Surds Extra (further work on learning objectives 1.1.9 & 1.1.10)

*All exercises should be attempted **without** using a calculator*.

1.1 Basic manipulation of surds

The use of numbers expressed in ***surd*** form when solving Mathematical problems is one of the hallmarks of a competent Mathematician. Manipulation of surds is an indispensable basic skill that can be used extensively in many different topic areas. Although surds have probably been in use for around 1200 years, there is no universally accepted definition of what a surd is. To suit our purposes, one way to view a surd, is simply to think of it as the positive ***root*** of a number, which we leave unevaluated. For this learning objective we will focus on square roots. The ***irrational*** number, $\sqrt{5}$ is a surd but if we evaluate this on a calculator we obtain $\sqrt{5}$ = 2.236067977… Clearly, regarding the accuracy of our calculations, it is better to work with the exact form of$\sqrt{5}$. Introducing rounded decimals at an early stage in a sequence of numerical calculations is inadvisable since any rounding errors will propagate as we progress through the sequence.

Surds can be manipulated according to the following basic rules:

$$\left(\sqrt{a}\right)^2 = \sqrt{a^2} = a$$

$$\sqrt{a}\sqrt{b} = \sqrt{ab}$$

$$\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$$

Here are some examples of simplifying expressions containing surds, using the first rule:

example 1.1a Simplify $\sqrt{200}$

We can write $\sqrt{200}$ in the form $p\sqrt{q}$ where $p, \sqrt{q} \in \mathbb{Z}$ (integers) by factorising 200:

$\sqrt{200} = \sqrt{100 \times 2} = \sqrt{100} \times \sqrt{2} = 10\sqrt{2}$ ▪

example 1.1b Simplify $\sqrt{38} \times \sqrt{2}$

We cannot simplify $\sqrt{2}$, as it is a prime number but we can write:

$\sqrt{38} = \sqrt{2} \times \sqrt{19}$ therefore, $\sqrt{38} \times \sqrt{2} = \left(\sqrt{2} \times \sqrt{19}\right) \times \sqrt{2}$

but $\sqrt{2} \times \sqrt{2} = 2$ so, $\sqrt{38} \times \sqrt{2} = 2\sqrt{19}$ ▪ $\left[\sqrt{a}\sqrt{a} = a\right]$

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example 1.1c express $\sqrt{32}+\sqrt{72}$ in the form $p\sqrt{q}$

In this case we need to look for the smallest common factor of 32 and 72:

$\sqrt{32} = \sqrt{16\times 2} = \sqrt{16}\sqrt{2}$ and $\sqrt{72} = \sqrt{36\times 2} = \sqrt{36}\sqrt{2}$

Hence, $\sqrt{32}+\sqrt{72} = \sqrt{16}\sqrt{2}+\sqrt{36}\sqrt{2} = 10\sqrt{2}$ ■ $\left[\sqrt{a^2}=a\right]$

Now try the following problems yourself.

EXERCISE 1.1a

Simplify the following surds:

a. $\sqrt{80}$ b. $\sqrt{300}$ c. $\sqrt{108}$ d. $\sqrt{3125}$ e. $7\sqrt{343}$

f. $\sqrt{20}+\sqrt{45}$ g. $\sqrt{18}-\sqrt{50}$ h. $\sqrt{27}+\sqrt{75}$ i. $\sqrt{54}+\sqrt{24}$ j. $\sqrt{77}-\sqrt{44}$

k. $\sqrt{8}\sqrt{6}$ l. $\sqrt{10}\sqrt{30}$ m. $\sqrt{15}\sqrt{35}$ n. $\sqrt{21}\sqrt{28}$ o. $\sqrt{132}\sqrt{84}$

In the next set of examples we will use the rule, for division of surds:

example 1.1d Simplify $\dfrac{\sqrt{108}}{\sqrt{75}}$

$$\sqrt{\frac{108}{75}} = \sqrt{\frac{9\times 12}{25\times 3}} = \sqrt{\frac{36}{25}} = \frac{6}{5} \ ■$$

example 1.1e simplify $\sqrt{\dfrac{40}{12}}(\sqrt{27}+\sqrt{90})$

$$\sqrt{\frac{40}{12}}(\sqrt{27}+\sqrt{90}) = \frac{\sqrt{10}}{\sqrt{3}}(3\sqrt{3}+3\sqrt{10})$$

$$\frac{\sqrt{10}}{\sqrt{3}}(3\sqrt{3}+3\sqrt{10}) = 3\sqrt{10}+\frac{30}{\sqrt{3}}$$ [*Note*: $30 = 10\sqrt{3}\sqrt{3}$]

$$3\sqrt{10}+\frac{30}{\sqrt{3}} = 3\sqrt{10}+10\sqrt{3} \ ■$$

EXERCISE 1.1b

Simplify the following:

a. $\frac{\sqrt{10}}{\sqrt{5}}$ b. $\frac{\sqrt{132}}{\sqrt{24}}$ c. $\frac{\sqrt{46}}{\sqrt{6}}$ d. $\frac{\sqrt{1}}{\sqrt{8}}$ e. $\frac{\sqrt{\frac{3}{16}}}{\sqrt{\frac{5}{8}}}$

f. $\frac{1}{\sqrt{2}}(\sqrt{10}+\sqrt{8})$ g. $\frac{\sqrt{15}\sqrt{18}}{\sqrt{3}}$ h. $\frac{\sqrt{27}}{\sqrt{8}}(\sqrt{12}-\sqrt{24})$

1.2 Rationalisation of expressions containing surds

Surds are ***irrational*** numbers. An irrational number cannot be expressed as a fraction p/q where p and q are both integers. Irrational numbers are ***infinite***, ***non-reccuring*** decimals. A very well known irrational number is π. The number $\sqrt{2}$ is irrational, but $\sqrt{0.81}$ is not. Since $\sqrt{0.81}$ = 0.9 and 0.9 = 9/10, which is a ***rational*** number. All fractions and terminating decimals are rational. Infinite recurring decimals are also rational. 1/3 = 0.333…. is rational as is 1/9 = 0.111...

When we have fractions containing surds in the denominator, we can often simplify the fraction by a process known as "***rationalising the denominator***". The idea is to convert an irrational denominator into a rational one, as shown in the following examples.

example 1.2a

Rationalise $\frac{1}{\sqrt{2}}$:

We multiply the fraction by $\frac{\sqrt{2}}{\sqrt{2}}$. This will not alter its value.

$\frac{1}{\sqrt{2}}.\frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{2}$ ■

[if the fraction has $\sqrt{a}$ in the denominator, multiply by $\frac{\sqrt{a}}{\sqrt{a}}$]

example 1.2b

Rationalise $\frac{3+\sqrt{3}}{\sqrt{6}}$: Multiply by $\frac{\sqrt{6}}{\sqrt{6}}$

$\frac{3+\sqrt{3}}{\sqrt{6}}.\frac{\sqrt{6}}{\sqrt{6}} = \frac{\sqrt{6}(3+\sqrt{3})}{6} = \frac{3\sqrt{6}+3\sqrt{2}}{6} = \frac{1}{2}(\sqrt{6}+\sqrt{2})$ ■

example 1.2c

Rationalise $\dfrac{2\sqrt{3}}{2-\sqrt{3}}$: Multiply by $\dfrac{2+\sqrt{3}}{2+\sqrt{3}}$

[Difference of squares $(a - b)(a + b) = a^2 - b^2$ will give a rational denominator.]

$$\frac{2\sqrt{3}}{2-\sqrt{3}}\cdot\left(\frac{2+\sqrt{3}}{2+\sqrt{3}}\right) = \frac{4\sqrt{3}+6}{4-3} = 4\sqrt{3}+6 \ \blacksquare$$

example 1.2d

Rationalise $\dfrac{1-2\sqrt{2}}{3\sqrt{3}-2\sqrt{2}}$: Multiply by $\dfrac{3\sqrt{3}+2\sqrt{2}}{3\sqrt{3}+2\sqrt{2}}$

$$\frac{1-2\sqrt{2}}{3\sqrt{3}-2\sqrt{2}}\cdot\left(\frac{3\sqrt{3}+2\sqrt{2}}{3\sqrt{3}+2\sqrt{2}}\right) = \frac{(3\sqrt{3}+2\sqrt{2})(1-2\sqrt{2})}{(3\sqrt{3})(3\sqrt{3})-(2\sqrt{2})(2\sqrt{2})} = \frac{(3\sqrt{3}+2\sqrt{2})(1-2\sqrt{2})}{19} \ \blacksquare$$

Note: Sometimes the numerator will simplify, but in this example we are probably better off leaving it in its original form.

EXERCISE 1.2

Rationalise the following, simplifying when appropriate:

a. $\dfrac{3}{\sqrt{3}}$

b. $\dfrac{1}{3\sqrt{2}}$

c. $\sqrt{\dfrac{8}{5}}$

d. $\dfrac{1-\sqrt{2}}{\sqrt{2}}$

e. $\dfrac{\sqrt{12}+\sqrt{8}}{\sqrt{6}}$

f. $\dfrac{4\sqrt{12}-3\sqrt{27}}{6\sqrt{3}}$

g. $\dfrac{4}{2+\sqrt{2}}$

h. $\dfrac{3\sqrt{5}}{5-3\sqrt{15}}$

i. $\dfrac{3\sqrt{7}}{6\sqrt{7}-2\sqrt{21}}$

j. $\dfrac{3+\sqrt{5}}{2-\sqrt{3}}$

k. $\dfrac{2\sqrt{2}-\sqrt{3}}{3\sqrt{3}-2\sqrt{6}}$

l. $\dfrac{a\sqrt{b}-b\sqrt{a}}{b\sqrt{b}+a\sqrt{a}}$

1.3 Equations containing surd coefficients

Consider a simple ***linear equation*** : $mx + n = 0$. The constant values, m and n are called ***coefficients***. The solution to this equation is $x = -n/m$. These coefficients can take any form, including surds.

example 1.3a

Solve the equation $2\sqrt{3}\,x - \sqrt{6} = 0$, giving your answer in a simplified form.

$2\sqrt{3}\,x - \sqrt{6} = 0 \Rightarrow 2\sqrt{3}\,x = \sqrt{6}$

[The symbol $\Rightarrow$ means "implies": "statement A $\Rightarrow$ statement B" or "B is implied by A"]

Thus, $x = \dfrac{\sqrt{6}}{2\sqrt{3}} = 2\sqrt{2}$ ■

Linear simultaneous equations can also be solved when the coefficients are surds.

example 1.3b

Solve this pair of linear simultaneous equations giving your answer in simplified surd form:

$2\sqrt{2}\,y - \sqrt{3}\,x + 2\sqrt{2} = 0$ (1)

$2\sqrt{2}\,x - 2y - 4 = 0$ (2)

Firstly, multiply equation (2) by $\sqrt{2}$: $4x - 2\sqrt{2}\,y - 4\sqrt{2} = 0$ (3)

Adding equations (1) and (3) gives: $4x - \sqrt{3}\,x + 2\sqrt{2} - 4\sqrt{2} = 0$

Simplifying: $(4 - \sqrt{3})\,x = 2\sqrt{2}$

Dividing both sides by $(4 - \sqrt{3})$: $x = \dfrac{2\sqrt{2}}{(4-\sqrt{3})}$

Now rationalise the denominator: $x = \dfrac{2\sqrt{2}}{(4-\sqrt{3})}\left(\dfrac{4+\sqrt{3}}{4+\sqrt{3}}\right) = \frac{2\sqrt{2}}{13}(4 + \sqrt{3})$

Making y the subject of equation (1) : $y = \dfrac{\sqrt{3}x - 2\sqrt{2}}{2\sqrt{2}} = \left(\dfrac{\sqrt{3}}{2\sqrt{2}}\right)x - 1$

Substituting $x = \frac{2\sqrt{2}}{13}(4 + \sqrt{3})$: $y = \left(\dfrac{\sqrt{3}}{2\sqrt{2}}\right)\frac{2\sqrt{2}}{13}(4 + \sqrt{3}) - 1$

$y = \frac{\sqrt{3}}{13}(4 + \sqrt{3}) - 1 = \frac{2}{13}(2\sqrt{3} - 5)$ ■

Quadratic equations may also have surd coefficients. The general form is $ax^2 + bx + c = 0$. There are always two solutions to a quadratic, called the ***roots of the equation***. (You may if you wish, depending on the level of your background skills, skip this section for now and come back to it after studying core unit 2.2 on quadratic functions.)

example 1.3c

Form a quadratic equation with the solutions $x = 2$ and $x = -3$:

We know that $x - 2 = 0$ and $x + 3 = 0$, therefore, we can write:

$(x - 2)(x + 3) = 0$, expanding the brackets gives: $x^2 + x - 6 = 0$ ■

example 1.3d

Form a quadratic equation with roots $x = (3 - \sqrt{3})$ and $x = (\sqrt{3} - 3)$:

$(x - (3 - \sqrt{3}))(x - (\sqrt{3} - 3)) = 0$,

expanding the brackets gives:

$$x^2 - (3 - \sqrt{3})x - (\sqrt{3} - 3)x + (3 - \sqrt{3})(\sqrt{3} - 3) = 0$$

finally, this simplifies to: $x^2 + 3(\sqrt{3} - 4) = 0$ ■

EXERCISE 1.3

1. Solve the following linear equations:

a. $4\sqrt{8} - 3\sqrt{24}\,t = 0$ b. $(6 - \sqrt{5})p - \sqrt{30} = 0$ c. $(4 + 2\sqrt{6}) - (2 - 4\sqrt{3})x = 0$

2. Solve these linear simultaneous equations:

a. $2y + \sqrt{3}x = 4$ (1)
$x + \sqrt{2}y = 1$ (2)

b. $2\sqrt{7}p - \sqrt{8}q - 5 = 0$ (1)
$4\sqrt{3}q + \sqrt{6}p + 2 = 0$ (2)

c. $\sqrt{5}r - 3\sqrt{2}s = 2\sqrt{5}$ (1)
$3\sqrt{5}s + 2\sqrt{2}r = \sqrt{2}$ (2)

3. Form the quadratic equations having these roots:

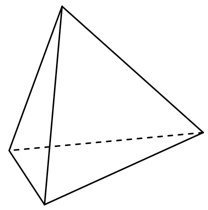
a. $x = 2$ or $x = 2\sqrt{5}$ b. $x = 3\sqrt{2}$ or $x = -2\sqrt{6}$ c. $x = -4\sqrt{5}$ or $x = -(5\sqrt{2} - 4\sqrt{3})$

CHALLENGE EXERCISE 1 – requires additional knowledge and skills from subsequent core units

Do not use a calculator. Express all answers in their simplest surd forms.

1. A regular tetrahedron has an edge length a. Show that the volume, V of the tetrahedron is given by:

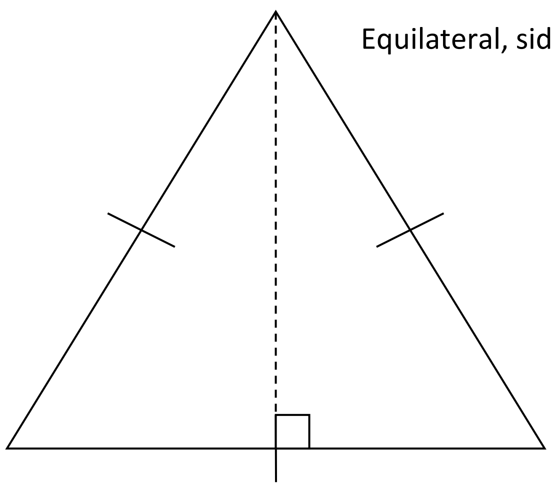
$$V = \frac{a^3\sqrt{3}}{12}$$



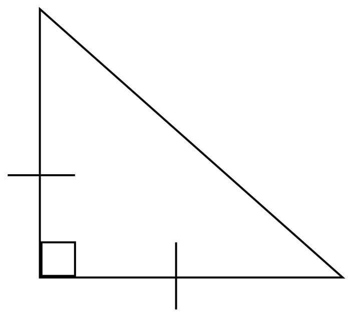
2. The matrix **A** = $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ has an inverse given by $\mathbf{A}^{-1} = \dfrac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

find the inverse of the matrix **A** = $\begin{pmatrix} \sqrt{2} & 1-\sqrt{3} \\ 1+\sqrt{3} & \sqrt{2} \end{pmatrix}$

3. Using the two triangles below, fill in the table with the exact values of the trigonometric ratios.



Equilateral, side length = 2



Right-Isosceles, length of equal sides = 1

Size of angle *A*	sin *A*	cos *A*	tan *A*
30°			
45°			
60°			

4. By choosing suitable values of A from your table in question 3, show that:

a) $2 \sin A \neq \sin 2A$ [*The symbol* $\neq$ *means "not equal to"*]

b) $\sin 2A = 2 \sin A \cos A$

c) $(\sin A)^2 + (\cos A)^2 = 1$

d) $\dfrac{\sin A}{\cos A} = \tan A$

e) $(\cos A)^2 - (\sin A)^2 = \cos 2A$

5. Expand and simplify: $(\sqrt{a} - \sqrt{b})(\sqrt{a} + \sqrt{b})$

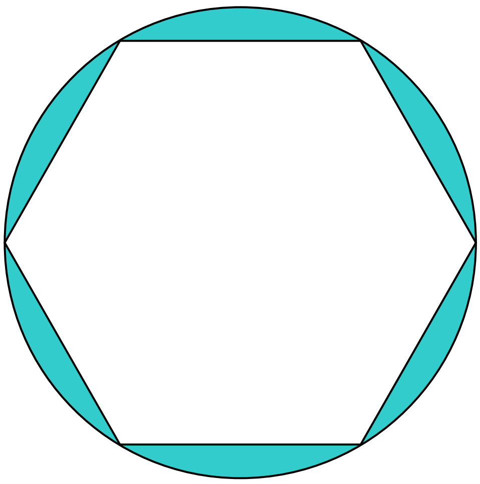
6. Form a quadratic equation in x, whose roots are $3 + 2\sqrt{3}$ and $2 - 3\sqrt{3}$

7. Show that the sum of the square roots of the first ten integers: $\sqrt{1} + \sqrt{2} + \cdots \sqrt{10}$ can be written as:

$$(\sqrt{a} + 1)(b\sqrt{a} + \sqrt{b} + \sqrt{c}) + \sqrt{d}$$

where a, b, c and d are consecutive prime numbers to be determined.

8. Find, in terms of r, the area of the shaded region contained between the hexagon inscribed within a circle of radius r, and the circumference of the circle.



9. Show that $\dfrac{x}{x - \sqrt{x}} \equiv \dfrac{x + \sqrt{x}}{x - 1}$

[*The symbol* $\equiv$ *means "identical to" it is true for all values of* x*. You have to show that the left hand side and right hand side of the expression are equivalent*.]

10. An equilateral triangle has vertices *ABC*, lying on the circumference of a circle, radius *r*. The mid-points of the sides of the triangle *AB*, *BC* and *AC* are *M*, *N* and *P* respectively. M, N and P are joined together to form another equilateral triangle *MNP*. Find the ratio of the area of *MNP* to the area contained between the circumference of the circle and the triangle *ABC*.