n.

0.000023 (23 millionths) can be written as 2.3×10^{-5}. It's the same as $\dfrac{23}{1\,000\,000}$. The '3' is in the 6th decimal place, since $\dfrac{1}{1\,000\,000} = 10^{-6}$. Small numbers have negative powers of n.

Converting an ordinary number to a number in standard form

E.g 1 Large number

Change 375 000 000 to standard form

3.75 000 000. ← decimal point

new decimal point

8 places
$n = 8$

3.75×10^8

E.g 2 Small number

Change 0.00000913 to standard form

0.000009.13

decimal point

6 places   new decimal point

$n = -6$

9.13×10^{-6}

Converting from standard form to an ordinary number

Example 1   Large number

Change 3.715×10^8 to an ordinary number

$n = +8$, so move the point 8 places to the right

3.7 1 5 0 0 0 0 0. $\therefore$ $3.715 \times 10^8 = \underline{371\ 500\ 000}$

Example 2   small number

Change 8.776×10^{-5} to an ordinary number

$n = -5$, so move the point 5 places to the left

.0 0 0 0 8. 7 7 6 $\therefore$ $8.776 \times 10^{-5} = \underline{0.00008776}$

Calculations with numbers in standard form

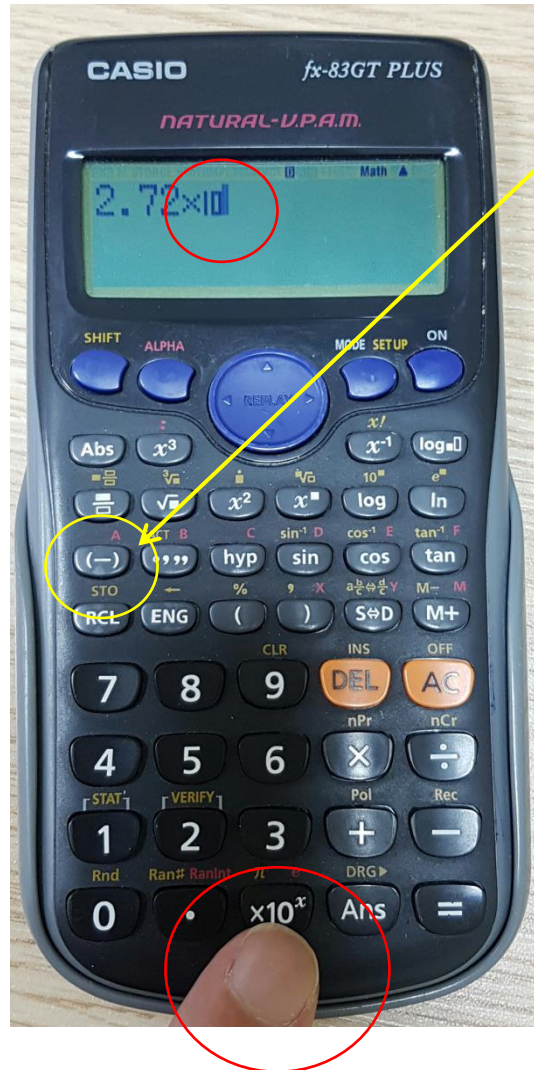You should use your calculators for this learning objective.

Example 1

Work out, $(2.72 \times 10^{-7}) \times (7.92 \times 10^{-7})$ as a number in standard form.

Solution

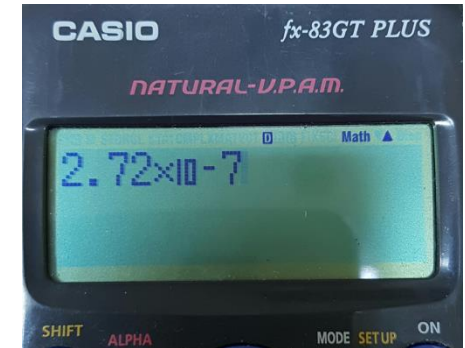**1.** Begin by entering the digits 2.72

2. Press the $x10^x$ button as shown

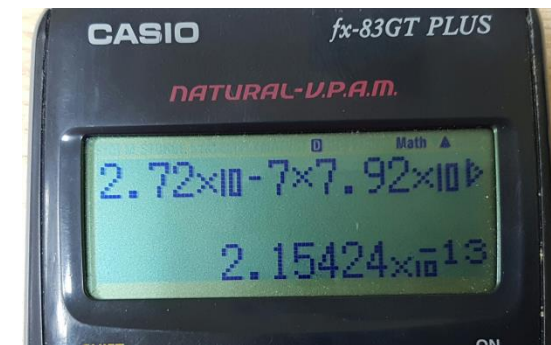**3.** You should see x10 appear in the display

4. Enter -7, using the (-) key

5. Enter the × symbol for multiplication then proceed to enter the next number 7.92×10^{-7} in the same way as before.

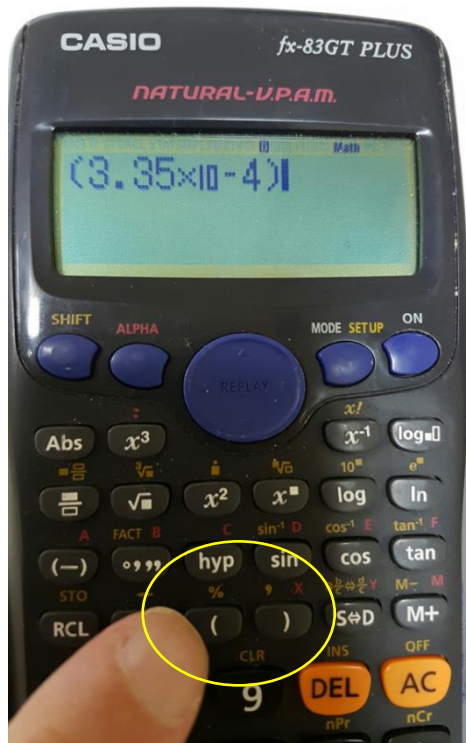**6.** Finally press the = key and you will see the answer 2.915424×10^{-13}
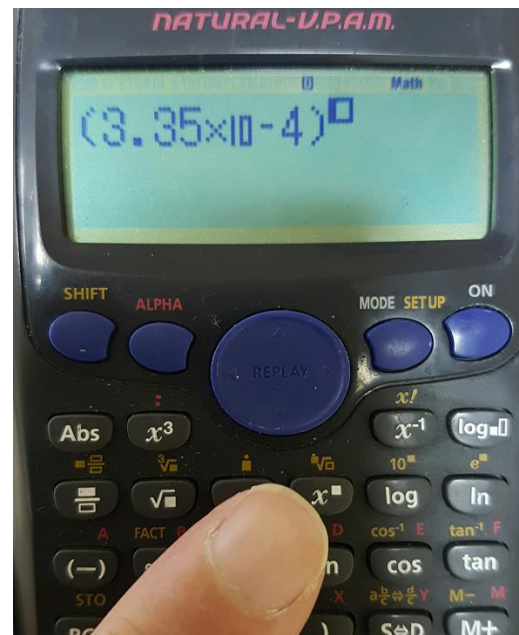
Example 2

Evaluate $(3.35 \times 10^{-4})^2 \div (2.27 \times 10^{-5})$, giving your answer in the form $A \times 10^n$ where A is given to 3 sf.

Solution

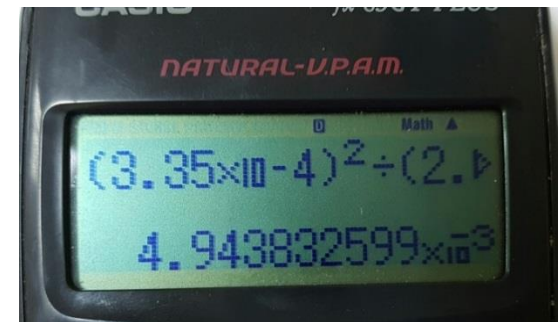**1**. Open parentheses then enter the first number in the same was as before. Close parentheses.

2. Use the x⬜ key to raise the bracket to a power, then enter 2.

3. Press the ÷ and enter the second number in parentheses as before.

4. Press the = key to get the answer

5. Lastly, write down the value of *A*, correct to 3 sf, giving the final answer
4.94×10^{-3}

Your turn!

1. Convert 23 000 000 000 to standard form

2. Convert 0.0000000000000165 to standard form

3. Work out $(2.37 \times 10^{-5})^5 \div (1.23 \times 10^5)$, giving your answer in the form $A \times 10^n$ where A is given to 2 dp.

Answers

1. Convert 23 000 000 000 to standard form

   2.3×10^{10}

2. Convert 0.000000000000165 to standard form

   1.65×10^{-13}

3. Work out $(2.37 \times 10^{-5})^5 \div (1.23 \times 10^5)$, giving your answer in the form $A \times 10^n$ where A is given to 2 dp.

   6.08×10^{-29}

Base dimensions

The base dimensions (base quantities) of Mechanics are internationally agreed dimensions for physical quantities. There are three base dimensions in Mechanics: Mass (M), Length (L) and Time (T).

Electric current, Thermodynamic temperature and luminous intensity are other examples of base quantities that arise in Science and Engineering.

Base units

Each one of the base dimensions has a corresponding base unit. These are called the Système International d'Unités (SI Units). The SI units in Mechanics are;

Mass: Kilogram (kg),

Length: Meter (m),

Time: second (s).

All other units in Mechanics can be derived from these three base dimensions and their corresponding SI units.

Relationship of other common units to base units

Mass kg	Length m	Time s
1000 grammes (g) = 1 kg	1000 millimetres (mm) = 1 m	1 minute (min) = 60 s
1000 kg = 1 ton (t)	1000 m = 1 kilometre (km)	1 hour (hr) = 60 min
1000 milligrammes (mg) = 1g	100 centimetres (cm) = 1 m	
1000 microgrammes (μ g) = 1mg		

Capacity (litres) l
1000 millilitres (ml) = 1 l
1000 microlitres (μ l) = 1 ml

Capacity is the amount of material that a container can hold. It is related to but not the same as volume. There is no base dimension or base unit for capacity

Examples

Area = L × L = L^2 (unit e.g., m^2)

Volume = L × L × L = L^3 (unit e.g., cm^3)

Density = mass/volume = M L^{-3} (unit e.g., kgm^{-3})

Acceleration = L/(T × T) = L T^{-2} (unit e.g., ms^{-2})

Force = mass × acceleration = ML T^{-2}
(unit $1kgms^{-2}$ = 1Newton (N))

1.1.6 Standardise (or convert) units in calculations

We need to be able to convert between units.

Example 1

Given that speed = distance/time calculate the speed of an object in ms^{-1} if the object travels a distance of 2.37 km in 1.13 hours. Give your final answer correct to 2 dp.

2.37 km = 2370 m

1.13 hours = 4068 s

Speed = 2370/4068 $\approx$ 0.58 ms^{-1}

Example 2

An object has a mass of 3 683 000 000 kg. What is the mass in tons (t)?

1 ton = 1000 kg. We are converting from a smaller unit (kg) to a bigger unit (t). The rule is to divide.

3 683 000 000/1000 = 3 683 000 t

Remember: small unit converted to big unit, always divide

Example 3

A bottle has a volume of 0.02 m³. Find the volume in cm³.

1 m³ = 100 × 100 × 100 cm³ = 1 000 000 cm³. We are converting from a bigger unit (m³) to a smaller unit (cm³). The rule is to multiply.

0.02 × 1 000 000 = 20 000 cm³

Remember: big unit converted to small unit, always multiply

Given that density = mass/volume find the density of a object in gm^{-3} which has a mass of 9.625 t and a volume of 500 000 cm^3.

The object is made from a block of pure metal. Why, in this case would it be inappropriate to round your answer to 1 dp if the density is used to identify the type of metal?

Given that density = mass/volume find the density of a object in gcm^{-3} which has a mass of 9.625 t and a volume of 500 000 cm^3.

Mass = 9 625 000 g

Volume = 500 000 cm^3

Density = 9 625 000/500 000 = 19.25 gcm^{-3}

The object is made from a block of pure metal. Why, in this case would it be inappropriate to round your answer to 1 dp if the density is used to identify the type of metal?

Density of pure W (tungsten) = 19.25 gcm^{-3}

Density of pure Au (gold) = 19.3 gcm^{-3}

If a measurement is given to a specified number of decimal places or rounded to the nearest whole unit, then the "true" value must lie within a **range** of values. The least value in the range is called the **lower bound** and the greatest value is called the **upper bound**.

Example 1

The length of a fence is measured as 16 m, to the nearest 1m. The "error" in this case is 1 m, centred on the measurement of 16 m. So we can write this as;

$15.5 \text{ m} \leq 16 \text{ m} < 16.5 \text{ m}.$

15.5 m is the lower bound and 16.5 m is the upper bound.

Example 2

A sphere has a radius of 8.73 cm, correct to 2 dp. Find to the nearest 1 cm³, the greatest and least volume of the sphere. $V_{sphere} = \frac{4\pi}{3} r^3$

We start by writing 8.725 cm $\leq$ 8.73 cm < 8.735 cm.

The lower bound is 8.725 cm and the upper bound, 8.735 cm. We use the lower bound to find the minimum volume and the upper bound to find the maximum volume.

$V_{min} = \frac{4\pi}{3} 8.725^3 \approx 2782$ cm³

$V_{max} = \frac{4\pi}{3} 8.735^3 \approx 2792$ cm³

Example 3

A solid cuboid has a volume of 3.2 m^3 correct to 1 dp.
The mass of the cuboid is 73 kg, to the nearest kg.
Given that density = mass/volume, find correct to 1 dp,
the least and greatest density of the cuboid.

We start by writing;

Volume: 3.15 m$^3 \leq$ 3.2 m^3 < 3.25 m^3.

Mass: 72.5 kg $\leq$ 73 kg < 73.5 kg.

Least density = 72.5/3.25 $\approx$ 22.3 kg m^{-3}

Greatest density = 73.5/3.15 $\approx$ 23.3 kg m^{-3}

A solid cone has a radius, r of 0.7 m correct to 1 dp. The height of the cone, h is 1.0m again, correct to 1 dp. The mass of the cone is 3 kg, to the nearest kg. Given that density = mass/volume, and the volume of the cone, $V_{cone} = \dfrac{\pi}{3}r^2h,$ find correct to 1 dp, the least and greatest density of the cone.

A solid cone has a radius, r of 0.7 m correct to 1 dp. The height of the cone, h is 1.0m again, correct to 1 dp. The mass of the cone is 3 kg, to the nearest kg. Given that density = mass/volume, and the volume of the cone, $V_{cone} = \frac{\pi}{3}r^2 h,$ find correct to 1 dp, the least and greatest density of the cone.

$$\rho_{max} = 3.5 \div \left(\frac{\pi}{3}(0.65)^2 \times 0.95\right) \approx 8.3 \text{ kgm}^{-3}$$

$$\rho_{min} = 2.5 \div \left(\frac{\pi}{3}(0.75)^2 \times 1.05\right) \approx 4.0 \text{ kgm}^{-3}$$

Do NOT round the volume first. Perform the whole operation without rounding. Only round the final answer.

Rational numbers

A **rational number** is any number that can be expressed in fractional form $\frac{p}{q}$ where p and q are integers and $q \neq 0$.

Rational numbers can be expressed as integers, finite (terminating) decimal numbers and decimals with a repeating sequence of digits.

Irrational numbers

An **irrational number** cannot be expressed in fractional form $\frac{p}{q}$ where p and q are integers and $q \neq 0$.

Irrational numbers are sometimes expressed in an exact, surd (root) form or in symbolic form. Their decimal equivalents are infinite with a non-repeating sequence.

Examples of rational and irrational numbers

Rational numbers

Whole numbers: 1, 2, 200500

Fractions: $\frac{1}{2}$, $\frac{25}{7}$, $3\frac{1}{5}$

Terminating decimals: 0.125, 0.9993785632

Recurring decimals (finite or infinite): $0.\dot{2}$ = 0.2222…

Square roots of square numbers: $\sqrt{225}$

Arithmetical operations on rational numbers will always produce another rational number.

Examples of rational and irrational numbers

Irrational numbers

Surds (roots): $\sqrt{2}$, $\sqrt[3]{3}$

Infinite non-recurring decimals: $\pi, \ e$

Arithmetical operations on irrational numbers can produce rational or irrational numbers.

E.g., π^2 is irrational. However, $\sqrt{2} \times \sqrt{8} = \sqrt{16} = 4$ which is a rational number.

1.1.9. Carry out operations of ASMD with simple surds

A surd is another name for a "root" of a number, often it refers to a square root and this is what we will be working with in this and the following learning outcome.

We assume that you can already factorise numbers (split bigger numbers into products of smaller numbers) and simplify (cancel) fractions but if you are not sure how to do this, you can ask your tutor.

Expressing a surd as the root of a prime number

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Consider the surd $\sqrt{80}$. We can express the surd part as the root of a prime number.

$$\sqrt{80} = \sqrt{16 \times 5} = \sqrt{16} \times \sqrt{5} = 4\sqrt{5}$$

On the other hand, with $\sqrt{90}$ we can't express this surd in prime form. However, we can still factorise and reduce the magnitude of the surd part, since,

$$\sqrt{90} = \sqrt{9 \times 10} = \sqrt{9} \times \sqrt{10} = 3\sqrt{10}$$

This technique is sometimes useful when it comes to arithmetical operations.

Basic arithmetic with surds

Like any other type of number we can carry out simple arithmetical operations with surds.

1. Addition and subtraction

We can't add directly $\sqrt{3} + \sqrt{7}$ because the surds are roots of different prime numbers.

If two surds can be factorised so as to contain the root of the same number, then we can add them.

Example 1

Express $\sqrt{27} + \sqrt{48}$ as a single surd

$$\sqrt{27} = \sqrt{9 \times 3} = 3\sqrt{3}$$
$$\sqrt{48} = \sqrt{16 \times 3} = 4\sqrt{3}$$
$$\therefore \ 3\sqrt{3} + 4\sqrt{3} = 7\sqrt{3}$$

Example 2

Express $\sqrt{1000} - \sqrt{360}$ as a single surd

$$\sqrt{1000} = \sqrt{100 \times 10} = 10\sqrt{10}$$
$$\sqrt{360} = \sqrt{36 \times 10} = 6\sqrt{10}$$
$$\therefore \ 10\sqrt{10} - 6\sqrt{10} = 4\sqrt{10}$$

2. Multiplication

Multiplication of surds can help us to "simplify" an expression.

Example 1

Express $2\sqrt{6} \times 3\sqrt{8}$ as a single surd

$$2\sqrt{6} \times 3\sqrt{8} = 6\sqrt{48}$$

$$\text{but } \sqrt{48} = \sqrt{16 \times 3} = 4\sqrt{3}$$

$$\therefore 2\sqrt{6} \times 3\sqrt{8} = 24\sqrt{3}$$

3. Division

Division by numbers in surd form may be done by rationalisation, covered in section 0.1.10. We will deal with simple cases here.

Example 1

Simplify $\dfrac{\sqrt{45}}{\sqrt{40}}$

$$\frac{\sqrt{45}}{\sqrt{40}} = \sqrt{\frac{45}{40}} = \sqrt{\frac{9}{8}} = \frac{3}{2\sqrt{2}}$$

Carry out the following operations, simplifying your answer where possible. "Simplifying" means expressing the surd in prime form of the smallest non-prime integer.

1. $\sqrt{24} + \sqrt{150} =$

2. $\sqrt{2} \times \sqrt{150} =$

3. $\dfrac{5\sqrt{96}}{4\sqrt{63}} =$

Answers

1. $\sqrt{24} + \sqrt{150} = 7\sqrt{6}$

2. $\sqrt{2} \times \sqrt{150} = 10\sqrt{3}$

3. $\dfrac{5\sqrt{96}}{4\sqrt{63}} = 5\sqrt{\dfrac{42}{21}}$

1.1.10. Rationalise the denominator of fractions containing surds

If a fraction has a surd in the denominator, it is usually considered to be "good practice" to "remove" the surd from the denominator so that it appears only in the numerator of the fraction.

The procedure for doing this is called rationalisation of the denominator as we are converting the denominator from an irrational to a rational number.

General procedure for rationalisation

A fraction in the form $\frac{a + \sqrt{b}}{c + \sqrt{d}}$ can be rationalised by multiplying both the numerator and denominator by $c - \sqrt{d}$. If the denominator were $c - \sqrt{d}$ we would multiply by $c + \sqrt{d}$.

Example

Rationalise $\dfrac{2}{2+\sqrt{7}}$

Multiply numerator and denominator by $2-\sqrt{7}$

$$\frac{2}{2+\sqrt{7}}\left(\frac{2-\sqrt{7}}{2-\sqrt{7}}\right)=\frac{4-2\sqrt{7}}{4-7}$$

$$\therefore \frac{2}{2+\sqrt{7}}=-\frac{4-2\sqrt{7}}{3}$$

Example 2

Rationalise $\dfrac{5}{\sqrt{7}}$

Multiply numerator and denominator by $-\sqrt{7}$

$$\frac{5}{\sqrt{7}}\left(\frac{-\sqrt{7}}{-\sqrt{7}}\right) = \frac{-5\sqrt{7}}{-7}$$

$$\therefore \ \frac{5}{\sqrt{7}} = \frac{5\sqrt{7}}{7}$$

Your turn!

Rationalise the following

1. $\dfrac{2}{\sqrt{35}}$

2. $\dfrac{3 - \sqrt{3}}{3 + \sqrt{3}}$

Answers

1. $\dfrac{2}{\sqrt{35}} = \dfrac{2\sqrt{35}}{35}$

2. $\dfrac{3 - \sqrt{3}}{3 + \sqrt{3}} = 2 - \sqrt{3}$

1.1 Summary

In this session you learned how to;

1.1.1   Identify different types of Numbers

1.1.2.   Round a number to a given number of decimal places

1.1.3.   Round a number to a given number of significant figures

1.1.4.   Convert a number to standard index form and vice-versa

1.1.5.   State the base dimensions and base units in mechanics

1.1.6.   Standardise units in calculations

1.1.7.   Carry out calculations within limits of accuracy

1.1.8.   State the difference between rational and irrational numbers

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