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UNIVERSITY OF SURREY

10.1 Introduction to complex numbers

Core Preparatory Topics
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FEPS Mathematics Support Framework

10.1 Introduction

The aim of this unit is to introduce you to some of the basic concepts associated with complex numbers.

While studying these slides you should attempt the ‘Your Turn’ questions in the slides.

After studying the slides, you should attempt the Consolidation Questions.

10.1 Learning checklist

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Learning resource	Notes	Tick when complete
Slides		
Your turn questions		
Consolidation questions		

10.1 Learning objectives

After completing this unit you should be able to

10.1.1 Apply the basic definition $i^2 = -1$ and find $Re(z)$ and $Im(z)$

10.1.2 Add, subtract and multiply complex numbers

10.1.3 Solve a quadratic equation where the discriminant $b^2 - 4ac < 0$

10.1.4 Find the conjugate of a complex number

10.1.5 Divide complex numbers in Cartesian form

10.1.6 Represent complex numbers on the Argand diagram

10.1.7 Find the modulus of a complex number

10.1.8 Find the argument (including principal argument) of a complex number

10.1.1 Apply basic definitions of a complex number

Any number, k such that $k^2 \geq 0$ is called a Real number and belongs to the set of real numbers, i.e., $k \in \mathbb{R}$.

To deal with a number k such that $k^2 < 0$ requires the introduction of a new symbol (number) i defined as $i^2 = -1$ (alternatively, $i = \sqrt{-1}$).

It is sometimes called an imaginary number but this should not be taken literally, as all numbers are imaginary!

The complex number, z

A complex number, z is an expression with two terms (or components), $z = a + ib$, where, $i^2 = -1$ and $a, b \in \mathbb{R}$.

The number z belongs to the set of complex numbers, $\mathbb{C}$, i.e., $z \in \mathbb{C}$.

It comprises a real part, $Re(z) = Re(a + ib) = a$
and an imaginary part, $Im(z) = Im(a + ib) = b$.

Complex numbers have some analogous properties to vectors, as we will see during the unit.

Your turn! (1)

Given two complex numbers, $z_1 = 2 + 3i$ and $z_2 = 2 - 3i$

Find the following;

1. $Im(z_1)$

2. $Re(z_2)$

3. $Re(z_1) - Re(z_2)$

4. $Im(z_1) + 2Re(z_2)$

Solutions

$z_1 = 2 + 3i$

$z_2 = 2 - 3i$

1. $Im(z_1) = 3$

2. $Re(z_2) = 2$

3. $Re(z_1) - Re(z_2) = 2 - 2 = 0$

4. $Im(z_1) + 2Re(z_2) = 3 + 2(2) = 7$

Roots of negative numbers

We can easily work with roots of negative numbers since $i = \sqrt{-1}$ is always a factor!

e.g.,

$$\sqrt{-81} = \sqrt{81}\sqrt{-1} = 9i$$

$$\sqrt{-12} = \sqrt{12}\sqrt{-1} = 2\sqrt{3}i$$

Your turn! (2)

Write in terms of i,

1. $\sqrt{-27}$

2. $\sqrt{-144}$

Solutions

1. $\sqrt{-27} = 3\sqrt{3}\, i$

2. $\sqrt{-144} = 12i$

10.1.2 Add, subtract and multiply complex numbers

Given $z_1 = a + ib$ and $z_2 = c + id$

1. **Addition & subtraction**

$$z_1 + z_2 = a + c + (b + d)i$$
$$z_1 - z_2 = a - c + (b - d)i$$

2. Multiplication

$$\begin{aligned} z_1 z_2 &= (a + ib)(c + id) \\ &= ac + (ad + bc)i + bdi^2 \\ &= ac - bd + (ad + bc)i \end{aligned}$$

Example

Given two complex numbers, $z_1 = 2 + 3i$ and $z_2 = 2 - 3i$ find the following;

1. $z_1 + z_2$
2. $z_1 - z_2$
3. $Im(z_1 + z_2)$
4. $2z_1 + 3z_2$

1. $z_1 + z_2 = 2 + 3i + 2 - 3i = 4$

2. $z_1 - z_2 = 2 + 3i - (2 - 3i) = 6i$

3. $Im(z_1 + z_2) = Im(4) = 0$

4. $2z_1 + 3z_2 = 2(2 + 3i) + 3(2 - 3i) = 10 - 3i$

Multiplication

Evaluate $(2 + i)(3 - i)$

$$\begin{aligned}(2 + i)(3 - i) &= 2(3) - 2i + 3i - i^2 \\ &= 6 + i - (-1) \\ &= 7 + i\end{aligned}$$

Evaluate $Im\{(3-2i)^2\}$

$$\begin{aligned}(3-2i)^2 &= (3-2i)(3-2i) \\ &= 3(3) - 3(2i) - 2i(3) + (2i)^2 \\ &= 9 - 12i - 4 \\ &= 5 - 12i\end{aligned}$$

$$Im(5-12i) = -12$$

Your turn! (3)

Evaluate the following;

1. i^3

2. $2i(3-2i)$

3. $(2-3i)(2+3i)$

Solutions

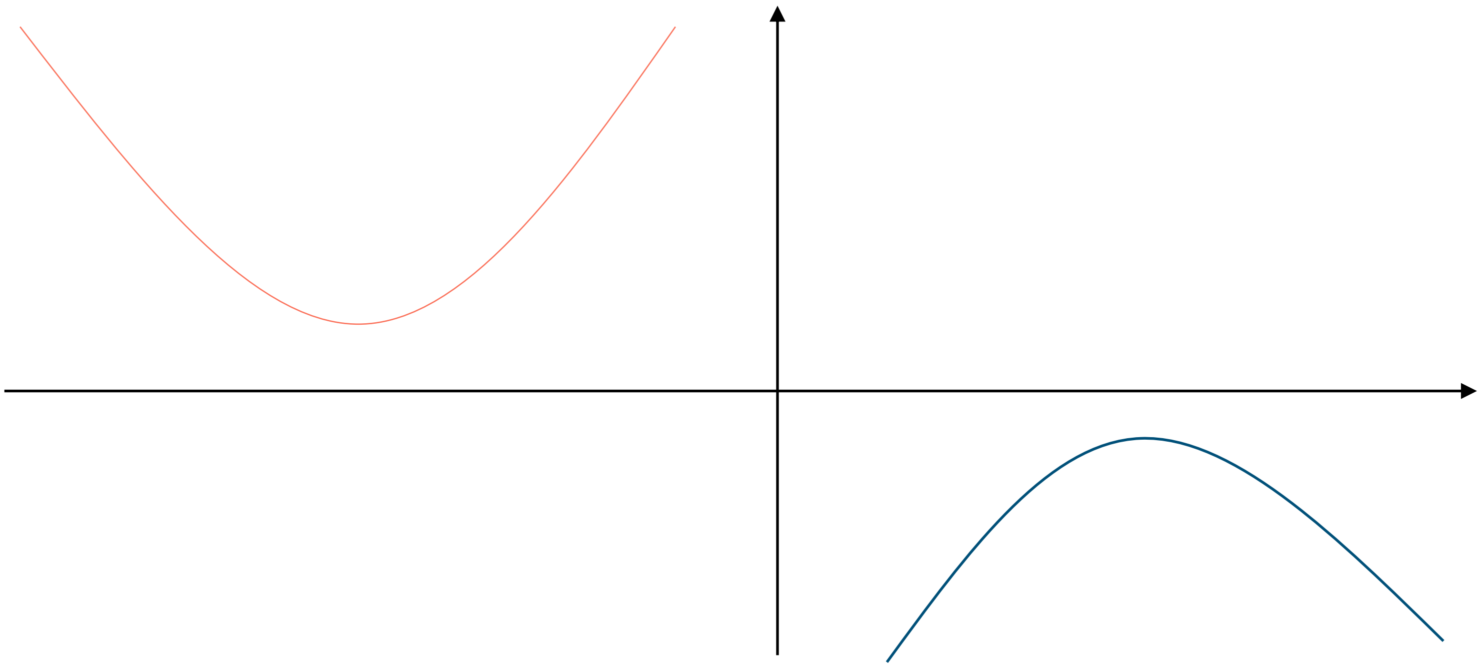
1 $i^3 = i(i^2) = -i$

2 $2i(3-2i) = 6i - 4i^2 = 4 + 6\mathrm{i}$

3 $(2-3i)(2+3i) = 4 - 9i^2 = 13$

10.1.3 Solve quadratics where $b^2 - 4ac < 0$

As your saw in unit 2.2 there are two distinct cases of quadratic functions with a vertical axis of symmetry where the graph of the corresponding function does not intercept the x-axis. These two cases lead to quadratic equations with complex roots.



Example

Solve $x^2 + 9 = 0$

$$x^2 = -9$$

$$x = \pm\sqrt{-9}$$

$$x = \pm 3i$$

Solve $x^2 - 4x + 7 = 0$

$$x = \frac{4 \pm \sqrt{16-28}}{2}$$

$$x = 2 \pm \frac{1}{2}\sqrt{-12}$$

$$x = 2 \pm \sqrt{3}\, i$$

$$x_1 = 2 + \sqrt{3}\, i, \;\; x_2 = 2 - \sqrt{3}\, i$$

Your turn!(4)

Solve the equation $x^2 + 2x + 5 = 0$

Have you noticed something about the pairs of roots in the earlier example and this problem?

Solution

$$x = \frac{-2 \pm \sqrt{4-20}}{2}$$

$$x = \frac{-2 \pm \sqrt{-16}}{2}$$

$$x = \frac{-2 \pm 4i}{2}$$

$$x_1 = -1 + 2i, \quad x_2 = -1 - 2i$$

The imaginary part of the two roots always seem to have the opposite sign. In fact this is true for any polynomial of any degree with real coefficients having complex roots. In this case, complex roots **always occur in pairs** with the only difference between the two roots being the sign of the imaginary part. This is called the **complex conjugate roots theorem**. The word conjugate refers to the difference in the signs.

10.1.4 find the conjugate of a complex number

For a complex number, $z = a + ib$, the complex conjugate is, $z^* = a - ib$ (the symbol $\bar{z}$ is sometimes used instead of z^*).

The product zz^* is always a real number,

$(a + ib)(a - ib) \equiv a^2 + b^2$

Your turn!(5)

Find $z + z^*$ and $z\, z^*$ for each of the following

1 $z = 6 - 3i$

2 $z = \sqrt{5} - 3i\sqrt{5}$

Solutions

1 $z = 6 - 3i$

$$z + z^* = (6 - 3i) + (6 + 3i) = 12$$
$$zz^* = (6 - 3i)(6 + 3i)$$

2 $z = \sqrt{5} - 3i\sqrt{5}$

$$z + z^* = \left(\sqrt{5} - 3i\sqrt{5}\right) + \left(\sqrt{5} + 3i\sqrt{5}\right) = 2\sqrt{5}$$
$$zz^* = \left(\sqrt{5} - 3i\sqrt{5}\right)\left(\sqrt{5} + 3i\sqrt{5}\right) = 5 + \left(3\sqrt{5}\right)^2 = 50$$

10.1.5 divide one complex number by another

$$\frac{z_1}{z_2} = \frac{z_1 z_2^*}{z_2 z_2^*}$$

To make denominator real we multiply denominator and numerator by the conjugate of the denominator.

$$\frac{a+ib}{c+id} = \frac{(a+ib)(c-id)}{(c+id)(c-id)} = \frac{ac+bd+(bc-ad)i}{c^2+d^2}$$

Examples

Evaluate $\frac{10+5i}{1+2i}$

Multiply the numerator and denominator by the conjugate of $1+2i$,

$$\frac{(10+5i)}{(1+2i)} \times \frac{(1-2i)}{(1-2i)} = \frac{10-20i+5i-10i^2}{5}$$

$$= 4-3i$$

Simplify i^{-1}

$$i^{-1} = \frac{1}{i}$$

$$\frac{(1)}{(i)}\frac{(-i)}{(-i)} = \frac{-i}{-i^2} = -i$$

The conjugate of i is $-i$, since $i = 0 + i$

10.1.6 The argand diagram

An Argand diagram is a convenient way of representing complex numbers graphically in a 2-D plane.

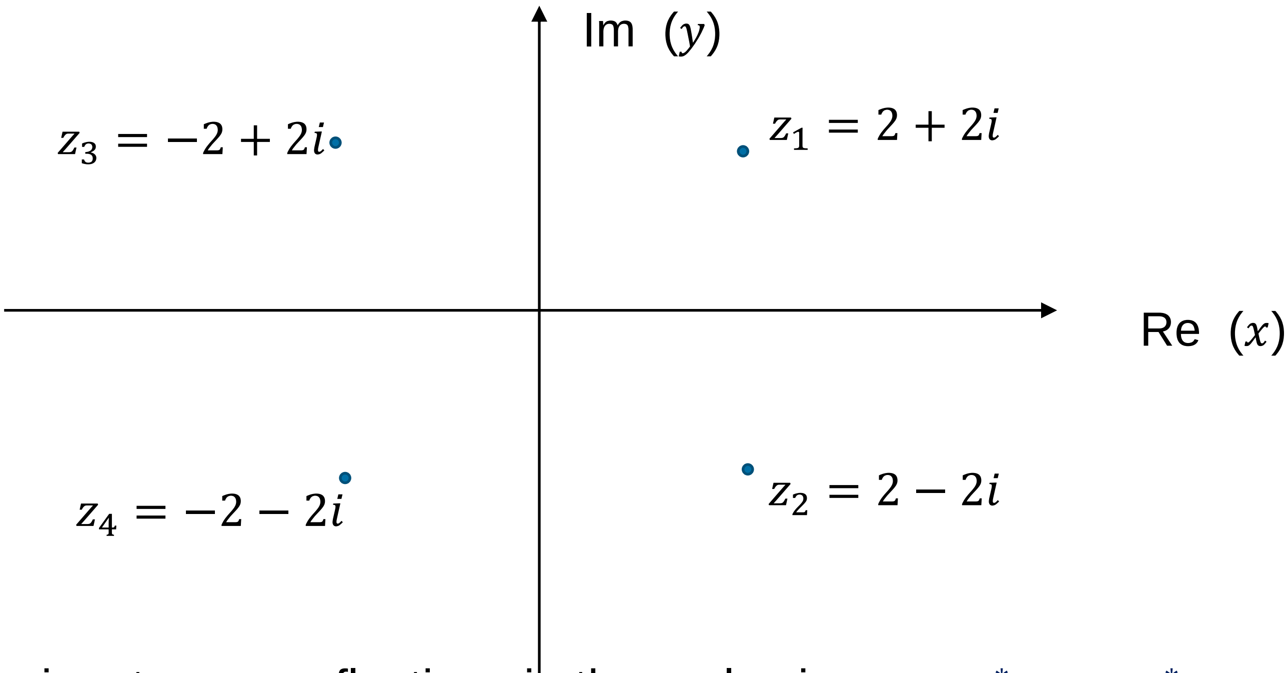
Complex numbers can be represented as points or vectors on the Argand diagram, since $z = x + iy$ where (x, y) is either a point or vector components

The x −axis in the Argand diagram is called the real (Re) axis and the y −axis is called the imaginary (Im) axis.

Points on the Argand diagram

Plot the following complex numbers as points on an Argand diagram;

$$z_1 = 2 + 2i$$
$$z_2 = 2 - 2i$$
$$z_3 = -2 + 2i$$
$$z_4 = -2 - 2i$$

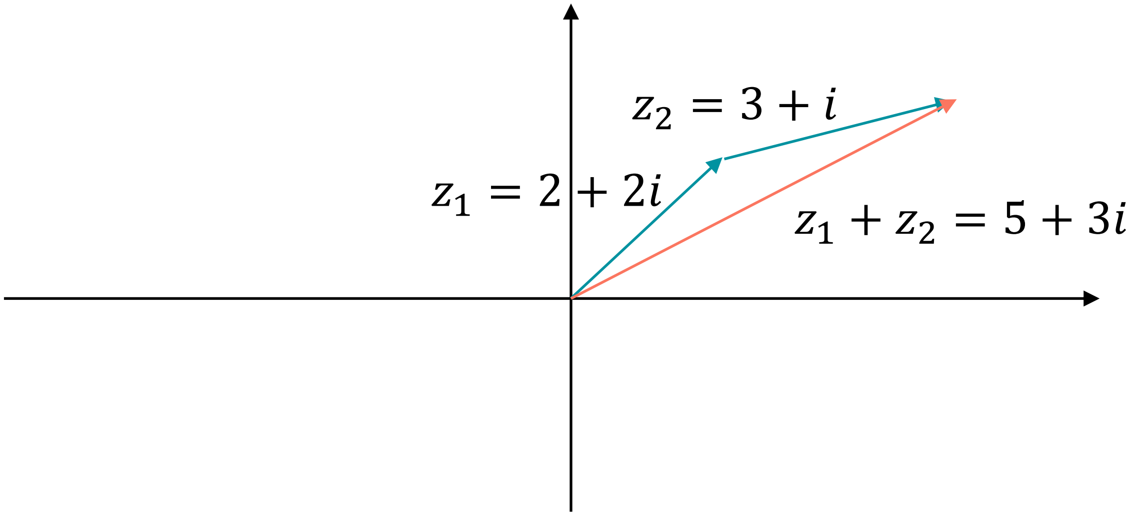


Note that the conjugates are reflections in the real axis: $z_2 = z_1^*, z_4 = z_3^*$

Addition on the Argand diagram

Some representations and operations with complex numbers are closely linked to those of vector components. A complex number on the Argand diagram can be represented as a point or a vector. The addition of two or more complex numbers can be represented in vector form on the Argand diagram, as shown in the following example:

Given $z_1 = 2 + 2i$ and $z_2 = 3 + i$ show the result as a vector addition on an Argand diagram

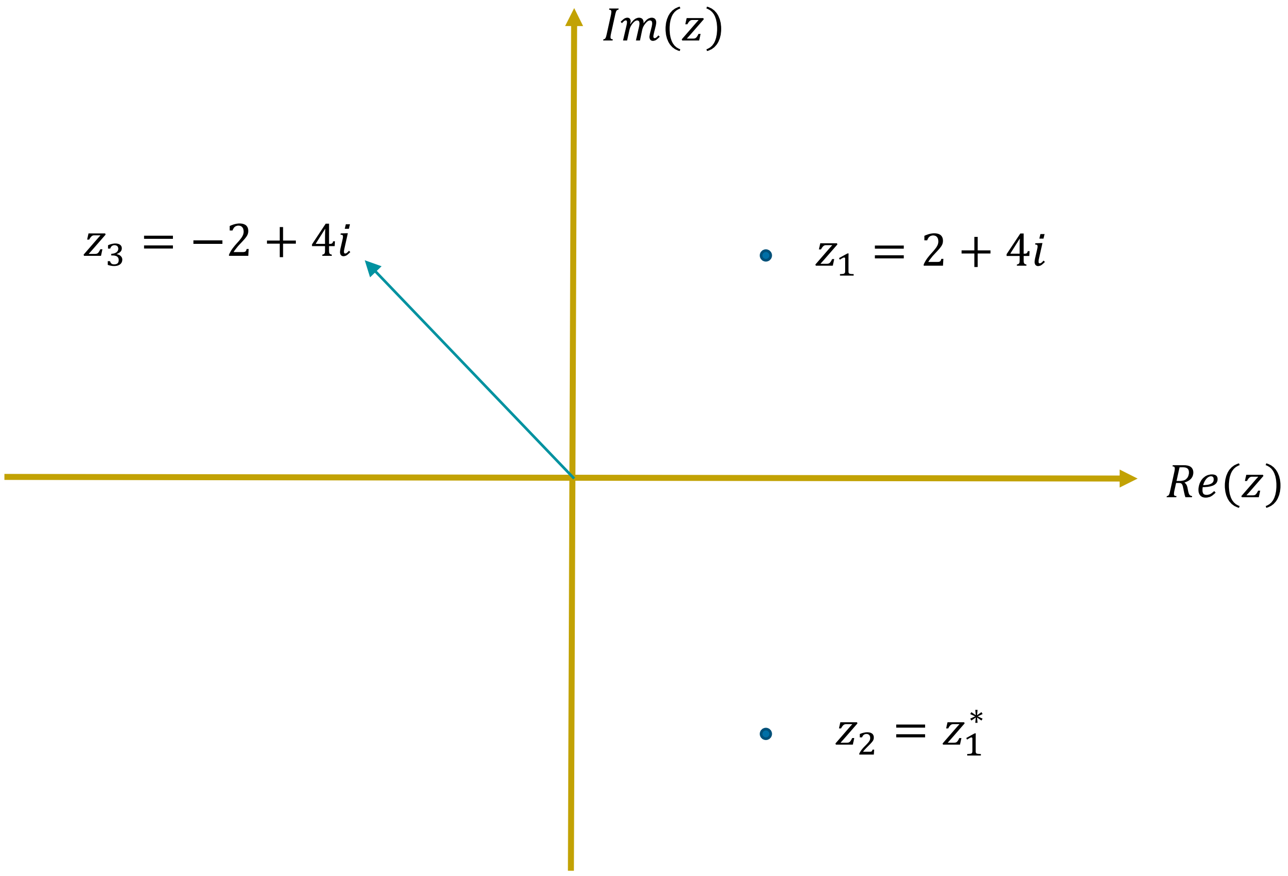


Your turn! (6)

Sketch an Argand diagram to represent the following;

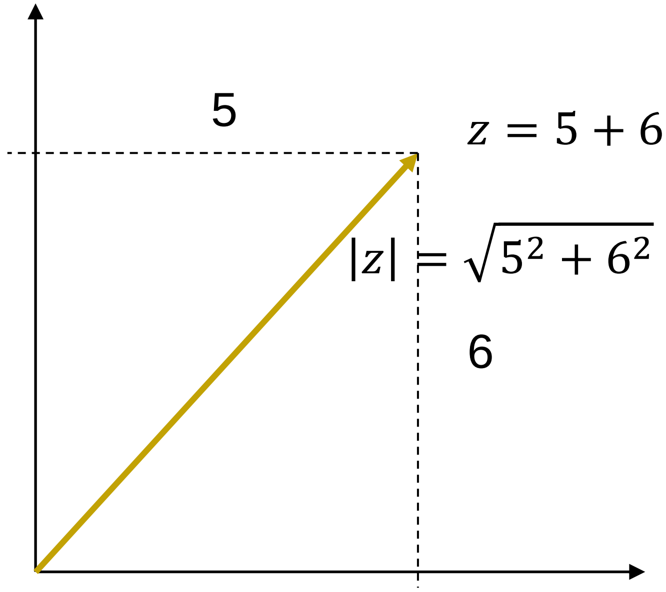
1. The point, $z_1 = 2 + 4i$
2. The point, $z_2 = z_1^*$
3. The vector, $z_3 = -2 + 4i$

Solution



10.1.7 Find the modulus of a complex number $|z|$

As we've already seen, complex numbers have two components that can be represented in a similar way to vectors. Vectors have a magnitude (modulus). The modulus of a complex number is computed in the same way as the magnitude of a 2-dimensional vector. The modulus $|z|$ is equivalent to the length of the vector.



$$z = 5 + 6i$$

$$|z| = \sqrt{5^2 + 6^2} = \sqrt{61}$$

10.1.8 Find the argument of a complex number

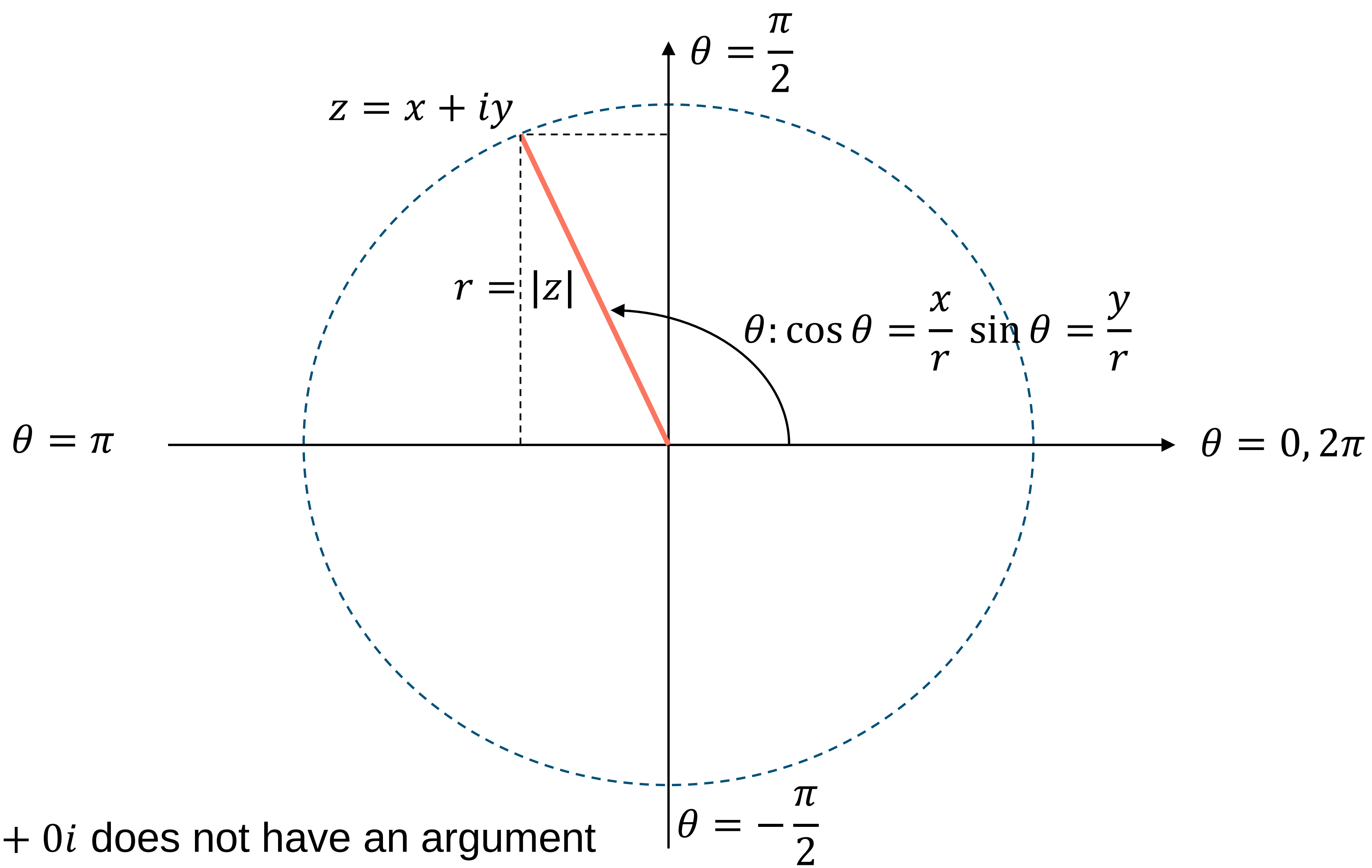
The argument of the complex number $z = x + iy$ is the angle θ between the positive real axis and the vector representing the complex number z on the Argand diagram.

An argument, θ of the complex number $z = x + iy$ is, $\theta = \tan^{-1}\frac{y}{x}$

If $-\pi \leq \theta \leq \pi$ or $-\pi \leq \theta \leq \pi$ it is called the **principal argument**

A complex number can have an infinite number of arguments but only one principal argument

An argument of a complex number $z = x + iy$



$z = 0 + 0i$ does not have an argument

Finding the principal argument

The principal argument

- The principal argument of any complex number is

$$\boxed{\begin{array}{c} -\pi < \theta \leq \pi \\ \text{or} \quad -180^\circ < \theta \leq 180^\circ \end{array}}$$

Example of converting an argument to the principal argument

Given an argument that is not a principal argument, it can be converted to a principal argument by adding or subtracting $2n\pi$ $(n \in \mathbb{Z})$

e.g. 1 convert $\frac{25\pi}{4}$ to a principal argument

$$6\pi = \frac{24\pi}{4} \qquad \frac{25\pi}{4} - \frac{24\pi}{4} = \pi/4$$

e.g. 2 convert $-\pi$ to a principal argument

$$\underline{-\pi} + 2\pi = \underline{\underline{\pi}}$$

Finding the principal argument: Remember this simple strategy

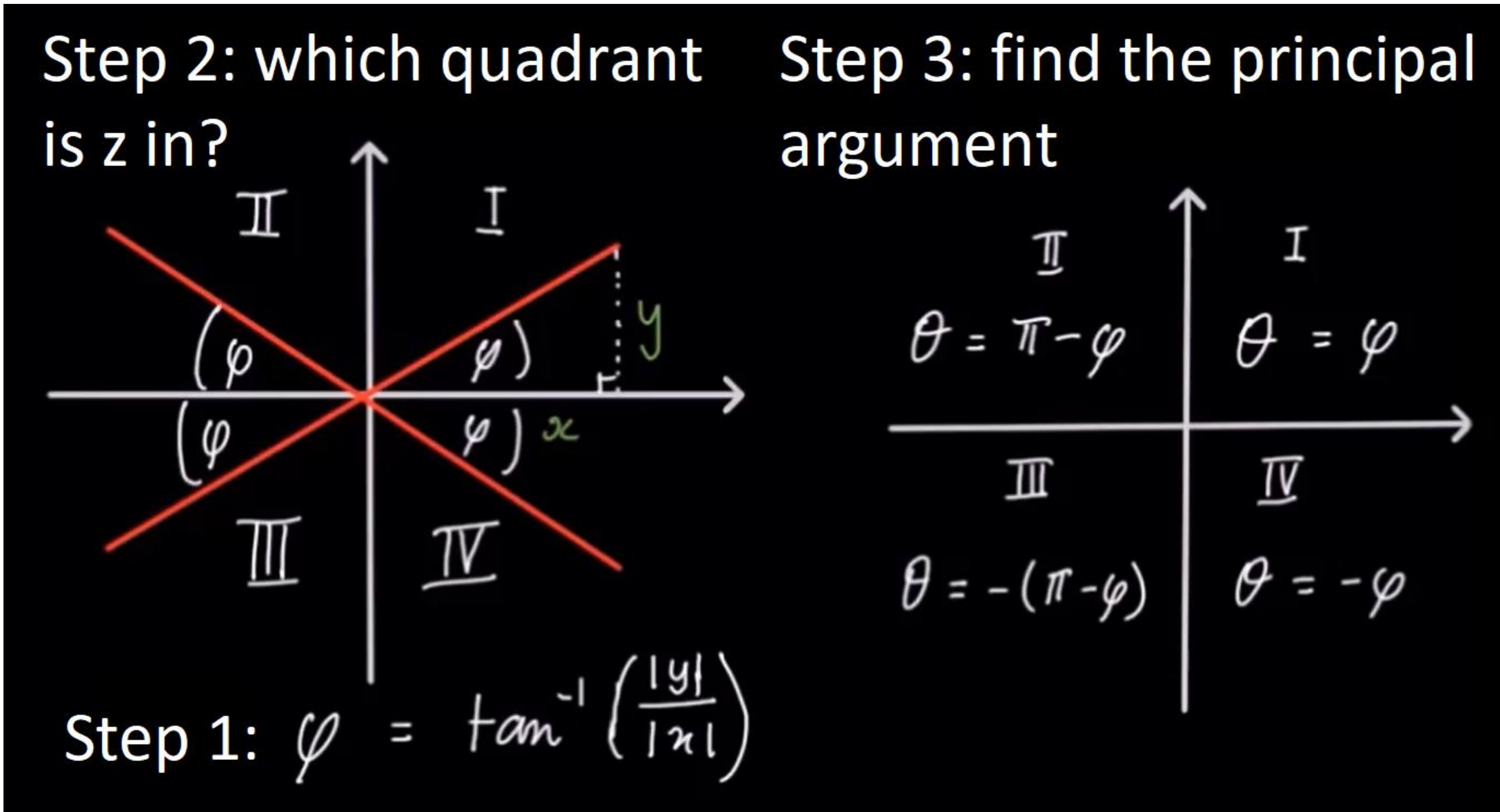
Step 1. Calculate

$$\varphi = tan^{-1}\left(\frac{|y|}{|x|}\right)$$

Step 2. Identify the quadrant which z lies in, by looking at the signs of x and y in $z = x + iy$.

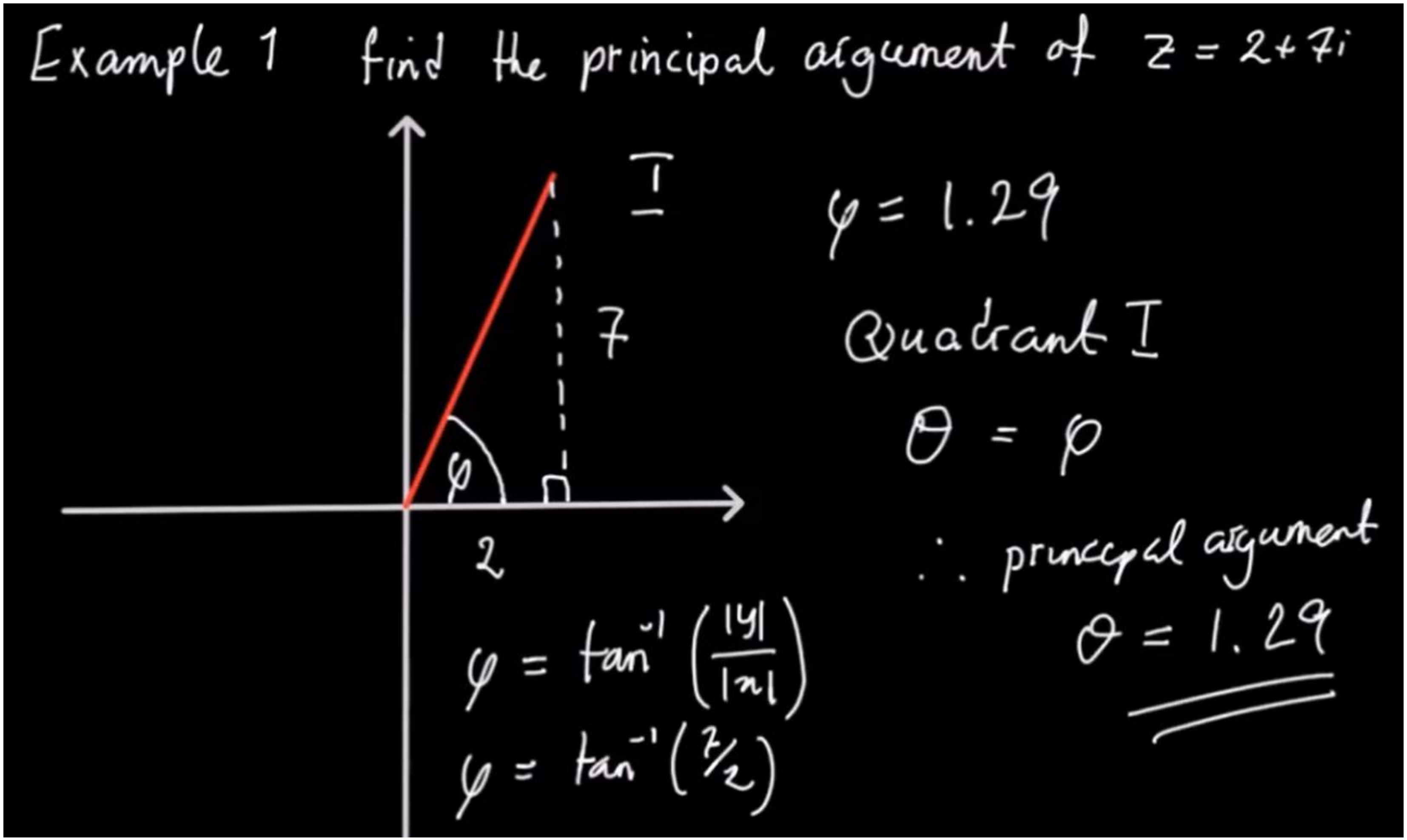
Step 3. Calculate the principal argument according to the quadrant rules below

Quadrant 2 $\boldsymbol{\theta = \pi - \varphi}$	Quadrant 1 $\boldsymbol{\theta = \varphi}$
Quadrant 3 $\boldsymbol{\theta = -(\pi - \varphi)}$	Quadrant 4 $\boldsymbol{\theta = -\varphi}$

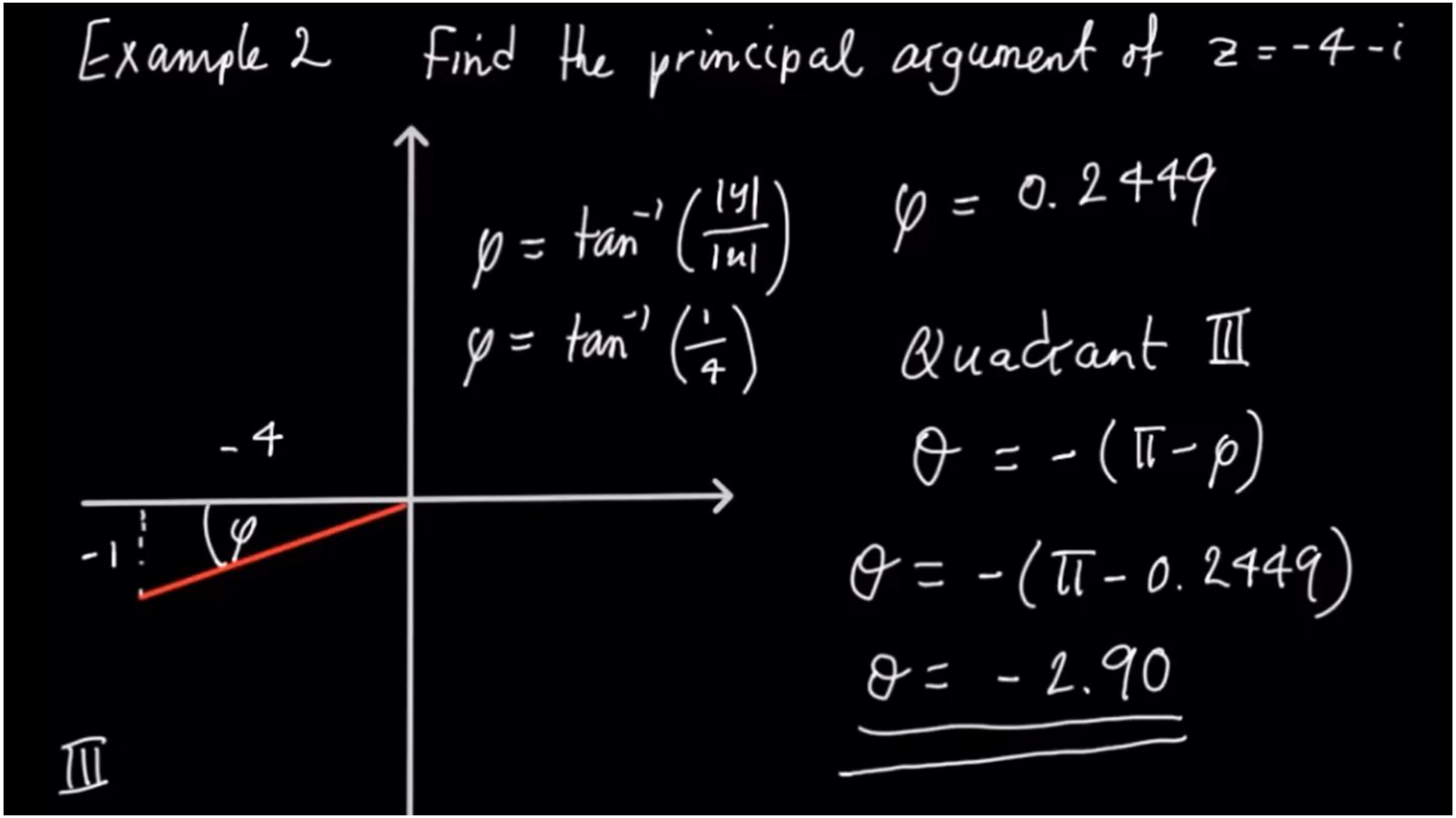


Step 2: which quadrant is z in?
II
I
III
IV
φ
y
x
Step 3: find the principal argument
II
$\theta = \pi - \varphi$
I
$\theta = \varphi$
III
$\theta = -(\pi - \varphi)$
IV
$\theta = -\varphi$
Step 1: $\varphi = \tan^{-1}\left(\frac{|y|}{|x|}\right)$

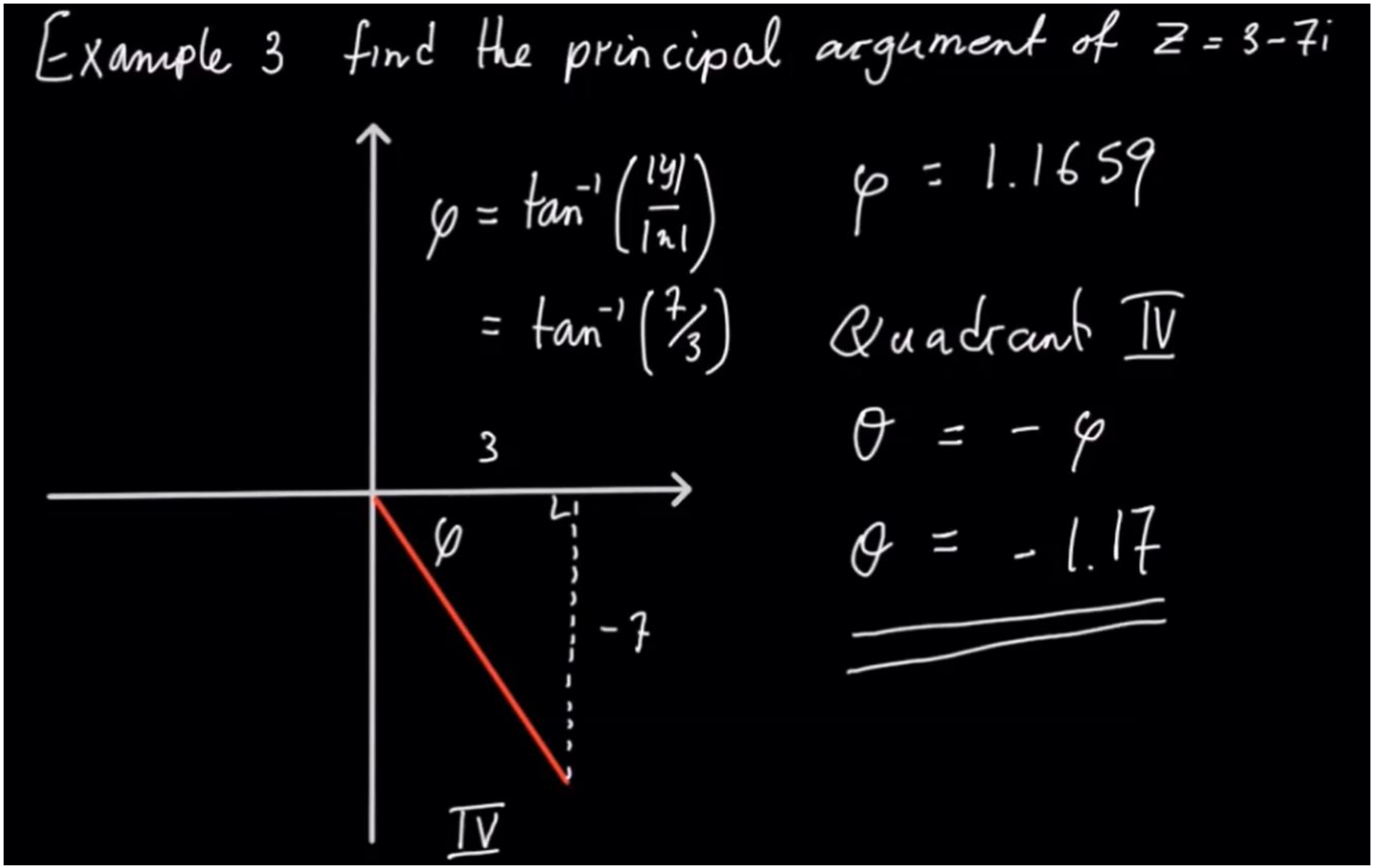
Example: Quadrant I



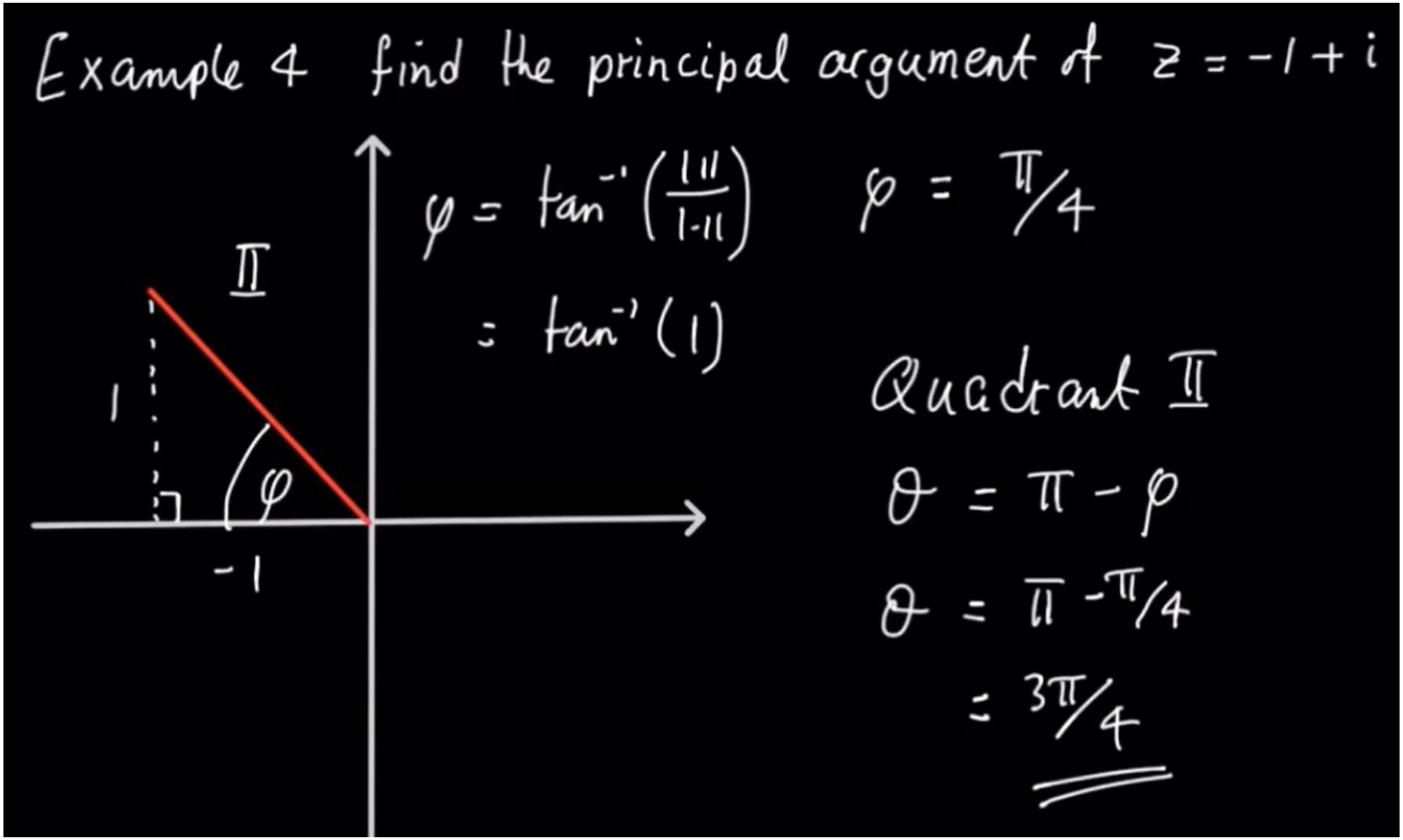
Example: Quadrant 3



Example: Quadrant 4



Example: Quadrant 2



Argument and principal argument

The principal argument is sometimes written as $\theta = \arg(z)$. $-\pi < \theta \leq \pi$

An argument is sometimes written as Arg(z). There is only one principal argument but infinitely many arguments.

An argument, θ of a complex number $z = x + iy$, is such that $\tan\theta = y/x$.

The Arg(z) and arg(z) notation can be confusing. It should not be used without first stating explicitly whether it refers to an argument or a principal argument.

Always use an Argand diagram to be sure you have worked out the argument correctly, according to the quadrant.

It's good practice to work with principal arguments

10.1 Summary

You should now be able to

10.1.1 Apply the basic definition $i^2 = -1$ and find $Re(z)$ and $Im(z)$

10.1.2 Add, subtract and multiply complex numbers

10.1.3 Solve a quadratic equation where the discriminant $b^2 - 4ac < 0$

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