Dr Richard Harrison

UNIVERSITY OF SURREY

11.5 Review of basic integration with applications

Core Preparatory Topics
1.1
1.2
2.1
2.2
2.3
3.1
5.1
5.2
5.3
9.1
10.1
11.1
11.5

FEPS Mathematics Support Framework

11.5 Introduction

The aim of this unit is to review some of the basic rules of integration together with some applications.

While studying these slides you should attempt the ‘Your Turn’ questions in the slides.

After studying the slides, you should attempt the Consolidation Questions.

11.5 Learning checklist

Learning resource	Notes	Tick when complete
Slides		
Your turn questions		
Consolidation questions		

11.5 Learning objectives

At the end of this lecture and seminar you should be able to;

11.5.1 Integrate functions of the form ax^n (n is rational, $n \neq -1$) using

$$\int ax^n dx = \frac{ax^{n+1}}{n+1} + C$$

11.5.2 Apply the fundamental theorem of calculus:

$$\int_a^b f(x)dx = [F(x)]_a^b = F(b) - F(a)$$

and use this to solve a variety of problems

11.5.3 Integrate basic functions containing $e^x, \sin x, \cos x$

11.5.4 Find the equation of a curve given $f'(x)$ and any point $P(x, y)$ on the curve

Integration as the reverse process of differentiation

Integration is the reverse of differentiation but consider the following functions and their derivatives:

Differentiation →

Function
x^2
$x^2 + 5$
$x^2 - 3$

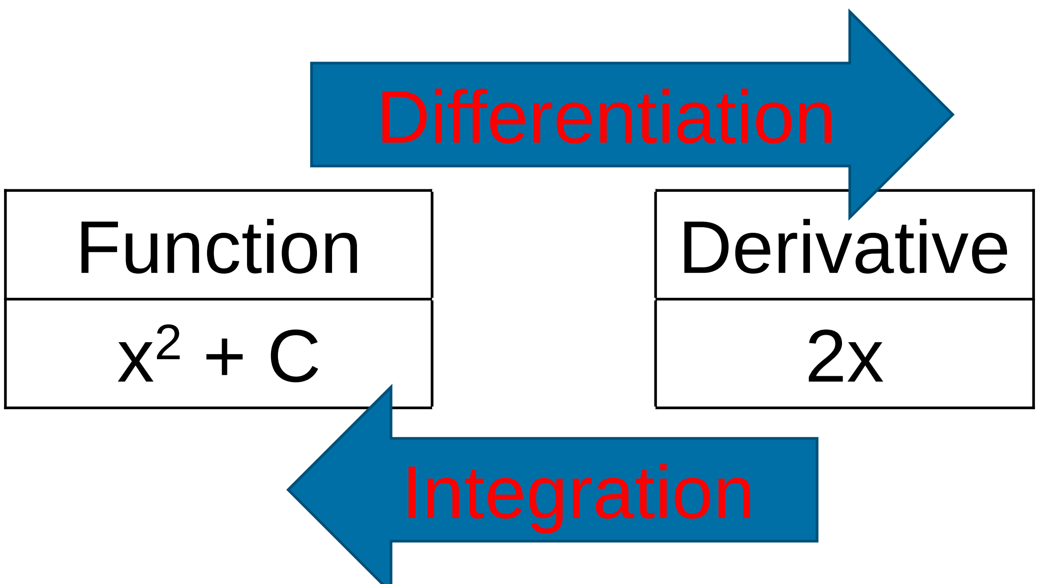
Derivative
2x
2x
2x

← Integration - but which one?

The constant of integration

We don't know what the original function was, only that it belongs to a family of functions containing x^2 + a constant!

So, we add a constant of integration, C, where c is any real number:



11.5.1 Integrate functions of the form ax^n

A general rule:

$$\text{If } \frac{dy}{dx} = ax^n, \text{ then } y = \frac{ax^{n+1}}{n+1}$$

But we must remember the constant of integration:

$$\text{If } \frac{dy}{dx} = ax^n, \text{ then } y = \frac{ax^{n+1}}{n+1} + C$$

Does this rule work for all values of n?

What happens when $n = -1$?

If we did try and use our rule for $n = -1$, think what would happen:

$$\text{If } \frac{dy}{dx} = ax^n, \text{ then } y = \frac{ax^{n+1}}{n+1} + C$$

Let $\frac{dy}{dx} = ax^{-1}$, then using the rule above:

$$y = \frac{ax^0}{0} + c$$

Cleary the division by zero is an undefined operation…

We need a special case!

Integration is the reverse of differentiation

We know that $\frac{d}{dx}(\ln x) = \frac{1}{x}$

So,

$$\boxed{\int \frac{dx}{x} = \ln x + C}$$

We must add a condition to our general rule for integration of algebraic expressions because it is not valid for every case:

$$\text{If } \frac{dy}{dx} = ax^n, \text{ then } y = \frac{ax^{n+1}}{n+1} + C, \; n \neq -1$$

It's mostly the case that we use the integration symbol,

$$\int ax^n dx = \frac{ax^{n+1}}{n+1} + C$$

Example

Given $\dfrac{dy}{dx} = -3x^2 + \dfrac{5}{x^3} + \sqrt[4]{x}$, find y as a function of x.

We want to find $\int -3x^2 + \dfrac{5}{x^3} + \sqrt[4]{x} \ dx$

Procedure: integrate each term separately using the rule:

$$\text{If } \frac{dy}{dx} = ax^n, \text{ then } y = \frac{ax^{n+1}}{n+1} + C, \ n \neq -1$$

Rewrite the expression we want to integrate using negative and fractional powers,

$$\frac{dy}{dx} = -3x^2 + \frac{5}{x^3} + \sqrt[4]{x}$$

becomes:

$$\frac{dy}{dx} = -3x^2 + 5x^{-3} + x^{1/4}$$

Then we use the rule for integration,

$$y = -3\frac{x^{2+1}}{2+1} + 5\frac{x^{-3+1}}{-3+1} + \frac{x^{1/4+1}}{1/4+1} + C$$

Notice that all constants of integration are consolidated into one constant of integration. Now simplify the coefficients and powers:

$$y = -x^3 - \frac{5}{2x^2} + \frac{4}{5}x^{5/4} + C$$

Example

Find,

$$\int \left(4x^2 + 2x^{-2/3}\right) dx$$

The rule for integration can be applied to individual terms. It's often easier to think of it as,

1. Add 1 to the exponent
2. Divide by the new exponent

$$\int \left(4x^2 + 2x^{-2/3}\right) dx = \frac{4x^3}{3} + 6x^{1/3} + C$$

Example

Find,

$$\int x^2\left(x^3 - x^{-2/3} + \frac{\sqrt{x}+2}{x}\right)dx$$

Before we integrate, we must simplify the expression into single terms in x.

$$\int\left(x^5 - x^{4/3} + x^{3/2} + 2x\right)dx$$

Then

$$\int f(x)dx = \frac{x^6}{6} - \frac{3x^{7/3}}{7} + \frac{2x^{5/2}}{5} + x^2 + C$$

Your turn! (1)

Given $f(x) = \sqrt{x^5} - \frac{4x}{\sqrt[3]{x^4}} + e^{\pi} - \frac{1}{\ln 2}$

Find $\int f(x)dx$

Solution

$$f(x) = \sqrt{x^5} - \frac{4x}{\sqrt[3]{x^4}} + e^{\pi} - \frac{1}{\ln 2}$$

$$f(x) = x^{\frac{5}{2}} - 4x^{-\frac{1}{3}} + e^{\pi} - \frac{1}{\ln 2}$$

$$\int f(x)dx = \frac{x^{\frac{7}{2}}}{\frac{7}{2}} - \frac{x^{\frac{2}{3}}}{\frac{2}{3}} + \left(e^{\pi} - \frac{1}{\ln 2}\right)x + C$$

$$\int f(x)dx = \frac{2x^{\frac{7}{2}}}{7} - \frac{3x^{\frac{2}{3}}}{2} + \left(e^{\pi} - \frac{1}{\ln 2}\right)x + C$$

11.5.2 Apply the fundamental theorem of calculus

$$\int_a^b f(x)dx = \left[F(x)\right]_a^b = F(b) - F(a)$$

The above expression is called the fundamental theorem of calculus

Example

Evaluate the following definite integral:

$$\int_{1}^{3}(x^2 - 2x^3)dx$$

The numbers at the top and the bottom of the integration symbol are the **limits of integration**

$$\int_{1}^{3}(x^2 - 2x^3)dx$$

Since we are integrating with respect to x, as indicated by the dx, then the limits are values of x, in this case, $x = 1$ is the lower limit and $x = 3$, the upper limit.

To evaluate this definite integral, we first integrate as usual,

$$\int_1^3 (x^2 - 2x^3)dx = \left[\frac{x^3}{3} - \frac{x^4}{2}\right]_1^3$$

Notice the use of square brackets after integration, and the limits of integration are written to the right of the square brackets. We do not need to include the constant of integration as it will cancel out.

Finally, evaluate the definite integral,

$$\int_1^3 (x^2 - 2x^3)\,dx = \left[\frac{x^3}{3} - \frac{x^4}{2}\right]_1^3$$

$$= \left(\frac{3^3}{3} - \frac{3^4}{2}\right) \quad - \quad \left(\frac{1^3}{3} - \frac{1^4}{2}\right) = -\frac{94}{3}$$

Substitute x = 3

Substitute x = 1

Your turn! (2)

Evaluate the definite integral

$$\int_1^e \left(\frac{1}{x} + 3x^2\right) dx$$

Solution

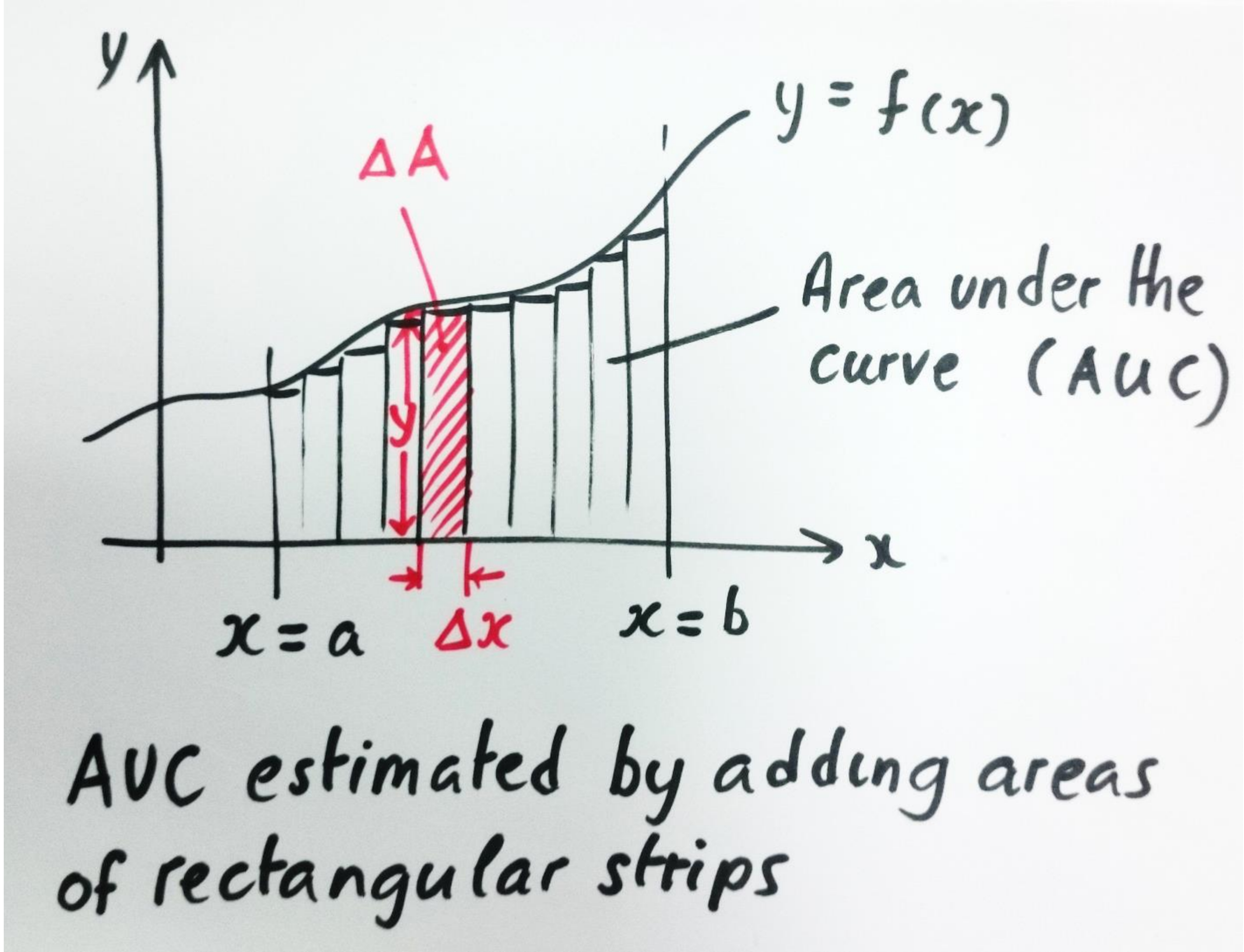
$$\int_1^e \left(\frac{1}{x} + x^2\right) dx = [\ln x + x^3]_1^e$$

$$= (\ln e + e^3) - (\ln 1 + 1)$$

$$= (1 + e^3) - (0 + 1)$$

$$= e^3$$

Definite integration and areas



Summing the area elements

- The width of a strip is Δx and the height is y.
- Each strip has an area $\Delta A \simeq y \Delta x$
- Total area $= \sum_{x=a}^{x=b} \Delta A$
- As $\Delta x \to 0$, the accuracy increases
- $A = \lim_{\Delta x \to 0} \sum_{x=a}^{x=b} y \Delta x$

Another way of thinking about it…

- We can also see that $y = \dfrac{\Delta A}{\Delta x}$
- But $\lim\limits_{\Delta x \to 0} \dfrac{\Delta A}{\Delta x}$ is $\dfrac{dA}{dx}$
- Thus, $\dfrac{dA}{dx} = y$ and $A = \int y\,dx$
- Boundary values of x are $x = a$ and $x = b$.
- total area $= \int_a^b y\,dx$

Definite integration as the limit of a sum

therefore

$$\lim_{\Delta x \to 0} \sum_{x=a}^{x=b} y \Delta x = \int_a^b y \, dx$$

b is the upper limit of integration and a is the lower limit.

Fundamental theorem of calculus again!

Let f be continuous on $[a, b]$

and $F(x) = \int f(x)\,dx$

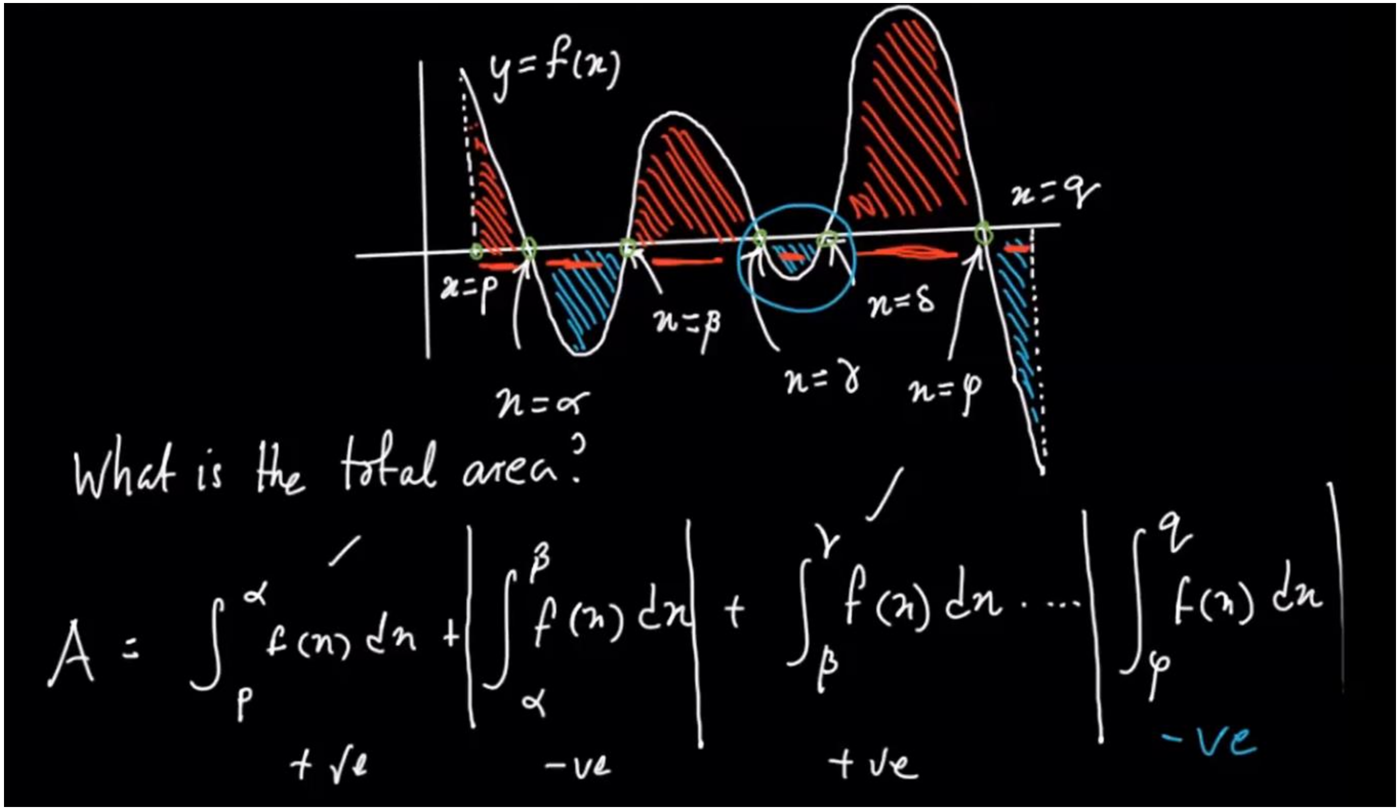
then

$$\boxed{\begin{aligned}\int_a^b f(x)\,dx &= \Big[F(x)\Big]_a^b \\ &= F(b) - F(a)\end{aligned}}$$

This is what you have to apply to do definite integration, it's fairly easy (some of the time and if it is possible!)

Finding areas by integration

For a continuous function, we must integrate areas between all roots of the function over the domain of the integration. In the illustration below we would need to carry out 6 separate integrations.

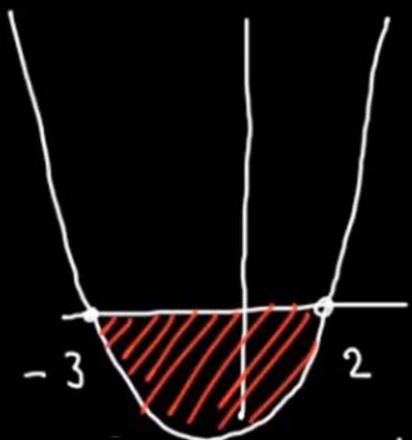


If we tried to integrate $\int_p^q f(x)dx$ across the entire domain we would not obtain the correct area.

A simple example to start with!

Find the area contained between the curve $y = x^2 + x - 6$ and the x-axis between the points where $f(x) = 0$

$$f(x) = x^2 + x - 6$$
$$= (x+3)(x-2)$$



$$A = \int_{-3}^{2} (x^2 + x - 6)\, dx$$

$$A = \left[\frac{1}{3}x^3 + \frac{x^2}{2} - 6x\right]_{-3}^{2}$$

$$= \left(\frac{1}{3}(2^3) + \frac{(2^2)}{2} - 6(2)\right) - \left(\frac{1}{3}(-3)^3 + \frac{(-3)^2}{2} - 6(-3)\right)$$

$$= \left(\frac{8}{3} - \frac{30}{3}\right) - \left(9 + 9/2\right) = \left|-\frac{125}{6}\right|$$

$$A = \frac{125}{6}$$

Your turn! (3)

Find the area of the finite region bounded by the curve $y = x(x - 3)$ and the $x-$axis.

Solution

The limits of integration are the x −intercepts, $x = 0$ and $x = 3$

Need to evaluate $\left|\int_0^3 x(x-3)dx\right|$

$$= \left|\int_0^3 (x^2-3x)dx\right| = \left|\left[\frac{x^3}{3}-\frac{3x^2}{2}\right]_0^3\right|$$

$$= \left|\left(\frac{27}{3}-\frac{3\times 9}{2}\right)-(0)\right| = \frac{9}{2}$$

Evaluating an integral – what are we doing?

Evaluate means just that – we don’t have to think about ‘positive’ and ‘negative’ areas or crossing points on the axis.

<u>Evaluate</u> $\int_{-4}^{3} (x^2 + x - 6)\,dx$

$$\int_{-4}^{3} (x^2 + x - 6)\,dx = \left[\frac{1}{3}x^3 + \frac{x^2}{2} - 6x\right]_{-4}^{3}$$

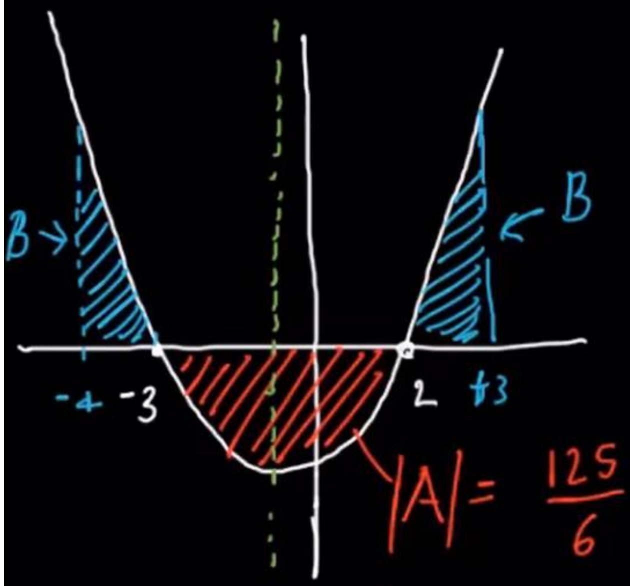
$$= \left(\frac{1}{3}(3^3) + \frac{(3^2)}{2} - 6(3)\right) - \left(\frac{1}{3}(-4)^3 + \frac{(-4)^2}{2} - 6(-4)\right)$$

$$= \left(9 + 9/2 - 18\right) - \left(-64/3 + 8 + 24\right)$$

$$= -9/2 - 32/3 = -91/6$$

Multiple enclosed regions

Find the area contained between the curve $y = x^2 + x - 6$ and the x-axis on the domain $-4 \le x \le 3$.



Total area $2B + |A|$

$$\int_{-4}^{3} (x^2 + x - 6)\,dx = -91/6$$

$A + 2B$

$$A + 2B = -91/6$$

$$\therefore\ 2B = 34/6$$

$$\text{Total} = |A| + 2B = 125/6 + 34/6 = 159/6$$

Your turn! (4)

Find the area of the finite region bounded by the curve $y = x(x - 1)(x + 3)$ and the x −axis.

Solution

$$\text{Area} = \int_{-3}^{0} (x^3 + 2x^2 - 3x)dx + \left|\int_{0}^{1} (x^3 + 2x^2 - 3x)dx\right|$$

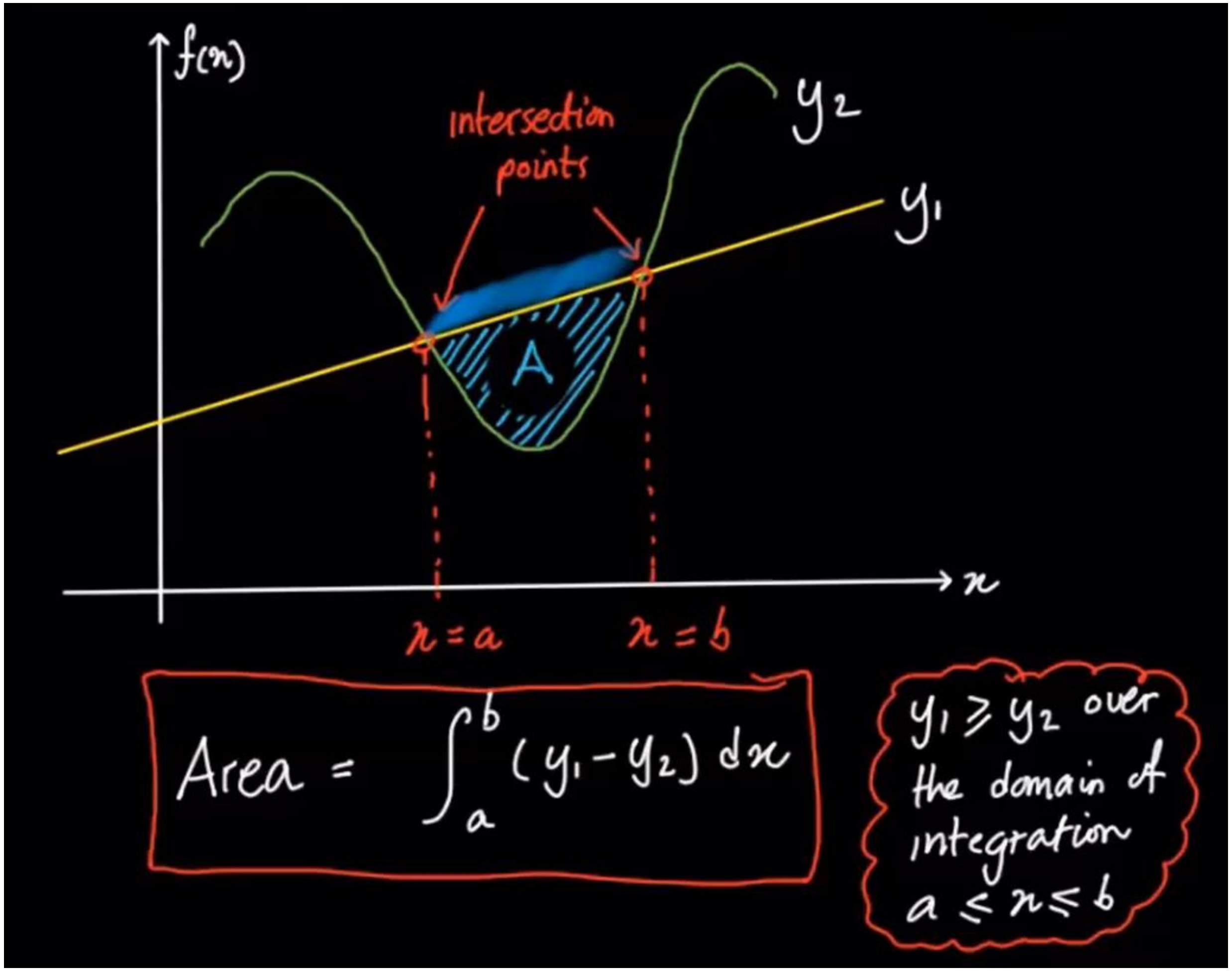
$$= \left[\frac{x^4}{4} + \frac{2x^3}{3} - \frac{3x^2}{2}\right]_{-3}^{0} + \left|\left[\frac{x^4}{4} + \frac{2x^3}{3} - \frac{3x^2}{2}\right]_{0}^{1}\right|$$

$$= (0) - \left(\frac{81}{4} - \frac{54}{3} - \frac{27}{2}\right) + \left|\left(\frac{1}{4} + \frac{2}{3} - \frac{3}{2}\right) - (0)\right|$$

$$= \frac{45}{4} + \frac{7}{12}$$

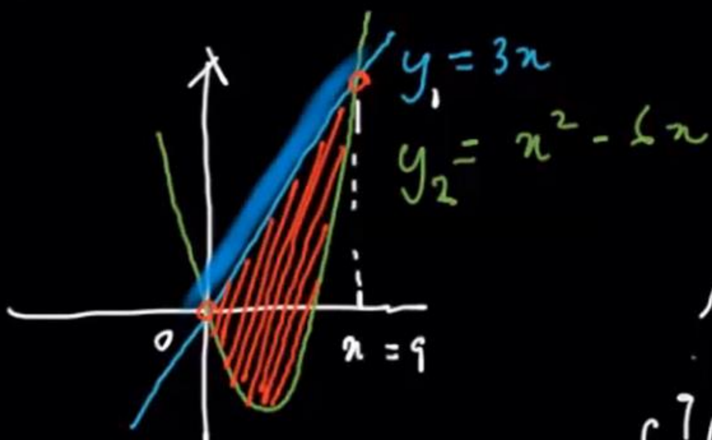
$$= \frac{71}{6}$$

Area between a curve and a straight line



Example

Find the area enclosed by the graph of $y = x(x-6)$ and the line $y = 3x$, between the intersection points of the curve and the line.



intersections

$3x = x^2 - 6x$

$\Rightarrow x^2 - 9x = 0$

$x(x-9) = 0 \quad \therefore \; x = 0, \; x = 9$

$$A = \int_0^9 (y_1 - y_2)\,dx$$

$$A = \int_0^9 (3x - (x^2 - 6x))\,dx$$

$$\int_0^9 (9x - x^2)\,dx = \left[\frac{9x^2}{2} - \frac{x^3}{3}\right]_0^9$$

$$= \frac{729}{2} - \frac{729}{3}$$

$$= 729/6 \quad = 243/2$$

11.5.3 Integrate functions containing e^x, $\sin x$, $\cos x$

Differentiation	Integration
$\frac{d}{dx}[\sin x] = \cos x$	$\int \cos x \, dx = \sin x + C$
$\frac{d}{dx}[-\cos x] = \sin x$	$\int \sin x \, dx = -\cos x + C$
$\frac{d}{dx}[e^x] = e^x$	$\int e^x \, dx = e^x + C$
$\frac{d}{dx}[\ln \lvert x \rvert] = \frac{1}{x}$	$\int \frac{1}{x} dx = \ln \lvert x \rvert + C$

Other useful results

Differentiation	Integration
$\frac{d}{dx}[\tan x] = \sec^2 x$	$\int \sec^2 x \, dx = \tan x + C$
$\frac{d}{dx}[-\cot x] = \operatorname{cosec}^2 x$	$\int \operatorname{cosec}^2 x \, dx = -\cot x + C$
$\frac{d}{dx}[\sec x] = \sec x \tan x$	$\int \sec x \tan x \, dx = \sec x + C$
$\frac{d}{dx}[-\operatorname{cosec} x] = \operatorname{cosec} x \cot x$	$\int \operatorname{cosec} x \cot x \, dx = -\operatorname{cosec} x + C$

Example

Find $\int (2\cos x + \frac{3}{x^2} - \sqrt{x})dx$

$$= \int 2\cos x dx + \int \frac{3}{x^2} dx - \int \sqrt{x} dx$$

$$= 2\sin x - \frac{3}{x} - \frac{2}{3} x^{\frac{3}{2}} + C$$

Example

$$\int \left(\frac{\cos x}{\sin^2 x} - 2e^x\right) dx$$

$$= \int \left(\frac{1}{\sin x}\frac{\cos x}{\sin x} - 2e^x\right) dx$$

$$= \int (\operatorname{cosec} x \cot x - 2e^x) dx$$

$$= -\operatorname{cosec} x - 2e^x + C$$

Your turn!(5)

Find the following definite integrals

$$a)\ \int \frac{5\tan^2 x}{1-\cos^2 x}\,dx$$

$$b)\ \int \left[\frac{1}{2t} - \sqrt{2}e^t\right] dt$$

Solutions(5)

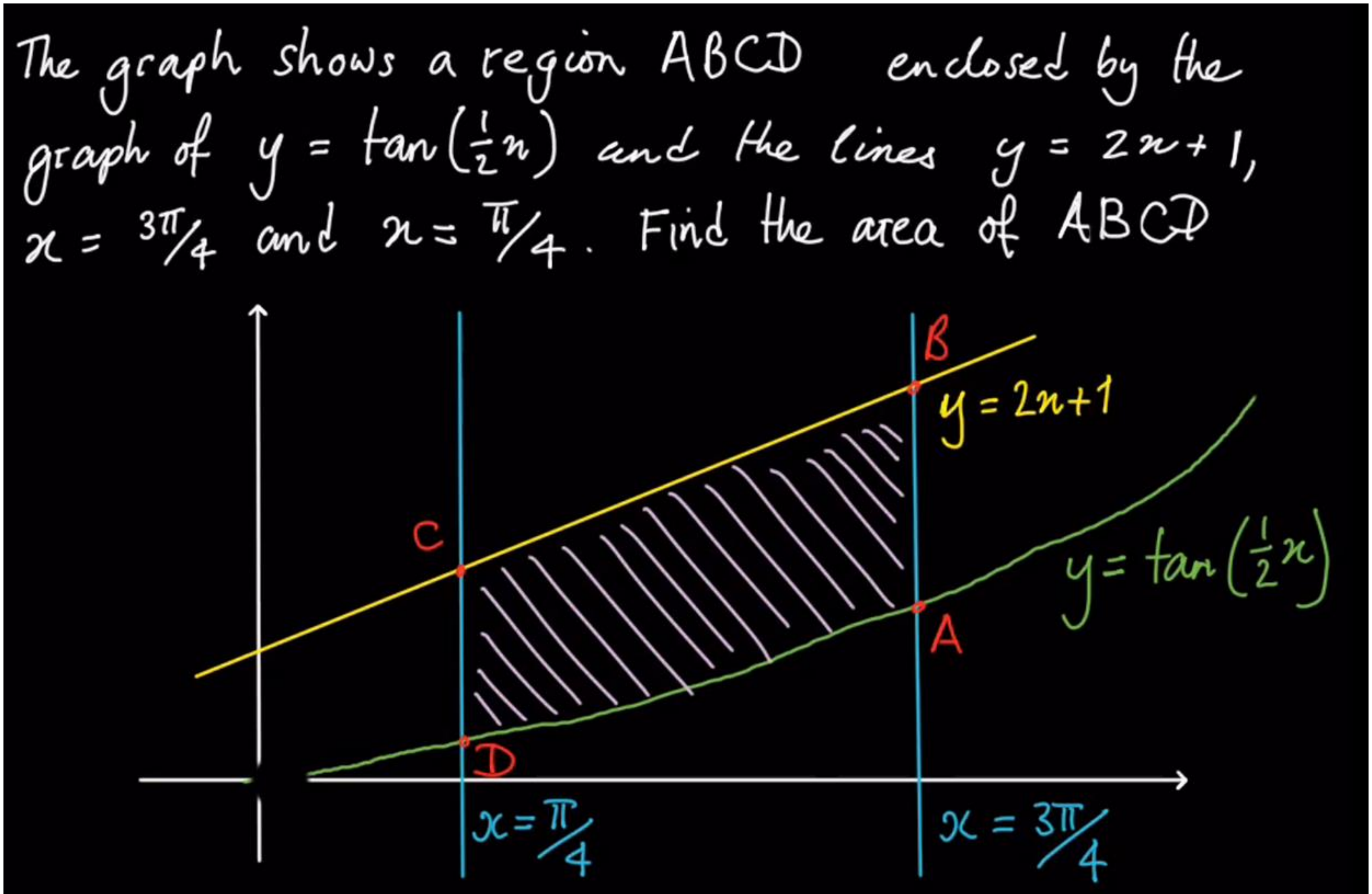
$$a)\ \int \frac{5\tan^2 x}{1-\cos^2 x}\,dx$$

$$= 5\int \frac{\tan^2 x}{\sin^2 x}\,dx = 5\int \frac{1}{\cos^2 x}\,dx$$

$$= 5\int \sec^2 x\,dx = 5\tan x + C$$

$$b)\ \int \left[\frac{1}{2t} - \sqrt{2}e^t\right] dt \qquad = \frac{1}{2}\ln|t| - \sqrt{2}e^t + C$$

Application examples



Solution

$$\left.\begin{aligned} y_1 &= 2x+1 \\ y_2 &= \tan\left(\frac{x}{2}\right) \end{aligned}\right\} \quad \int_{\pi/4}^{3\pi/4} (y_1 - y_2)\, dx$$

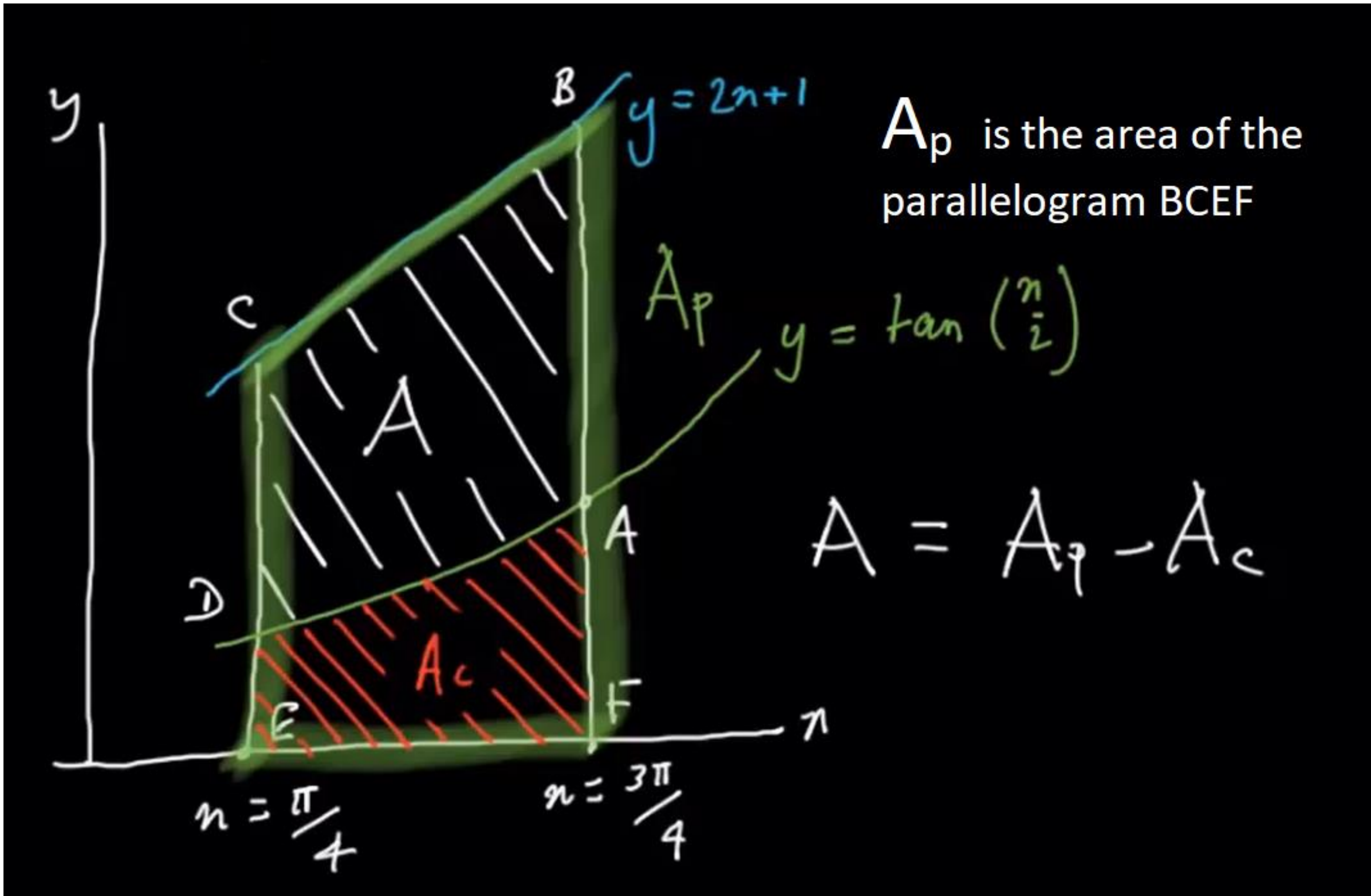
$$A = \int_{\pi/4}^{3\pi/4} \left(2x+1-\tan\left(\frac{x}{2}\right)\right) dx = \left[x^2 + x - 2\ln\left|\sec\left(\frac{x}{2}\right)\right|\right]_{\pi/4}^{3\pi/4}$$

$$= \left[x^2 + x + \ln\left(\cos^2\left(\frac{x}{2}\right)\right)\right]_{\pi/4}^{3\pi/4}$$

$$A = \left(\frac{9\pi^2}{16} + \frac{3\pi}{4} + \ln\cos^2\left(\frac{3\pi}{8}\right)\right) - \left(\frac{\pi^2}{16} + \frac{\pi}{4} + \ln\left(\cos^2\left(\frac{\pi}{8}\right)\right)\right)$$

$$A = \frac{\pi^2}{2} + \frac{\pi}{2} + \ln\left(\frac{\cos^2\left(\frac{3\pi}{8}\right)}{\cos^2\left(\pi/8\right)}\right)$$

Alternative solution



$$A = A_p - A_c$$

$A_p = \frac{1}{2} \times$ sum of parallel sides $\times$ distance between the parallel sides

$y = 2x + 1$

$x = \frac{\pi}{4}$ $\quad x = \frac{3\pi}{4}$

$$A_p = \frac{1}{2}\left\{\left(\frac{\pi}{2}+1\right) + \left(\frac{3\pi}{2}+1\right)\right\} \times \frac{\pi}{2}$$

$y = 2x+1$ $\quad y = 2x+1$

$$A_p = \frac{1}{2}(2\pi + 2)\frac{\pi}{2} = \frac{\pi^2}{2} + \frac{\pi}{2}$$

$$A_c = \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \tan\left(\frac{x}{2}\right) dx$$

$$= \left[2\ln\left|\sec\left(\frac{x}{2}\right)\right|\right]_{\pi/4}^{3\pi/4}$$

$$= \left[-\ln\left(\cos^2\left(\frac{x}{2}\right)\right)\right]_{\pi/4}^{3\pi/4} = -\ln\left(\frac{\cos^2(3\pi/8)}{\cos^2(\pi/8)}\right)$$

$$A = A_p - A_c$$

$$= \frac{\pi^2}{2} + \frac{\pi}{2} + \ln\left(\frac{\cos^2\left(\frac{3\pi}{8}\right)}{\cos^2(\pi/8)}\right)$$

11.5.4 Find the equation of a curve given $f'(x)$ and any point $P(x, y)$ on the curve $f(x)$.

UNIVERSITY OF SURREY

Find the equation of a curve, given $f'(x)$ and any point $p(x,y)$ on the curve $f(x)$.

consider the function $f(x) = x^2 + c$. Its graph is a parabola with a vertex located at $x = 0$ (on the y-axis)

If we differentiate this function, we get $f'(x) = 2x$.

This can tell us the gradient of $f(x)$ for any value of x

All these functions have the same gradient function

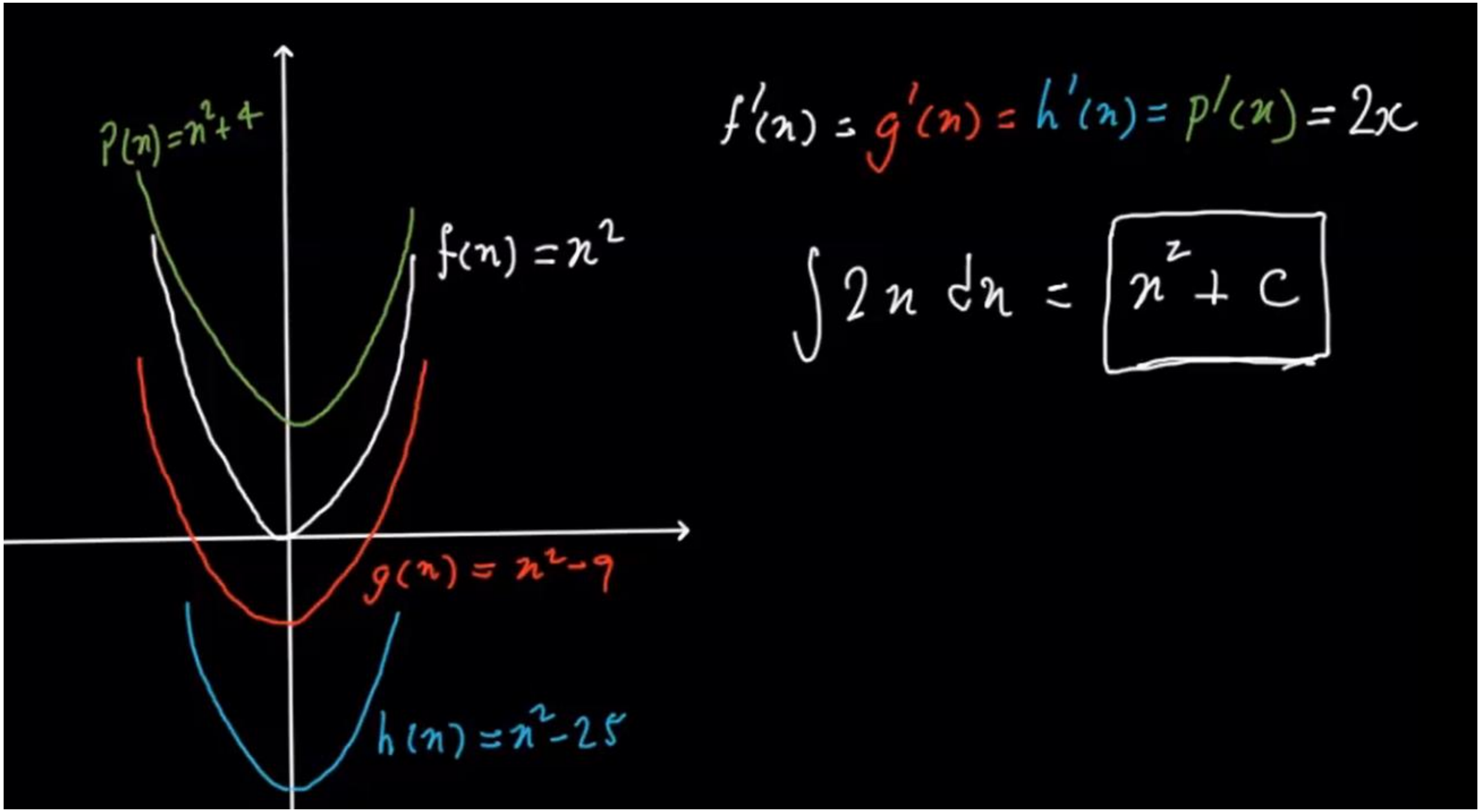
$f(x) = x^2$

$g(x) = x^2 - 9$

$h(x) = x^2 - 25$

$p(x) = x^2 + 4$

We can differentiate any of these functions but we can't recover the original function by integration alone



Example

1. Find y in terms of x, given $y' = 2$ and $(3,7)$ is a point on the graph of y

$$f'(x) = 2$$

$$f(x) = \int 2 \, dx = 2x + c$$

$$y = 2x + c \quad \text{at } (3,7)$$

$$7 = 2(3) + c \Rightarrow c = 1$$

$$\boxed{y = 2x + 1}$$

Example

2. Given $f'(x) = \frac{1}{x} + 2$ and the point $(1, 5)$ lies on the graph of $f(x)$, find $f(x)$.

$$f'(x) = \frac{1}{x} + 2$$

$$f(x) = \int \left(\frac{1}{x} + 2\right) dx$$

$$= \ln x + 2x + c$$

$y = \ln x + 2x + c$ at $(1, 5)$

$5 = \ln(1) + 2(1) + C$ $\quad \therefore \; y = \ln x + 2x + 3$

$5 = 2 + C \Rightarrow C = 3$

Example

3. Given $t' = \cot 3\theta$ and $\theta = \pi/12$ is a root of t, find t in terms of θ.

$$t' = \cot 3\theta$$

$$t = \int \cot 3\theta \, d\theta$$

$$= \frac{1}{3}\ln|\sin 3\theta| + c$$

$\left(\frac{\pi}{12}, 0\right)$

$$\frac{1}{3}\ln\left|\sin\frac{\pi}{4}\right| + c = 0$$

$$\therefore \quad c = -\frac{1}{3}\ln\left(\sin\frac{\pi}{4}\right)$$

$$c = -\frac{1}{3}\ln\left(\frac{\sqrt{2}}{2}\right)$$

$$c = \frac{1}{3}\ln\sqrt{2} = \frac{1}{6}\ln 2$$

$$\therefore \quad t = \frac{1}{3}\ln|\sin 3\theta| + \frac{1}{6}\ln 2$$

Example

Suppose that a curve with equation $y = f(x)$ passes through the point (4, 5). Given that $f'(x) = \frac{x^2-2}{\sqrt{x}}$, find the equation of the curve.

1) Write $f'(x)$ in a form suitable for integration: $f'(x) = x^{\frac{3}{2}} - 2x^{-\frac{1}{2}}$

2) Integrate: $f(x) = \int\left(x^{\frac{3}{2}} - 2x^{-\frac{1}{2}}\right)dx$

$$= \frac{x^{\frac{5}{2}}}{\frac{5}{2}} - 2\frac{x^{\frac{1}{2}}}{\frac{1}{2}} + C$$

$$= \frac{2}{5}x^{\frac{5}{2}} - 4x^{\frac{1}{2}} + \mathrm{C}$$

3) Use the fact that point (4,5) is on the curve, substitute for x and y:

$$5 = \frac{2}{5} \times 2^5 - 4 \times 2 + C$$

$$5 = \frac{64}{5} - 8 + C \Rightarrow C = \frac{1}{5}$$

The equation of the curve is $f(x) = \frac{2}{5}x^{\frac{5}{2}} - 4x^{\frac{1}{2}} + \frac{1}{5}$.

Your turn!(6)

The gradient of a particular curve is given by $\frac{dy}{dt} = -\sin t \, - 1$. Given that $y\left(\frac{\pi}{3}\right) = \frac{1}{2}$, find an equation of the curve $y = f(t)$.

Solution

1) $f(t) = \int(-\sin t - 1)dt = \cos t - t + C$

2) Point $\left(\dfrac{\pi}{3}, \dfrac{1}{2}\right)$ is on the curve $\Rightarrow$

$$\frac{1}{2} = \cos\frac{\pi}{3} - \frac{\pi}{3} + C \Rightarrow C = \frac{\pi}{3}$$

$$\Rightarrow f(t) = \cos t - t + \frac{\pi}{3}$$

Summary

You should now be able to;

11.5.1 Integrate functions of the form ax^n (n is rational, $n \neq -1$) using

$$\int ax^n dx = \frac{ax^{n+1}}{n+1} + C$$

11.5.2 Apply the fundamental theorem of calculus:

$$\int_a^b f(x)dx = [F(x)]_a^b = F(b) - F(a)$$

and use this to solve a variety of problems

11.5.3 Integrate basic functions containing $e^x, \sin x, \cos x$

11.5.4 Find the equation of a curve given $f'(x)$ and any point $P(x, y)$ on the curve

UNIVERSITY OF SURREY