Integration formulae and techniques

Standard integrals

Standard integrals are results of common integrations. You should be familiar with these results and be able to recognise when the results can be used to solve an integration problem.

$$\int 0 \, \mathrm{d}x = c$$

$$\int 1 \, \mathrm{d}x = x + c$$

$$\int ax^n \, \mathrm{d}x = \frac{ax^{n+1}}{n+1} + c$$

$$\int (ax+b)^n \, \mathrm{d}x = \frac{(ax+b)^{n+1}}{a(n+1)} + c; n \neq -1$$

$$\int \frac{1}{x} \mathrm{d}x = \ln |x| + c$$

$$\int \frac{1}{ax+b} \mathrm{d}x = \frac{1}{a} \ln |ax+b| + c$$

$$\int e^x \mathrm{d}x = e^x + c$$

$$\int a^x \mathrm{d}x = \frac{1}{\ln a} a^x + c$$

$$\int \cos x \, \mathrm{d}x = \sin x + c$$

$$\int \sin x \, \mathrm{d}x = -\cos x + c$$

$$\int \tan x \, \mathrm{d}x = \ln |\sec x| + c$$

$$\int \operatorname{cosec} x \cot x \, \mathrm{d}x = -\operatorname{cosec} x + c$$

$$\int \sec x \tan x \, \mathrm{d}x = \sec x + c$$

$$\int \operatorname{cosec}^2 x \, \mathrm{d}x = -\cot x + c$$

$$\int \sec^2 x \, \mathrm{d}x = \tan x + c$$

$$\int \cos(ax+b) \, \mathrm{d}x = \frac{1}{a} \sin(ax+b) + c$$

$$\int \sin(ax+b)\,\mathrm{d}x = -\frac{1}{a}\cos(ax+b) + c$$

$$\int \mathrm{cosec}x\,\mathrm{d}x = -\ln|\mathrm{cosec}x + \cot x| + c$$

$$\int \sec x\,\mathrm{d}x = \ln|\sec x + \tan x| + c$$

$$\int \cot x\,\mathrm{d}x = \ln|\sin x| + c$$

$$\int \sec^2(ax+b)\,\mathrm{d}x = \frac{1}{a}\tan(ax+b) + c$$

$$\int \sec(ax+b)\tan(ax+b)\,\mathrm{d}x = \frac{1}{a}\sec(ax+b) + c$$

$$\int \mathrm{cosec}(ax+b)\cot(ax+b)\,dx = -\frac{1}{a}\mathrm{cosec}(ax+b) + c$$

$$\int \mathrm{cosec}^2(ax+b)\,dx = -\frac{1}{a}\cot(ax+b) + c$$

$$\int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a} + c$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a} + c; \quad |x| < a$$

$$\int e^{ax+b}\,dx = \frac{1}{a}e^{ax+b} + c$$

Rules for indefinite integration

Integration has many different rules and methods but for this lecture we will consider;

1. $\int a\,f(x)\,\mathrm{d}x = a\int f(x)\,\mathrm{d}x$, where a is a constant.
2. $\int\{f(x) \pm g(x)\}\,\mathrm{d}x = \int f(x)\,\mathrm{d}x \pm \int g(x)\,\mathrm{d}x$

Important techniques

Trigonometric substitutions

Case I	The integrand contains, $\sqrt{a^2 - x^2}$	substitute $x = a\sin\theta$
Case II	The integrand contains,$\sqrt{a^2 + x^2}$, or $\frac{dx}{a^2+x^2}$	substitute $x = a\tan\theta$
Case III	The integrand contains, $\sqrt{x^2 - a^2}$	substitute $x = a\sec\theta$

Integration of quotients

There are three cases to consider where functions appear in fractional form and the way in which we carry out the integration will depend on the form of the integrand.

Case I	$\int \frac{f'(x)}{f(x)}\mathrm{d}x$	Solve by direct integration
Case II	$\int \frac{u'(x)}{f(u(x))}\mathrm{d}x$	Solve by changing the variable
Case III	$\int \frac{f(x)}{g(x)}\mathrm{d}x$	Solve by partial fractions

Integration by parts

Integration by parts is a method that can be used to integrate certain products of functions. The formula for integration by parts is,

$$\int u\frac{\mathrm{d}v}{\mathrm{d}x} = uv - \int v\frac{\mathrm{d}u}{\mathrm{d}x}\mathrm{d}x$$

where u and v are both functions of x. It is usually written in a more compact form as,

$$\int u\,\mathrm{d}v = uv - \int v\,\mathrm{d}u$$

The fundamental theorem of calculus (definite integration)

$$\int_a^b f(x)\,dx = [F(x)]_a^b = F(b) - F(a)$$

In the case of integration by parts

$$\int_q^p u\,\mathrm{d}v = [uv]_q^p - \int_q^p v\,\mathrm{d}u$$