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1.2 Algebraic processing skills

Core Preparatory Topics
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FEPS Mathematics Support Framework

1.2 Introduction

The aim of this unit is to assist you in consolidating and developing some fundamental algebraic processing skills.

While studying these slides you should attempt the ‘Your Turn’ questions in the slides and the ‘Quick Questions’ in the separate document.

After studying the slides and completing the short questions, you should attempt the Consolidation Questions.

1.2 Learning checklist

Learning resource	Notes	Tick when complete
Slides		
Your turn questions		
Quick Questions		
Consolidation questions		

1.2 Learning objectives

At the end of the following unit, you should be able to

1.2.1 Apply the laws of indices to algebraic and numerical terms
1.2.2 Add, subtract and multiply algebraic expressions
1.2.3 Evaluate algebraic expressions
1.2.4 Manipulate algebraic fractions
1.2.5 Rearrange (transpose formulae)
1.2.7 Apply some basic concepts of sequences
1.2.8 Solve problems with recurrence relationships
1.2.9 Solve problems involving arithmetic series

Some basic algebraic operations

Simplifying (a word that is sometimes ambiguously used),

Expanding (multiplying out brackets),

Factorising (taking out a common factor or term from an expression),

Combining **algebraic fractions** (finding the Lowest Common Multiple (LCM) of the denominator),

Working with **indices** (know the laws of indices),

Evaluating an expression (substituting a numerical value and working out the answer),

Apply the **order of operations** correctly (BIDMAS),

Change the subject of a formula (**transposition** of the formula)

Simplifying

Means to make an algebraic expression ‘simpler’ (cancelling, collecting terms etc)

E.g simplify, $2ab + 3a^2b + 4ab - a^2b$

“Like terms”

The simplified result is: $6ab + 2a^2b$ it has fewer terms compared to the original expression

Note, it can also be factorised and written as $2ab(3 + a)$

1.2.1 Apply the laws of indices to algebraic and numerical terms

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Laws of indices

$$a^m \times a^n = a^{m+n}$$

$$\frac{a^m}{a^n} = a^{m-n}$$

$$(a^m)^n = a^{mn}$$

$$a^{m/n} = \sqrt[n]{a^m}$$

$$a^0 = 1$$

Indices - examples

Example 1

Simplify $\dfrac{x^3y^2}{xy^3} = x^{3-1}y^{2-3} = x^2y^{-1}$

$= \dfrac{x^2}{y}$

simplify $\dfrac{(x^{-1/3})^{-2}}{(x^2y^{-2/3})^{-3}}$

$= x^{(-1/3)(-2)}(x^2y^{-2/3})^3$

$= x^{2/3}x^6y^{-2} = \dfrac{x^{20/3}}{y^2}$

Indices - examples

Example 2

Evaluate $9^{3/2} \times 16^{3/4}$

Roots first: $(\sqrt{9})^2(\sqrt[4]{16})^3 = (3^2)(2^3)$

$$= 9 \times 8 = \underline{72}$$

Evaluate $\left(2^{-4/7} + 243^{2/3}\right)^0$

$$(\text{some number})^0 = \underline{1}$$

Indices – your turn!

The short ‘Your turn’ questions help you to check your progress and develop your skills as you work through the slides.

Simplify the expression $\dfrac{p^3q^{-2}+p^{-1}q}{p^4q^{-3}}$

Simplify $\dfrac{81^{\frac{2}{3}}}{27^{-\frac{3}{2}}}$

1.2.2. Add, subtract and multiply algebraic expressions

Example 1

Expand $3xy(x^2 + 2x - 8)$

$$= (3xy \underset{1}{\cdot} x^2) + (3xy \underset{2}{\cdot} 2x) + (3xy \underset{3}{\cdot} (-8))$$

$$= 3x^3y + 6x^2y - 24xy$$

Multiplication examples

Example 2

Expand $(2y^2+3x)^3$

Use binomial coefficients (Pascal's triangle):

$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$

1
1 2 1
1 3 3 1
1 4 6 4 1

subst. $a = 2y^2$, $b = 3x$

$$\therefore (2y^2+3x)^3 = (2y^2)^3 + 3(2y^2)^2(3x) + 3(2y^2)(3x)^2 + (3x)^3$$

$$= 8y^6 + 36y^4x + 54y^2x^2 + 27x^3$$

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Multiplication examples

Example 3

Expand $(2x^2 + 3x - 6)(3x^2 - 2x + 1)$

Easy way (fewer errors!)

$$= 2x^2(3x^2 - 2x + 1) + 3x(3x^2 - 2x + 1) - 6(3x^2 + 2x + 1)$$

$$= 6x^4 - 4x^3 + 2x^2 + 9x^3 - 6x^2 + 3x - 18x^2 - 12x - 6$$

$$= 6x^4 + 5x^3 - 22x^2 - 9x - 6$$

An expression containing 3 terms is called a trinomial

Multiplication – your turn!

Expand and simplify the following expression

$$2xy^2(3x^2 - y^2)(3 - x) - 18x^3y^2 + 6xy^4$$

1.2.3. Evaluate algebraic expressions

Example 1

Given $x = -1,\ y = 2,\ z = -\tfrac{1}{2}$, evaluate

$$\frac{x^2 \cdot y}{(z+2)^2}$$

$$= \frac{(-1)^2(2)}{(2-\tfrac{1}{2})^2} = \frac{2}{(\tfrac{3}{2})^2}$$

$$= \frac{2}{(\tfrac{9}{4})} = \underline{\frac{8}{9}}$$

Evaluation examples

Example 2

Evaluate $f(x,y) = x^2y^2 - 2xy + x^2 - 4$

when $x = 1$ and $y = -2$

$$f(1,-2) = (1)^2(-2)^2 - 2(1)(-2) + (1)^2 - 4$$

$$= 4 + 4 + 1 - 4 = \underline{5}$$

Your turn!

Evaluate $f(x, y, z, t) = \frac{3xy - 4z}{5t^2}$

given, $x = -1, y = -3, z = 4, t = 5$

1.2.4. Manipulate algebraic fractions

Example 1

Simplify $\dfrac{2x + 2y}{4z}$

Factorise then cancel: $\dfrac{\cancel{2}(x+y)}{\cancel{2}(2z)} = \dfrac{x+y}{2z}$

or $\dfrac{1}{2}\left(\dfrac{x+y}{z}\right)$

Fraction examples

Example 2

Write $\dfrac{3}{2x^2} + \dfrac{6}{5x}$ as a single fraction

Method 1:

$$\frac{3}{x}\left(\frac{1}{2x} + \frac{2}{5}\right) = \frac{3}{x}\left(\frac{5\times 1 + 2\times 2x}{5\times 2x}\right)$$

$$= \frac{3}{x}\left(\frac{5+4x}{10x}\right)$$

$$= \frac{3}{x^2}\left(\frac{5+4x}{10}\right)$$

Fraction examples

Method 2:

$$\frac{3}{2x^2} + \frac{6}{5x} = \frac{3 \times 5x + 6 \times 2x^2}{2x^2 \times 5x}$$

$$= \frac{15x + 12x^2}{10x^3}$$

$$= \frac{\cancel{x}}{\cancel{x}}\left(\frac{15 + 12x}{10x^2}\right)$$

$$= \frac{3}{x^2}\left(\frac{5 + 4x}{10}\right)$$

Fraction examples

Example 3:

Write $\frac{1}{2x} - \frac{1}{3x} + \frac{1}{5x}$ as a single fraction

$$LCM = 2 \times 3 \times 5 = 30$$

use $30x$ as common denominator

$$\therefore \frac{1}{2x} - \frac{1}{3x} + \frac{1}{5x} = \frac{15}{30x} - \frac{10}{30x} + \frac{6}{30x}$$

$$= \underline{\frac{11}{30x}}$$

Fraction examples

Example 4

Write $\dfrac{3}{x^2+1} - \dfrac{2x}{x-4} + \dfrac{x^2}{x+1}$ as a single fraction

LCM = $(x^2+1)(x-4)(x+1)$

$$\therefore \frac{3(x-4)(x+1)}{(x^2+1)(x-4)(x+1)} - \frac{2x(x^2+1)(x+1)}{(\quad)(\quad)(\quad)} + \frac{x^2(x^2+1)(x-4)}{(\quad)(\quad)(\quad)}$$

Very tedious!

Your turn!

Simplify the following writing your answer as a single fraction;

$$\frac{3t}{2t-1} - \frac{2t-1}{2t+1}$$

1.2.5. Rearrange (transpose formulae)

Example 1

Make x the subject of $x^2+c^2+d^2=(x-c)(x-d)$

$$\cancel{x^2}+c^2+d^2=\cancel{x^2}-cx-dx+cd$$

$$c^2+d^2=-x(c+d)+cd$$

$$x(c+d)=cd-c^2-d^2$$

$$x=\frac{cd-c^2-d^2}{c+d}$$

Examples - transposition

Example 2

Make q the subject of $\dfrac{q+a}{q+b} = \dfrac{q+c}{q+d}$

$$\left(\frac{q+a}{q+b}\right)\cdot\left(\frac{q+d}{q+c}\right) = 1$$

$$(q+a)(q+d) = (q+b)(q+c)$$

$$\cancel{q^2} + qa + qd + ad = \cancel{q^2} + qb + qc + bc$$

$$qa + qd - qb - qc = bc - ad$$

$$\therefore \quad q = \frac{bc - ad}{a + d - b - c}$$

1.2.7 Apply some basic concepts of sequences

A **sequence** is a set of numbers (terms) generated according to some rule.

A **progression** is an ordered set of terms with each term in the sequence is generated by the same mathematical formula, e.g., 2, 4, 6, 8, 10,… In this case, if n is the position number in the sequence, starting with $n = 1$, the value of each term is just $2n$.

The ordered set of primes, 2, 3, 5, 7, 11, 13,... is not a progression, since successive terms cannot be generated from a mathematical formula. It is therefore a sequence.

Your turn!

Work out the next term in each of the following sequences and state the rule for generating the term.

1. $3,\ 9,\ 81, 6561, \ldots$
2. $1, 4, 7, 10, \ldots$

Solution

1 43 046 721, each successive term is the square of the previous term (recurrence relation.)

2 13, add 3 to the previous term or 3n - 2

The n^{th} term of a sequence

The n^{th} term of a sequence is often called the general term and is given a symbol such as u_n

If we know the formula for the general term it is easy to find the value of any term in the sequence.

Eg if $u_n = 5n - 4$ the 27th term is $5 \times 27 - 4 = 131$

We may also need to establish a formula for the general term given some values in the sequence and the term numbers.

Your turn!

1. Find the 1st and 15th term in the sequence defined by $u_n = 3n^2 - 6$

2. Find a possible formula for the nth term of the sequence $0, 3, 8, 15$ ...

3. The nth term in a sequence is given by

$$u_n = n^2 - n - 6, \quad n \geq 1.$$

find the value of n such that $u_n = 0$

Solutions

1. $u_n = 3n^2 - 6$ $\quad u_1 = 3(1)^2 - 6 = -3$

$$u_{15} = 3(15)^2 - 6 = 669$$

2. $0, 3, 8, 15 \ldots$

$$\begin{array}{lcccc} n = & 1 & 2 & 3 & 4 \\ u_n = & 0 & 3 & 8 & 15 \end{array} \quad \therefore\ u_n = n^2 - 1$$

3. $u_n = n^2 - n - 6,\ \ n \geq 1$ $\quad (n-3)(n+2) = 0, n \geq 1\ \therefore n = 3$

1.2.6 Solve problems with recurrence relationships

When the next term in a sequence can be defined in terms of operations on the previous term (or terms), it is known as a recurrence relationship.

The connection between the terms is defined by a recurrence formula linking together the next term and the previous term(s), together with an initial condition (first term or initial terms values.) For example,

$$u_j = 2u_{j-1} + 3u_{j-2}^2 + 6, \qquad u_0 = 5, u_1 = 8$$

The next term depends on the previous two terms

$$u_{k+1} = 2u_k - 7, \; u_0 = 2$$

The next term depends only on the previous term

Example

Find the next term in the following two sequences

1. $u_{n+1} = 3u_n - 7,\ u_1 = 3$

2. $u_{n+2} = 3u_{n+1} + u_n + 2,\ u_1 = 1, u_2 = 4$

Solutions

1. $u_{n+1} = 3u_n - 7,\ u_1 = 3$

$u_2 = 3(3) - 7 = 2$

2. $u_{n+2} = 3u_{n+1} + u_n + 2,\ u_1 = 1, u_2 = 4$

$$u_3 = 3(4) + 1 + 2 = 15$$

Another example

A sequence is generated by the recurrence relation,

$u_{n+1} = 2u_n - 5k, \;\; u_1 = 4$

given that $u_3 = 12$, find the value of k.

Solution

$$u_1 = 4$$
$$u_2 = 2(4) - 5k = 8 - 5k$$
$$u_3 = 2(8 - 5k) - 5k = 12$$
$$16 - 15k = 12 \;\Rightarrow k = \frac{4}{15}$$

1.2.9 Solve problems involving arithmetic series

A finite sequence is written as $u_1, u_2, \ldots u_{n-1}, u_n$
A finite **series** is written as $u_1 + u_2 + \cdots u_{n-1}, +u_n$

In an arithmetic series the next term is found by adding (or subtracting) a constant number.
This number is called **the common difference** it is usually given the symbol ***d*.**

The **first term** of an arithmetic series is usually represented by the symbol ***a*.**

Arithmetic Series

An arithmetic sequence can be written as follows

$n =$	1	2	3	…	$n-1$	n
$u_n =$	a	$a+d$	$a+2d$	…	$a+(n-2)d$	$a+(n-1)d$

Look at the relationship between the term number n and the coefficient of d in the value of u_n. You can see that the coefficient of d is always one less than the term number, n.

It is easy to write down the n[th] term of an arithmetic series (or sequence).

$$\boxed{u_n = a + (n-1)d}$$

Your turn!

1. The first term of the arithmetic sequence is $a = \frac{2}{3}$ and the common difference is $d = -\frac{2}{3}$. Which term k of this sequence has the value $u_k - \frac{20}{3}$?

2. The 3rd term of an arithmetic series is $u_3 = 27$ and the 8th term is $u_8 = -27$. Find an expression for the nth term.

Solutions

1. $u_k = a + (k-1)d$

$$-\frac{20}{3} = \frac{2}{3} + (k-1)\left(-\frac{2}{3}\right)$$

$$-8 = -\frac{2k}{3} \quad \therefore k = 12$$

Solutions

2. The 3rd term of an arithmetic series is $u_3 = 27$ and the 8th term is $u_8 = -27$. Find an expression for the nth term.

First need to form two linear simultaneous equations to find a and d.

$$u_3: a + 2d = 27 \quad (1)$$

$$u_8: a + 7d = -27 \quad (2)$$

$$(2) - (1): 5d = -54 \therefore d = -\frac{54}{5}$$

From (1): $a - \frac{108}{5} = 27 \quad \therefore a = \frac{243}{5}$

$$u_n = a + (n-1)d$$

$$u_n = \frac{243}{5} - \frac{54}{5}(n-1)$$

$$u_n = \frac{1}{5}(297 - 154n)$$

The sum of an arithmetic series from first principles

Consider the sum, S_n of the first n terms, of an arithmetic series, with last term, L.

$$S_n = a + (a+d) + (a+2d) + ... + (L-d) + L$$

$$S_n = L + (L-d) + (L-2d) + ...(a+d) + a \qquad \text{(in reverse)}$$

$$2S_n = (a+L) + (a+L) + ... + (a+L) + (a+L) = n(a+L)$$

$$\boxed{S_n = \frac{n}{2}(a+L)}$$

n^{th} term, $L = a + (n-1)d$

$$\boxed{\therefore S_n = \frac{n}{2}[2a + (n-1)d]}$$

You don't need to know how to derive this but you must be able to apply the formulae

Example of using the sum

Find the sum of the first 50 positive integers which are exactly divisible by 3

The sum we want to find is $3 + 6 + 9 + 12 + \cdots$

The first term $a = 3$ and the common difference is $d = 3$. There are 50 terms, so $n = 50$. Once we've recognised this, it's a straightforward case of substituting the values into the summation formula.

$$S_n = \frac{n}{2}[2a + (n - 1)d]$$

$$S_{50} = \frac{50}{2}[2(3) + (50 - 1)(3)] = 3825$$

Your turn!

An arithmetic series has a first term of 5 and a common difference of 7. Find the sum of the first 8 terms of the series.

Solution

$$s_n = \frac{n}{2}[2a + (n-1)d]$$

$a = 5, d = 7$ and $n = 8$

$$s_8 = \frac{8}{2}[2(5) + 7(8-1)] = 236$$

Another example

Find the least number of terms required for sum of $4 + 9 + 14 + 19 +$ ... to exceed 2000.

Solution

$4 + 9 + 14 + 19 + \dots > 2000$

Using $S = \frac{n}{2}[2a + (n-1)d]$

$\frac{n}{2} \times [2 \times 4 + (n-1) \times 5] > 2000$

$n \times [8 + 5n - 5] > 4000$

$5n^2 + 3n - 4000 > 0$

Solve the equation:

$5n^2 + 3n - 4000 = 0, n \in R$

$n = \frac{-3 \pm \sqrt{9+80000}}{10}$

$n \approx 27.98$ or $n \approx -28.58$

Since n is a whole number and it has to be positive, we need 28 terms.

Using the sigma Σ notation

You will frequently encounter the sigma $\sum$ notation. The symbol $\sum$ is an instruction to add some values together. There will be a formula which generates the values and further information to indicate where to start and stop the summation.

For example,

$$\sum_{n=2}^{7} 5n - 3$$

Is really just a shorthand way of asking you to find $7 + 12 + 17 + 22 + 27 + 32$.

Note, the coefficient of n which is 5 in this case is identical to the common difference $d = 5$ in the corresponding arithmetic series.

Example

Evaluate the following sum,

$$\sum_{r=20}^{r=50} 2r - 5$$

The easiest way to solve this problem is to evaluate,

$$\sum_{r=1}^{r=50} 2r - 5 - \sum_{r=1}^{r=19} 2r - 5$$

It is an arithmetic series with a common difference $d = 2$ and first term $a = -3$. So, we have,

$$\sum_{r=20}^{r=50} 2r - 5 = \frac{50}{2}[2(-3) + 2(50 - 1)] - \frac{19}{2}[2(-3) + 2(19 - 1)] = 2015$$

Further example

Oil drums containing 250 litres are stacked with 25 drums in the first row. The stack is 7 rows high. What is the total volume of oil contained in the stack?

One less in each row

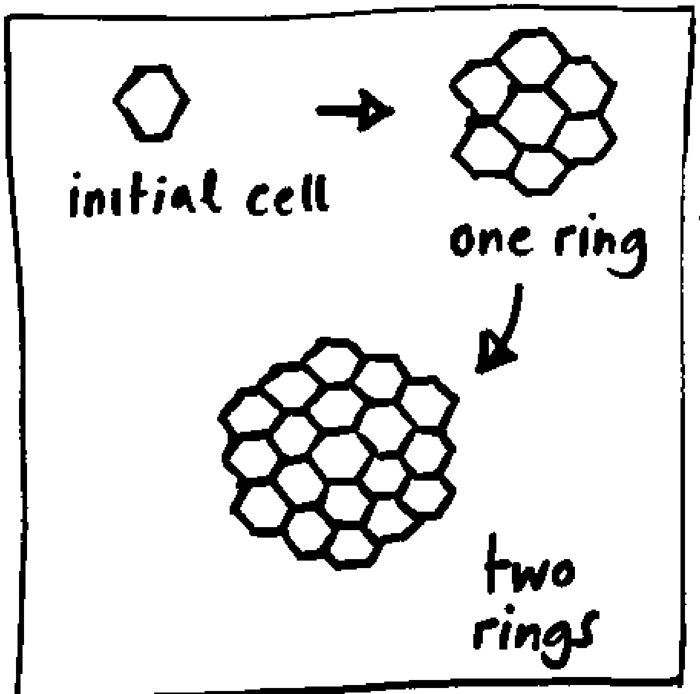
$$\text{Total drums} = 7\times 25 - \sum_{r=1}^{6} r$$

$$= 175 - \frac{6}{2}[2\times 1 + (6-1)\times 1]$$

$$= 154 \qquad 154\times 250 = 38500\,l$$

Your turn!

Honeybees construct honeycombs from hexagonal cells in an arithmetic progression



1. How many cells are in the n^{th} ring?
2. What is the total number of cells in a honeycomb with 9 rings?

Solution

n^{th} ring: $a_n = a + (n-1)d$

$= 6 + (n-1)6$

$= 6n$ $\blacksquare$

$$S_n = \frac{9}{2}[2 \times 6 + (9-1)6] = 270$$

Total cells $= 270 + 1 = 271$ $\blacksquare$

1.2 Summary

You should now be able to do the following

1.2.1 Apply the laws of indices to algebraic and numerical terms
1.2.2 Add, subtract and multiply algebraic expressions
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