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2.1 Linear equations & straight line graphs

Core Preparatory Topics
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FEPS Mathematics Support Framework

2.1 Introduction

The aim of this unit is to assist you in consolidating and developing your knowledge and skills in working with linear equations and straight line graphs.

While studying these slides you should attempt the 'Your Turn' questions in the slides.

After studying the slides, you should attempt the Consolidation Questions.

2.1 Learning checklist

Learning resource	Notes	Tick when complete
Slides		
Your turn questions		
Quick Questions		
Consolidation questions		

2.1 Learning objectives

After studying this unit you should be able to

2.1.1. Find the gradient of a straight line.

2.1.2. Use the gradient-intercept form and general equation of a straight line.

2.1.3. Carry out calculations with parallel and perpendicular lines.

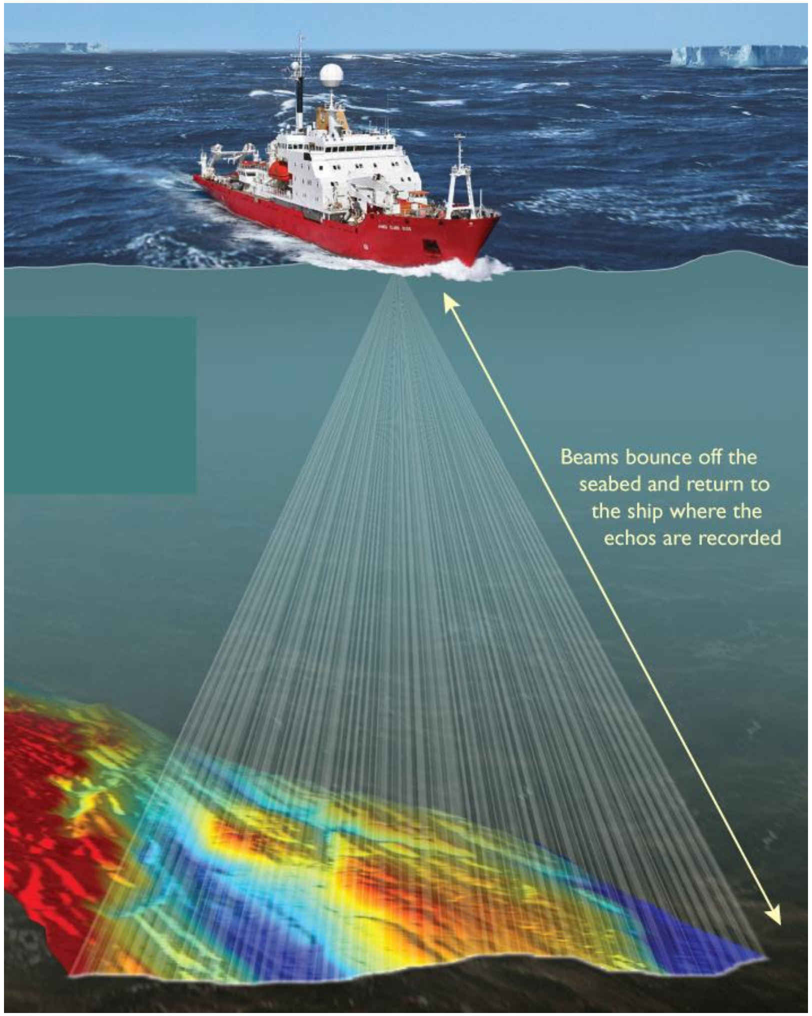
2.1.4. Find the distance between two points in the $x - y$ plane and the find the mid-point of a line segment.

2.1.5. Solve linear equations presented in different forms.

Straight lines have many applications in complex models

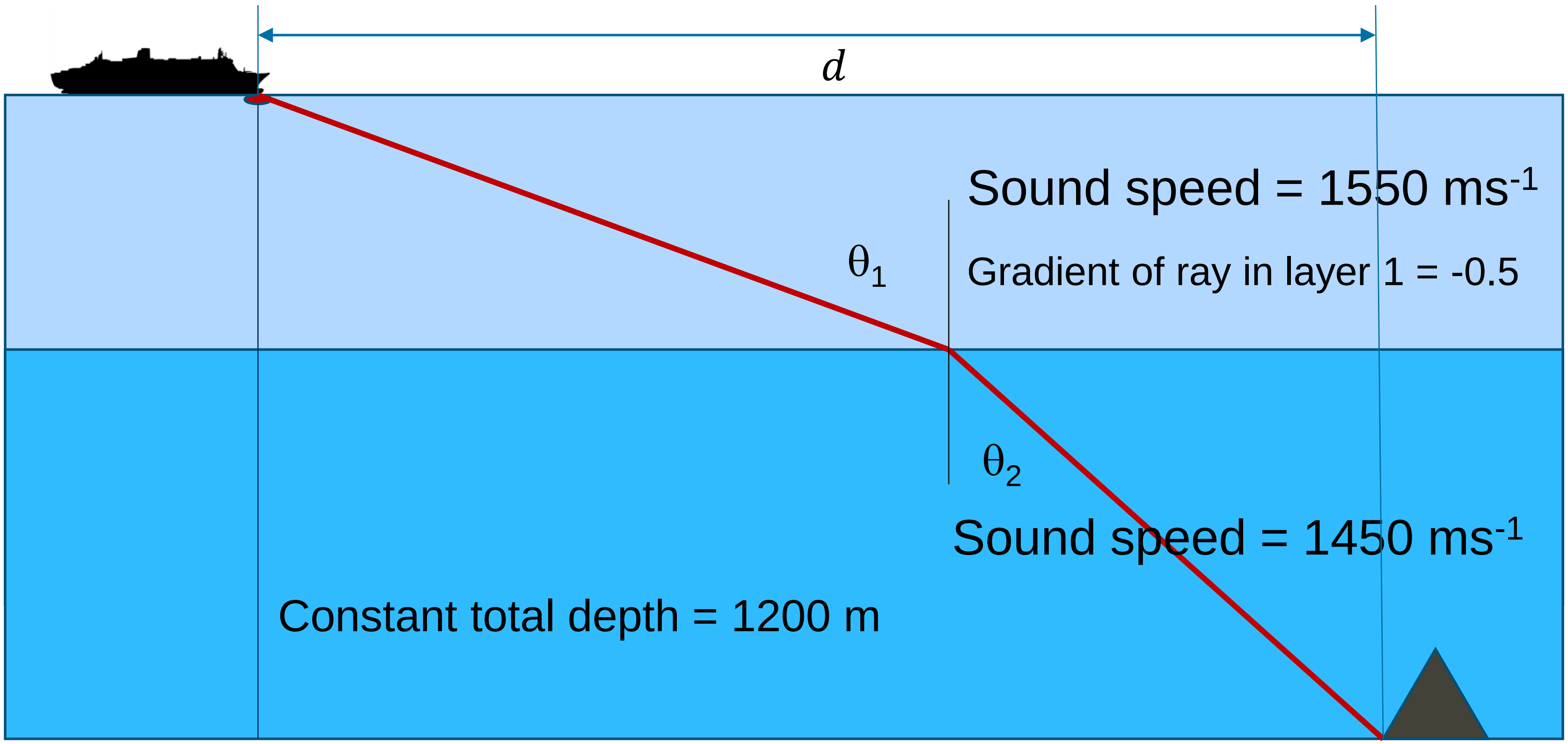
Ray tracing – wide scope of applications

When we use straight lines in models like these, we need to design a computational implementation



Here's an interesting problem!

What is the horizontal distance d, given the total time taken for an acoustic signal (ray) from the ship to hit the target and return to the ship is 2.8 seconds?



Snell’s law

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$$\frac{\sin\theta_1}{\sin\theta_2} = \frac{v_1}{v_1}$$

Can you solve the problem with this information?

Basic properties of lines

We can specify a straight line as a function, provided we know either;

(1) any two non-coincident (x, y) points on the line

Or

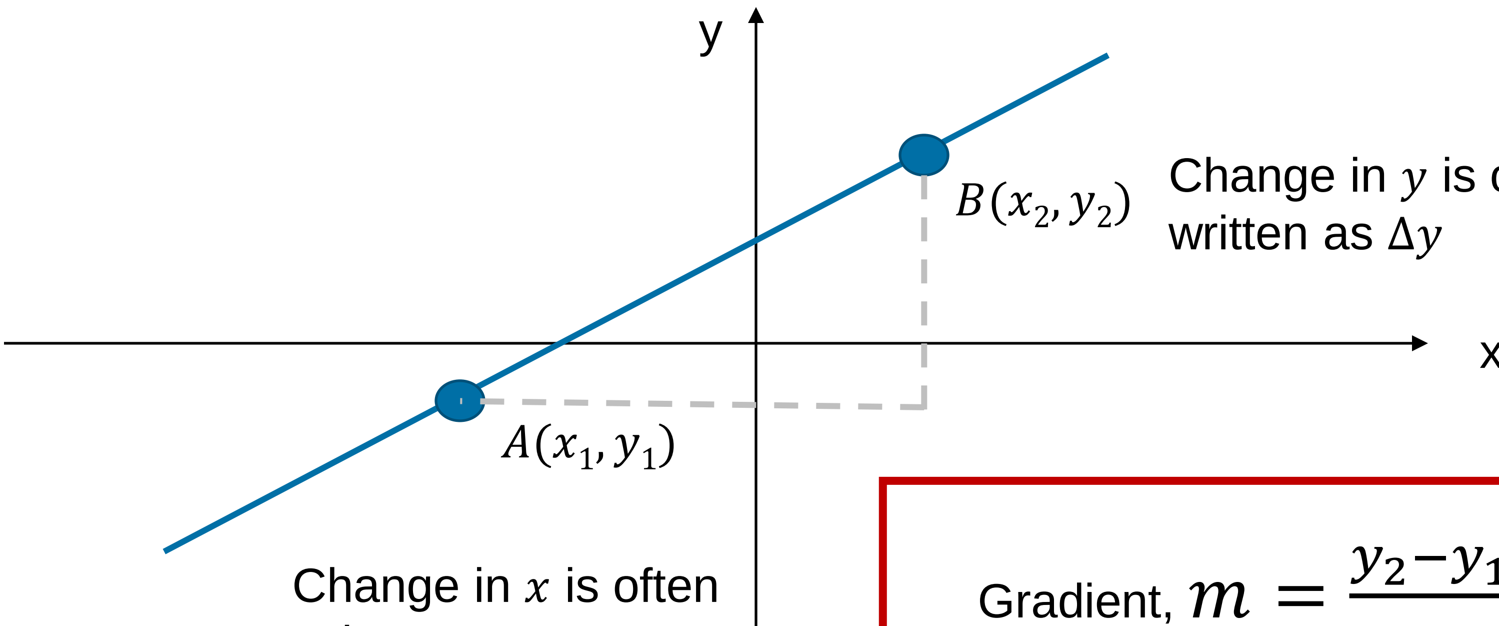
(2) any one (x, y) point on the line, together with the slope (gradient) of the line.

Since the gradient can be computed from any two points we will look at this first

2.1.1 Find the gradient of a straight line

Gradient = $\dfrac{\text{change in } y}{\text{change in } x}$

The gradient of a straight line graph is constant



Change in y is often written as Δy

Change in x is often written as Δx

Gradient, $m = \dfrac{y_2 - y_1}{x_2 - x_1}$

Gradient - Example

A straight line passes through the points $A(2,7)$ and $B(5,13)$ calculate the gradient of the line.

Gradient, $m = \frac{y_2 - y_1}{x_2 - x_1}$

$$m = \frac{13 - 7}{5 - 2} = 2$$

Your turn ! (1)

Find the gradient of the line segment AB between the points A(3, 5) and B(8, -1)

Solution

Gradient = $\dfrac{-1-5}{8-3} = \dfrac{-6}{5}$

2.1.2. Use the gradient-intercept form and general equation of a straight line

The general equation of a straight line is,

$$ax + by + c = 0$$

This is not particularly useful if we want to quickly gain an idea of the geometrical properties of the line, because the information is implicit.

What is the gradient of the line?

Where does it cut the axes?

We can't answer these directly just by looking at the equation

Gradient-intercept form

The gradient-intercept form is,

$$y = mx + c$$

m is the gradient of the straight line
c is the value of y where the line crosses (intercepts) the y-axis

Just by looking at the equation we know the gradient and the intercept!

It is useful to be able to convert the general equation to the gradient-intercept form.

Example - from general equation to gradient intercept form

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Convert the equation of the line $2x - 2y + 5 = 0$ into gradient-intercept form

It's really just a case of rearranging to make y the subject.

$$2x - 2y + 5 = 0$$

$$2y = 2x + 5$$

$$y = x + \frac{5}{2}$$

We can see straight away that the gradient, $m = 1$ and the y-intercept is at $\left(0,\ \frac{5}{2}\right)$

Your turn !(2)

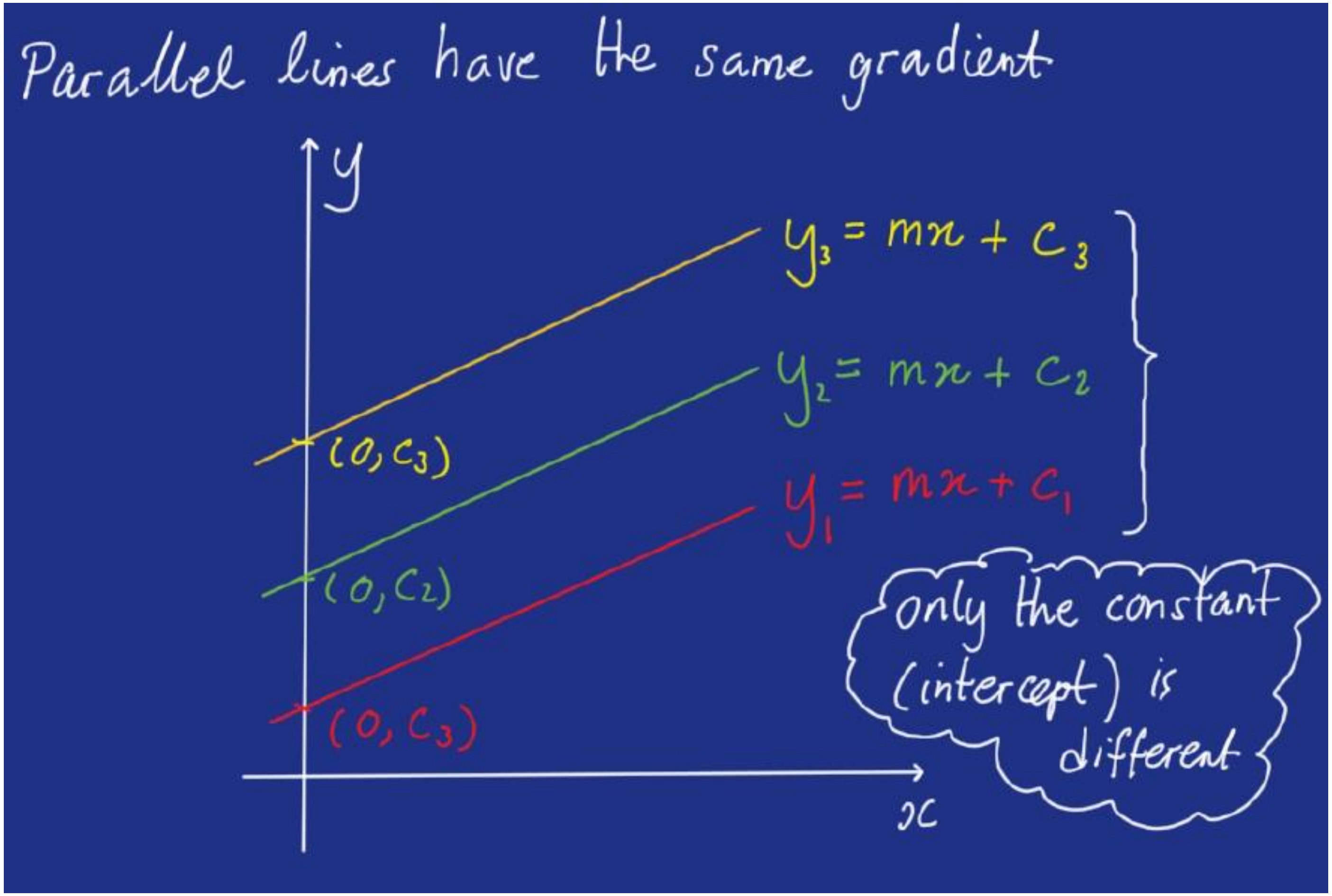
Convert $3y - 6x - 5 = 0$ to gradient-intercept form

Solution

$$3y - 6x - 5 = 0$$

$$\Rightarrow y = 2x + \frac{5}{3}$$

2.1.3. Carry out calculations with parallel and perpendicular lines



Example - parallel lines

Find the equation of the line passing through the point $(2, 4)$ parallel to the line $2x - 3y + 6 = 0$.

The first step is to write the given equation in gradient intercept form:

$$2x - 3y + 6 = 0 \;\therefore\; y = \frac{2}{3}x + 2$$

We want to find the equation of the line with a gradient of $m = \frac{2}{3}$ passing through the point $(2, 4)$.

At the point $(2, 4)$, $y = mx + c$ so,

$$4 = \frac{2}{3}(2) + c \Longrightarrow c = \frac{8}{3}$$

The equation of the line is therefore,

$$y = \frac{2}{3}x + \frac{8}{3}$$

Perpendicular lines

If two lines l_1, l_2 are perpendicular, the product of their gradients is $m_1 m_2 = -1$

Example

Find the equation of the line passing through the origin which is perpendicular to the line $4x - 2y + 5 = 0$

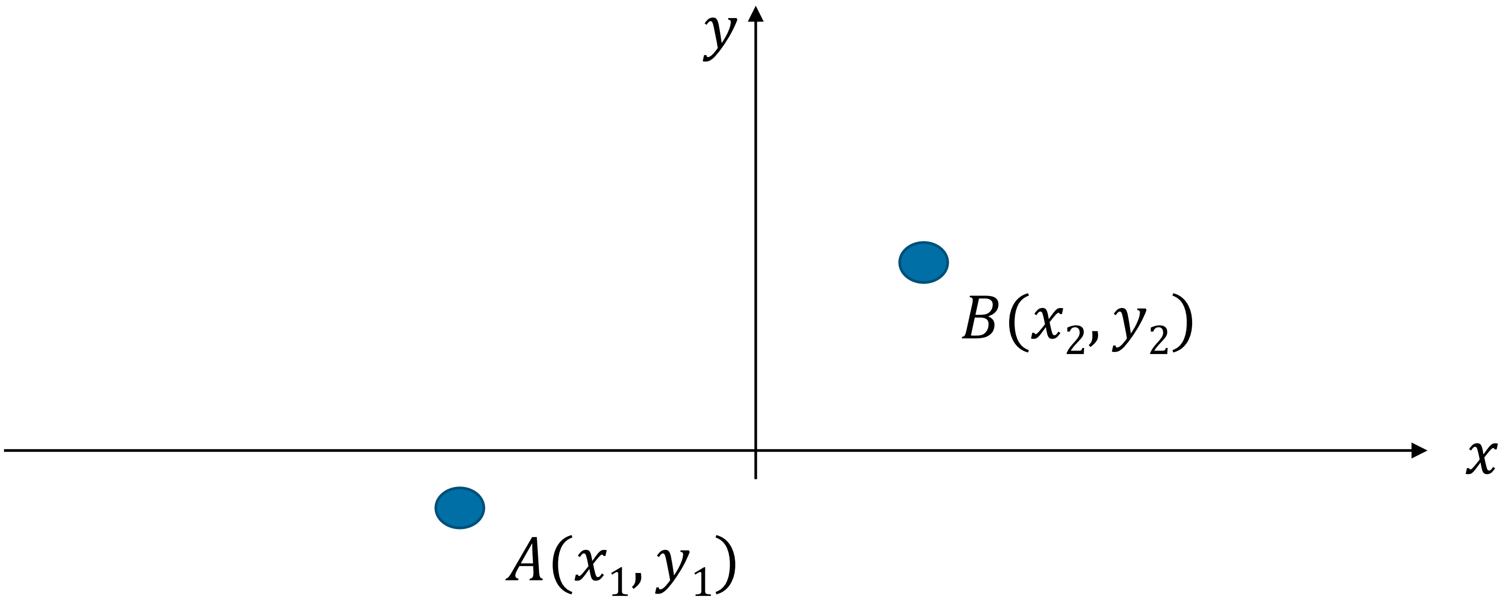
$$4x - 2y + 5 = 0 \Longrightarrow y = 2x + \frac{5}{2}$$

The gradient is +2 so the gradient of the perpendicular line must be $-\frac{1}{2}$

The equation of the required line is therefore $y = -\frac{1}{2}x$

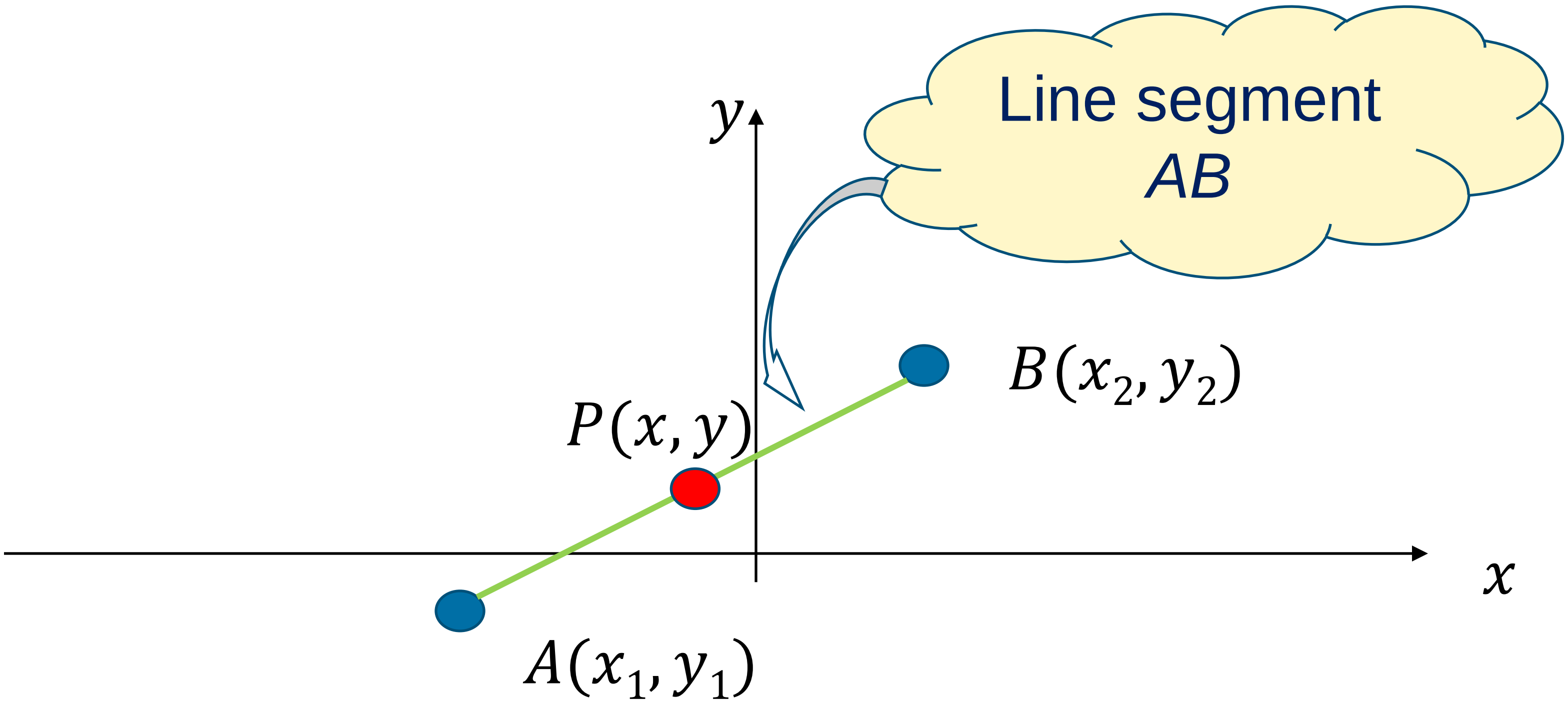
In this case we do not have to do any further calculation to find the intercept since we are told that the line passesthrough the origin.

Distance between two points



Distance, $|AB| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

Midpoint of any two points (line segment) in the plane



Midpoint of AB: $P(x, y) = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$

Distance and midpoints - Your turn! (4)

For the two points A(-12, -15) and B(-3, -20) find

(i) |AB|.

(ii) The midpoint of AB.

Solution

Distance, $|AB| = \sqrt{(-3-(-12))^2 + (-20-(-15))^2}$

$= \sqrt{(9)^2 + (-5)^2}$

$= \sqrt{106}$

Midpoint of AB: $\left(\frac{-3+(-12)}{2}, \frac{-20+(-15)}{2}\right)$

$$\left(-\frac{15}{2}, -\frac{35}{2}\right)$$

Circles

The distance between any point (x, y) on the circumference of a circle C and the centre (a, b) of the circle is constant (clearly, the value of the radius, r).

The equation of the circle can therefore be written as

$$(x-a)^2+(y-b)^2=r^2 \quad (1)$$

Alternatively as

$$x^2+y^2-2ax-2by+c=0 \quad (2)$$

$$\text{Where} \quad c=a^2+b^2-r^2$$

To convert from form (2) back to form (1) we can complete the square for the x and the y terms.

Note that a line tangential to the circle is perpendicular to the radial line at the point of tangency

Example

Find the equation of a circle which has a line segment from A(-5, 4) to B(1, 2) as a diameter.

The centre is the mid point,M of AB, M(-2, 3)

Radius = |AM| = $\sqrt{10}$

Equation of circle : $(x+2)^2+(y-3)^2=10$

Your turn! (5)

Find the equation of the tangent to the circle $(x+2)^2 + (y-3)^2 = 10$ at the point $(1,2)$

Solution

There are various ways of solving this but here is one quick way;

Centre of circle C(-2, 3)

Point of tangency T(1, 2)

Gradient of the radial line CT = -1/3

Tangent line perpendicular to radial line, so gradient of tangent = 3

At the point of tangency T(1, 2), $y = mx + c$

$$y = 3x + c$$
$$2 = 3(1) + C \Rightarrow C = -1$$
$$\therefore y = 3x - 1$$

2.1.5. Solve linear equations presented in different forms

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$y = mx + c$ can also be written in the form $f(x) = mx + c$ both of these forms are **functions**. Mathematically speaking, they are not equations (although we often refer to them as equations or equations of a straight line)

On the other hand, $f(x) = 0$ is an **equation** (mathematically speaking). For instance, there is one **solution** to the equation $2x - 4 = 0$.

$2x - 4 = 0$ is called a **linear equation** because the unknown quantity x that we are trying to solve for has a power of 1, it is x^1. This is a definition of a linear term.

$2x - 3y + 5 = 0$ is also a linear equation since both x and y are linear terms. However, $xy - 5 = 0$ is **non-linear**. Why is that? Both x and y are linear terms.

If we write y in terms of x, we have $y = \frac{5}{x}$ or $y = 5\,x^{-1}$. This is definitely not linear because the power of x is -1.

Difference between a function and an equation

A function specifies a relationship between variables on a particular domain,

E.g

$$y = 2x + 3, \ x \in \mathbb{R}$$

$$f(x) = 2x + 3, x \in \mathbb{R}$$

y is the independent variable, x is the dependent variable

The domain in this example is the set of real numbers. Both these functions are valid on this domain. The range is the set of y values generated by the function, operating on the corresponding domain.

When we solve an equation we are finding specific values (solutions). Often we want to find specific value(s) on the domain, i.e., find x values, given a specific value of y.

Solving linear equations - examples

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1. Solve $2x + 3 = 0$

$$2x = -3$$

$$x = -3/2$$

2. Find the value of x that satisfies,

$$3x + 5 = 2 - 4x$$

$$3x + 4x + 5 = 2$$

$$7x = -3$$

$$x = -3/7$$

Solving linear equations - examples

3. Solve $2(x-5) = 3(4-2x)$

$$2x - 10 = 12 - 6x$$

$$8x = 22 \Rightarrow \underline{x = 11/4}$$

4. Solve $\dfrac{1}{2x} = -\dfrac{4}{3}$

$$1 = -\frac{8x}{3}$$

$$\underline{x = -\frac{3}{8}}$$

2.1 Summary

You should now be able to

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2.1.4. Find the mid-point of a line segment and the distance between two points in the $x - y$ plane

2.1.5. Solve linear equations presented in different forms

References

Attwood, G., Macpherson, A., Moran, B., Petran, J., Pledger, K., Staley, G. and Wilkins, D. (2008), Edexcel AS and A Level Modular Mathematics series C1-C4, Pearson, Harlow, UK.