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2.2 Quadratic equations & graphs of quadratic functions

Core Preparatory Topics
1.1
1.2
2.1
2.2
2.3
3.1
5.1
5.2
5.3
9.1
10.1
11.1
11.5

FEPS Mathematics Support Framework

2.2 Introduction

The aim of this unit is to assist you in consolidating and developing your knowledge and skills in working with quadratic equations, functions and graphs of quadratic functions.

While studying these slides you should attempt the ‘Your Turn’ questions in the slides.

After studying the slides, you should attempt the Consolidation Questions.

2.2 Learning checklist

Learning resource	Notes	Tick when complete
Slides		
Your turn questions		
Consolidation questions		

2.2 Learning objectives

After studying this unit you should be able to

2.2.1 Solve a quadratic equation by factorisation
2.2.2 Complete the square on a quadratic function
2.2.3 Solve a quadratic equation using the quadratic formula
2.2.4 Apply the quadratic discriminant in problem solving
2.2.5 Sketch the graph of a quadratic function
2.2.6 Solve a bi-quadratic equation
2.2.7 Interpret graphically the nature of the roots of a polynomial equation

Solving quadratic equations

Quadratic equations are non-linear as they contain an "x^2" term.

There are several ways of solving a quadratic but the three ways we need to be familiar with are;

1. Factorisation
2. Completing the square
3. Using the quadratic formula

General form of quadratic equations

A quadratic equation has the general form

$$ax^2 + bx + c = 0$$

The constants, a, b and c are called coefficients, c is often called the "constant" term, a and b are coefficients of x^2 and x, respectively.

There may be one, two or three terms in a quadratic equation but one of the terms must contain x^2 for it to be a quadratic.

2.2.1. Solve a quadratic equation by factorisation

1. factorising when $a = 1$

- re arrange into the form $ax^2 + bx + c = 0$
- write the brackets $(x \quad)(x \quad) = 0$
- Find the factors of 'c' which sum to give 'b'.
- write the numbers in the brackets with the correct sign (+ or −)

Example: Solving quadratic equations by factorisation when $a = 1$

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Solve $x^2 - 8 = 2x$

rewrite as $x^2 - 2x - 8 = 0$

$$x^2 - 2x - 8 = (x \quad)(x \quad)$$

find factors of -8 that sum to -2:

Factors of -8: $-8, -4, -2, -1, 1, 2, 4, 8$

$-4 + 2 = -2$ ✓

$\therefore x^2 - 2x - 8 = (x+2)(x-4)$

So, for $(x+2)(x-4) = 0$, $x = -2$ or $x = 4$.

$x + 2 = 0$, $x - 4 = 0$

Some other examples of factorisation

Solve $x^2 + 3x = 0$

$x(x+3) = 0 \quad \therefore \ x = 0, -3$.

Solve $x^2 - 9 = 0$

$x^2 = 9 \quad \therefore \ x = \pm 3$.

Solve $3x^2 + 12 = 0$

$x^2 + 4 = 0$

$x^2 = -4 \quad \therefore$ No real solution

$ax^2 - c = 0$

$(x + \sqrt{c/a})(x - \sqrt{c/a}) = 0$

Factorising a general quadratic when $a \neq 1$

Is there a rule we can follow?

Factorise $6x^2 - x + 15$

Carry out your own investigation to find a strategy to do this (trial and error is an acceptable approach but there are other ways.)

2.2.2. Complete the square on a quadratic function

Completing the square is a useful technique when the equation cannot be factorised.

It also provides other useful information about the quadratic graph, as we will see later in the lecture.

It involves rewriting the quadratic as,

$$ax^2 + bx + c = p(x + q)^2 + r$$

Where p, q and r are constants to be determined.

Example: Completing the square when $a = 1$

Rewrite $x^2 + 6x + 3$ by completing the square

$$x^2 + 6x + 3 = (x + \text{something})^2 + r$$

when $a = 1$, 'something' $= b/2$

in this case, $b = 6$

$$\therefore \quad x^2 + 6x + 3 = (x+3)^2 + r$$

$$x^2 + 6x + 3 = x^2 + 6x + 9 + r$$

LHS must = RHS, so, $9 + r = 3 \therefore r = -6$

$$x^2 + 6x + 3 = (x+3)^2 - 6$$

Example: Completing the square when $a \neq 1$

Rewrite $2x^2 + 3x - 5$ by completing the square

$$2x^2 + 3x - 5 = p(x+q)^2 + r$$

$$2x^2 + 3x - 5 = 2(x+q)^2 + r$$

(The constant outside the bracket is the 'a' coefficient)

$$2x^2 + 3x - 5 = 2\left(x + \frac{3}{4}\right)^2 + r$$

(When $a \neq 1$ this number is always $b/2a$)

$$2x^2 + 3x - 5 = 2\left(x^2 + \frac{3}{2}x + \frac{9}{16}\right) + r$$

$$2x^2 + 3x - 5 = 2x^2 + 3x + 9/8 + r$$

Example: Completing the square when $a \neq 1$, continued

$$\tfrac{9}{8} + r = -5 \quad \therefore \quad r = -\tfrac{49}{8}$$

$$2x^2 + 3x - 5 = 2\left(x + \tfrac{3}{4}\right)^2 - \tfrac{49}{8} \quad \blacksquare$$

Alternatively, you may find it easier to take the ‘2’ out as a factor

$$2x^2 + 3x - 5 = 2(x^2 + \frac{3}{2}x - \frac{5}{2})$$

You can then complete the square on $x^2 + \frac{3}{2}x - \frac{5}{2}$ and multiply the result by 2.

Always a good idea to check your result is consistent with the original expression as it’s easy to make arithmetical errors in this process!

Your turn!

Complete the square on $x^2 + 4x - 1$, writing your answer in the form $p(x + q)^2 + r$

Solution

$$x^2 + 4x - 1 = (x + 2)^2 - 4 - 1$$

$$= (x + 2)^2 - 5$$

Extending to more variables...

The equation of a circle with centre (a, b) and radius r can be written as,

$$(x-a)^2+(y-b)^2=r^2$$

Example

Find the area of the circle given by the equation,

$$x^2 + y^2 - 4x - 6y - 12 = 0$$

$$(x - a)^2 + (y - b)^2 = r^2$$

$$x^2 - 4x + y^2 - 6y = 12$$

$$(x - 2)^2 - 4 + (y - 3)^2 - 9 = 12$$

$$(x - 2)^2 + (y - 3)^2 = 25$$

$\therefore area = 25\pi$ square units

If we can find r^2 its easy to get the area as it's just $A = \pi r^2$

2.2.3. Solve a quadratic equation using the quadratic formula

Solve the quadratic equation $2x^2 = 4x + 3$ leaving your answer in the form $x = p \pm q\sqrt{r}$ where p, q, r are whole numbers or fractions.

$$2x^2 = 4x + 3 \quad \therefore \quad 2x^2 - 4x - 3 = 0$$

$$a = 2, \quad b = -4, \quad c = -3$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-(-4) \pm \sqrt{(-4)^2 - 4.(2).(-3)}}{2(2)}$$

$$= \frac{4 \pm 2\sqrt{10}}{4} \quad \therefore \quad x = 1 \pm \frac{1}{2}\sqrt{10} \quad \blacksquare$$

2.2.5. Sketch the graph of a quadratic function

A graph of a function is a diagrammatic representation of the function in a 2 dimensional plane.

It shows the behavior of the function $y = f(x)$.

Points on the "curve" have coordinates $(x, f(x))$.

Values on the horizontal axis (x-axis) are the "inputs" to the function. Usually called the domain. The domain may be finite, infinite or made up of different intervals.

The y values represent the "output" of the function. These are called the range or codomain.

Each value of x must have a corresponding value of $y = f(x)$. If there is no value of y for a given value of x, then we must say the function is not defined for that value of x and exclude the particular value(s) of x from the domain.

A good example is $f(x) = 1/x$. The domain is $x \in R, x \neq 0$.

General form of quadratic functions

A quadratic function has the general form

$$y = ax^2 + bx + c$$

The domain is usually $x \in \mathbb{R}$ unless specified otherwise

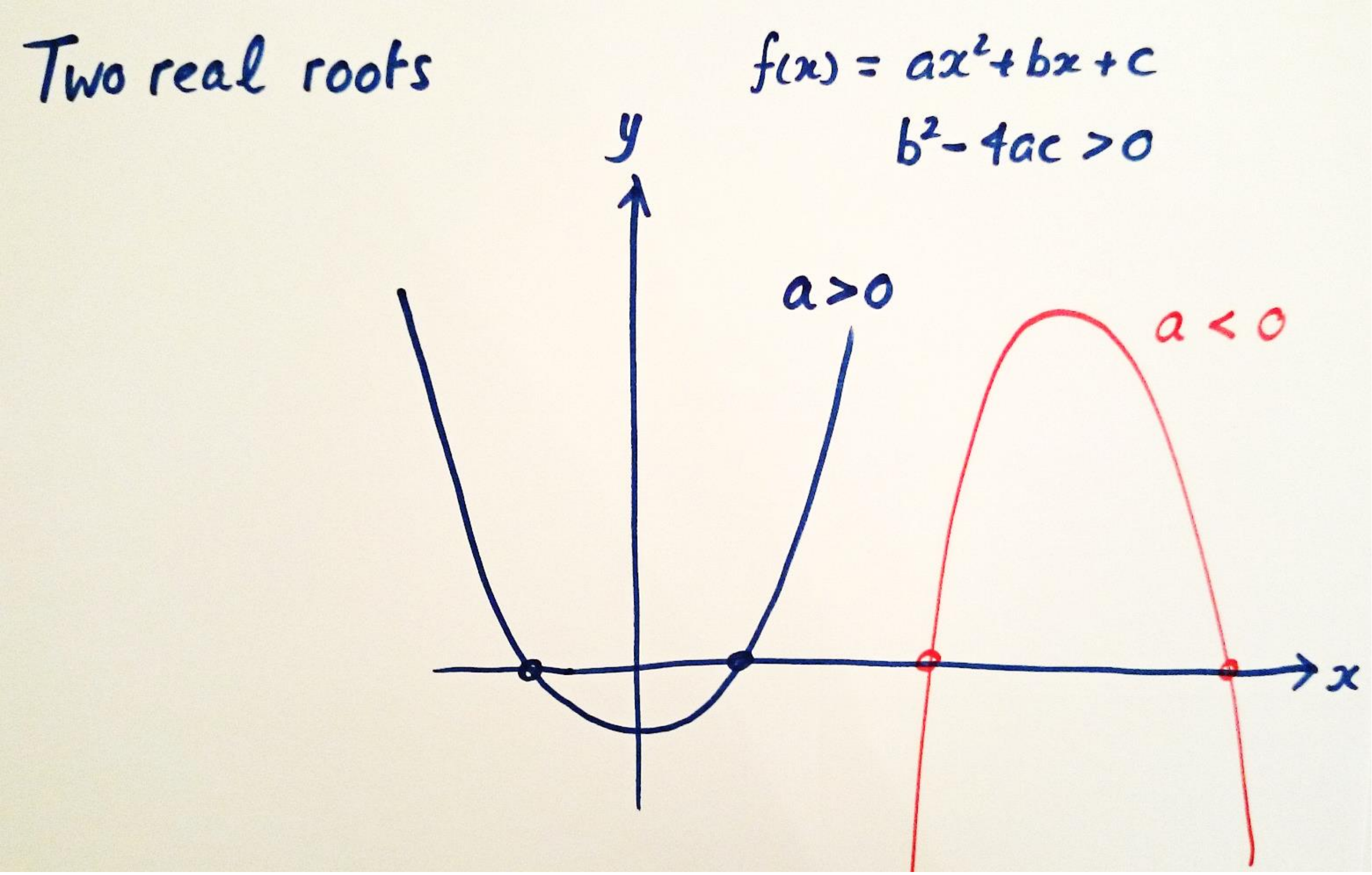
The quadratic discriminant, $b^2 - 4ac$

$b^2 - 4ac > 0$ real and distinct (different) roots

$b^2 - 4ac = 0$ repeated (equal) roots $(x = -b/2a)$

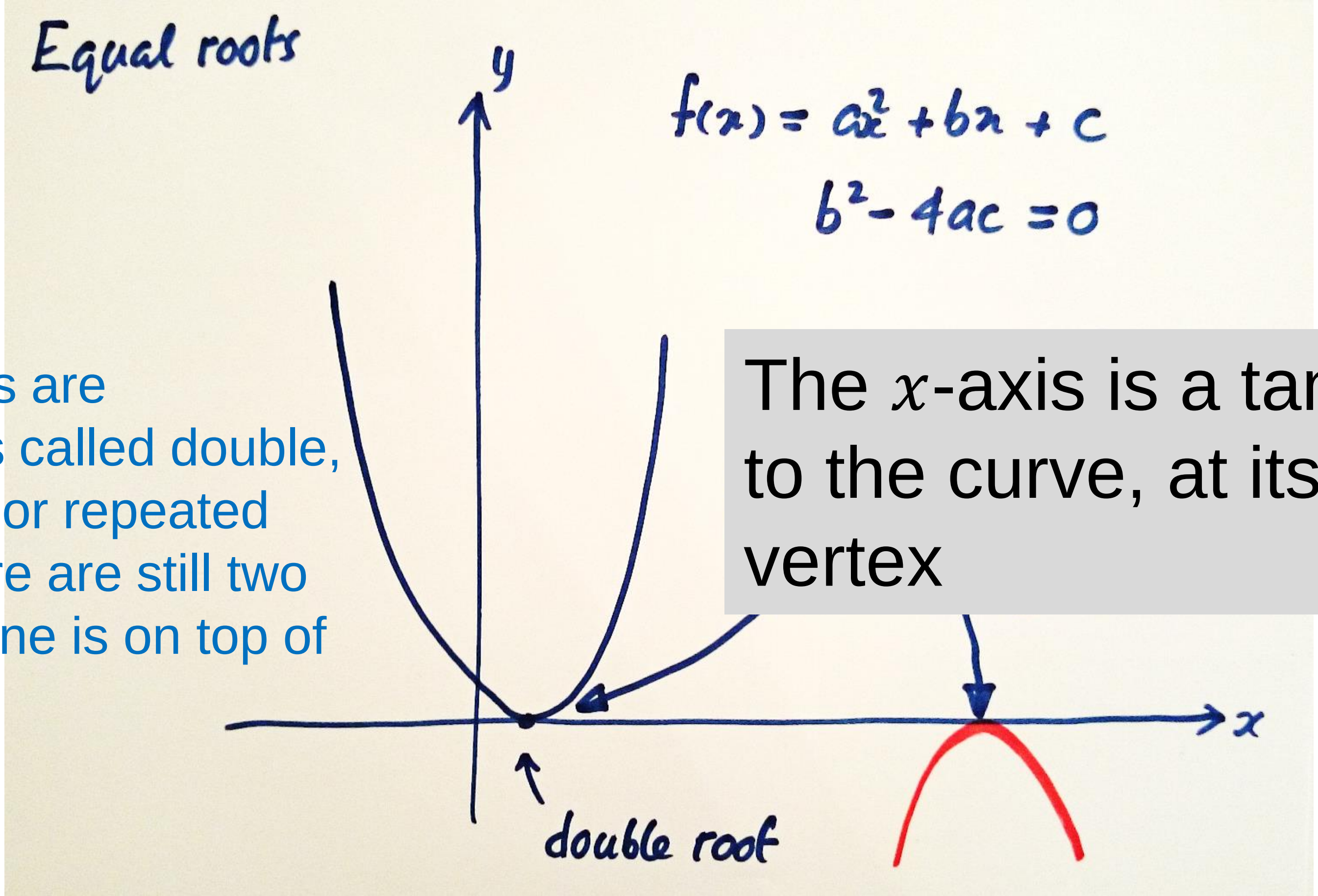
$b^2 - 4ac < 0$ No real roots

The quadratic graph: Two real roots



The quadratic graph is a called a parabola

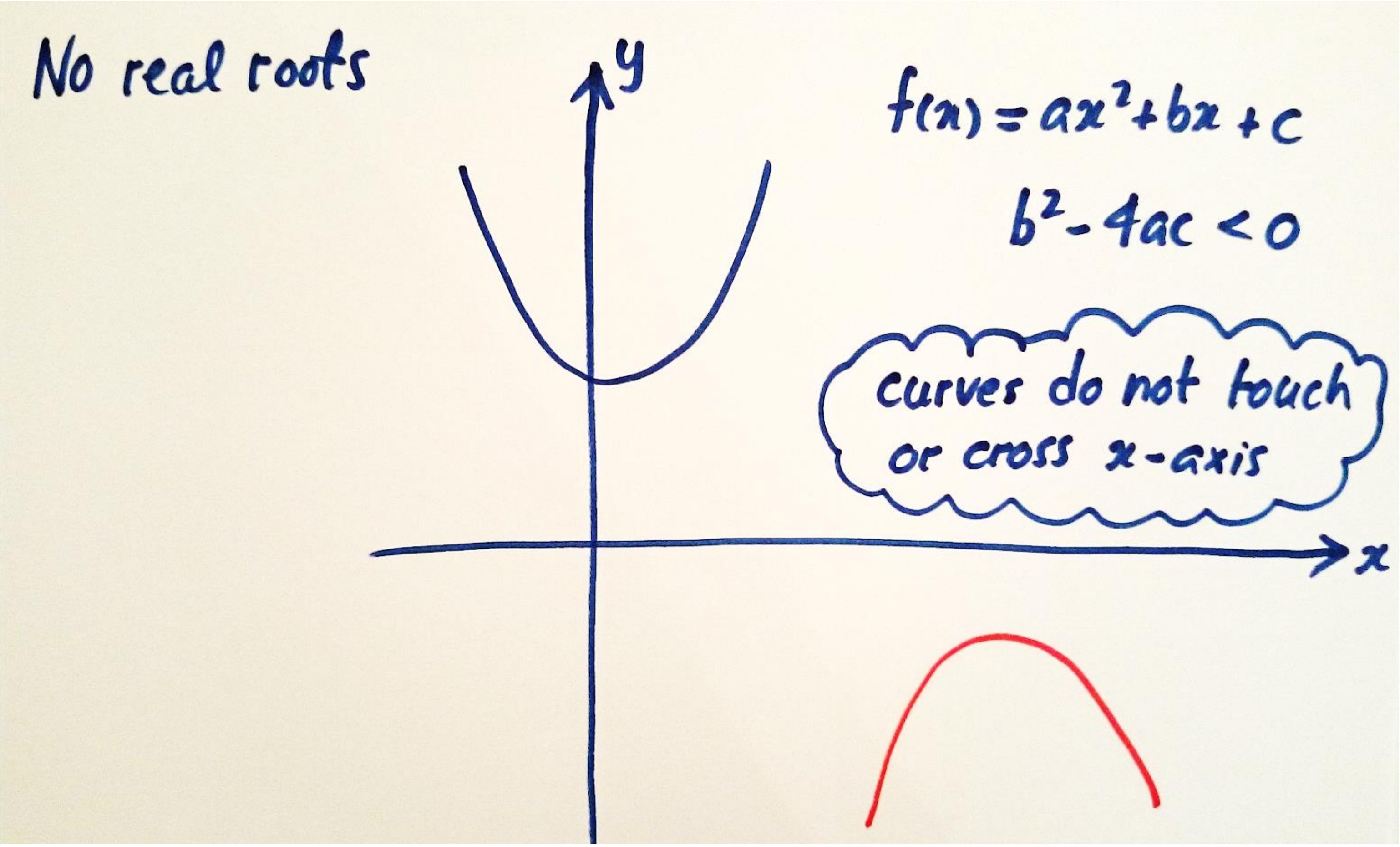
The quadratic graph: equal roots, x-axis is a tangent to the curve at the vertex



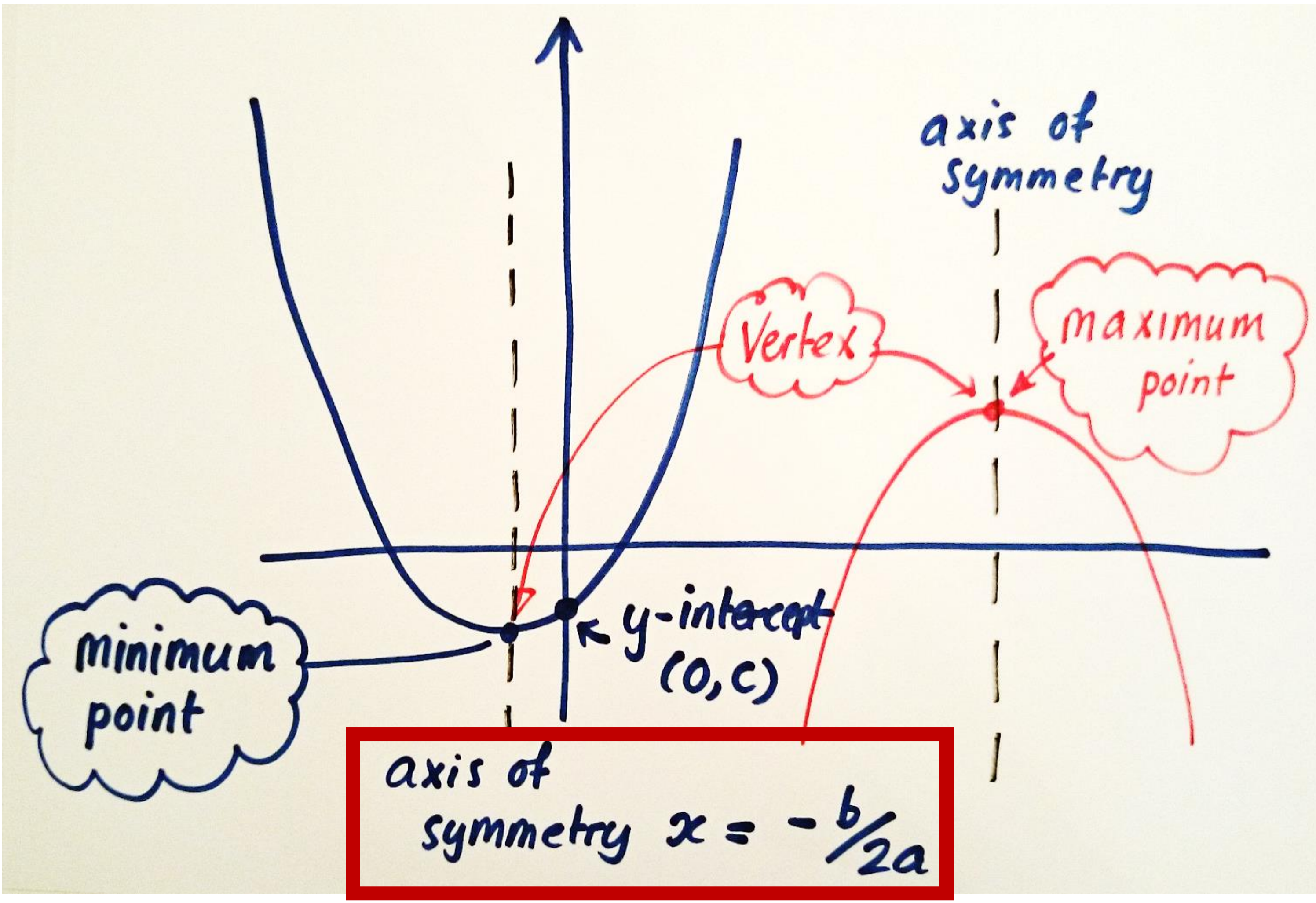
Equal roots are sometimes called double, coincident or repeated roots. There are still two roots but one is on top of the other.

The x-axis is a tangent to the curve, at its vertex

The quadratic graph: No real roots



Important properties of the quadratic graph – vertex on axis of symmetry



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Summary of important properties

- The maximum or minimum point must be on the axis of symmetry
- The coordinates of the maximum (minimum) are $\left(-\frac{b}{2a}, f\left(-\frac{b}{2a}\right)\right)$

- If $a > 0$ there is a minimum point
- If $a < 0$ there is a maximum point

- The y-intercept is at $(0, c)$
- The x-intercepts (if they exist) are the solutions to the quadratic, x_1, x_2. i.e., $(x_1, 0), (x_2, 0)$

Example: Sketching a quadratic graph

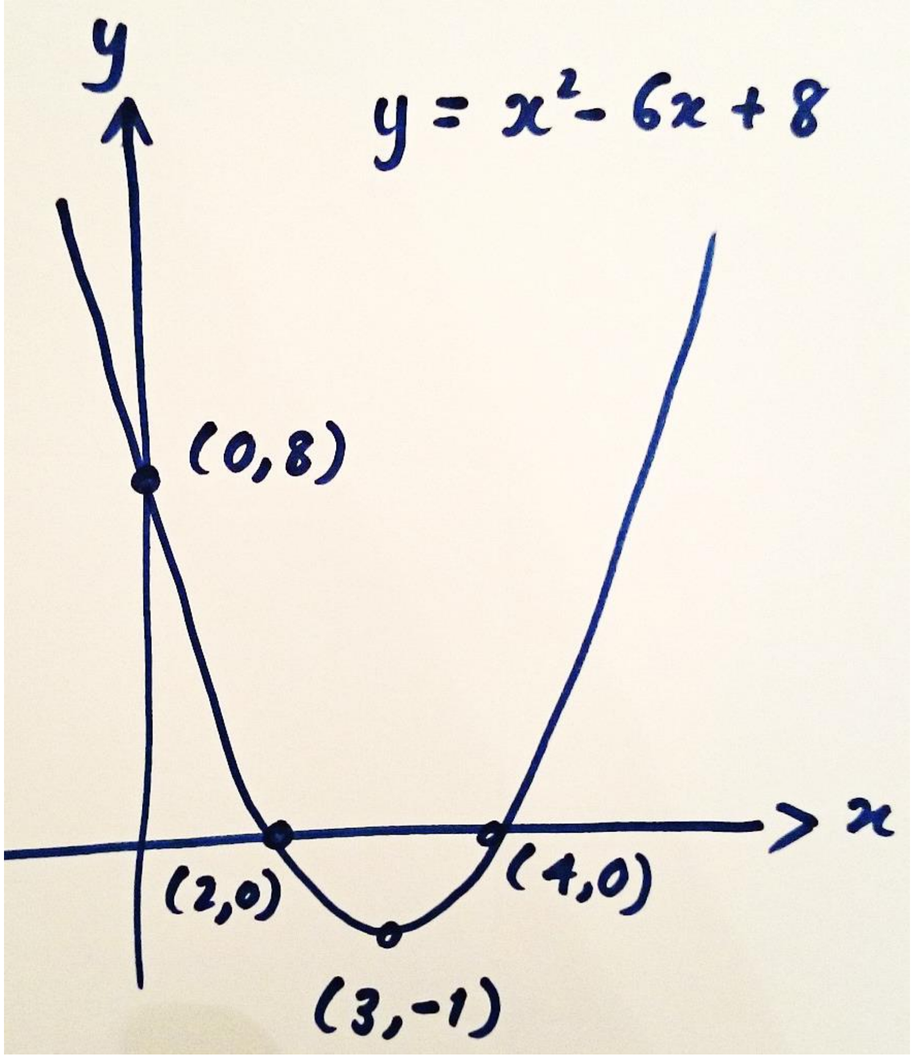
Sketch the graph of $y = x^2 - 6x + 8$

- y-intercept : $x = 0$, $\therefore (0, 8)$
- x-intercepts: $y = 0$, $\therefore (x-2)(x-4)$

  $x_1 = 2$, $x_2 = 4$
- minimum point $(a < 0)$

$$\left.\begin{aligned} x &= -\frac{b}{2a} = 3 \\ y &= f(3) = -1 \end{aligned}\right\} \therefore (3, -1) \text{ is the minimum.}$$

Example: Sketching a quadratic graph



Maximum (or minimum) points can also be found by completing the square

$$x^2 - 6x + 8 = (x-3)^2 + r$$
$$= x^2 - 6x + 9 + r$$

$9 + r = 8 \therefore r = -1$

$f(x) = (x-3)^2 - 1$

has a minimum because $a > 0$

The minimum value of $f(x) = (x-3)^2 - 1$ occurs when the bracket is zero i.e., $x = 3$ and $f(3) = -1 \therefore (3, -1)$ is the minimum point.

It's much quicker (and easier) just to find the axis of symmetry

Your turn!

Sketch the graph of

$$y = x^2 - 5x + 6$$

Solution

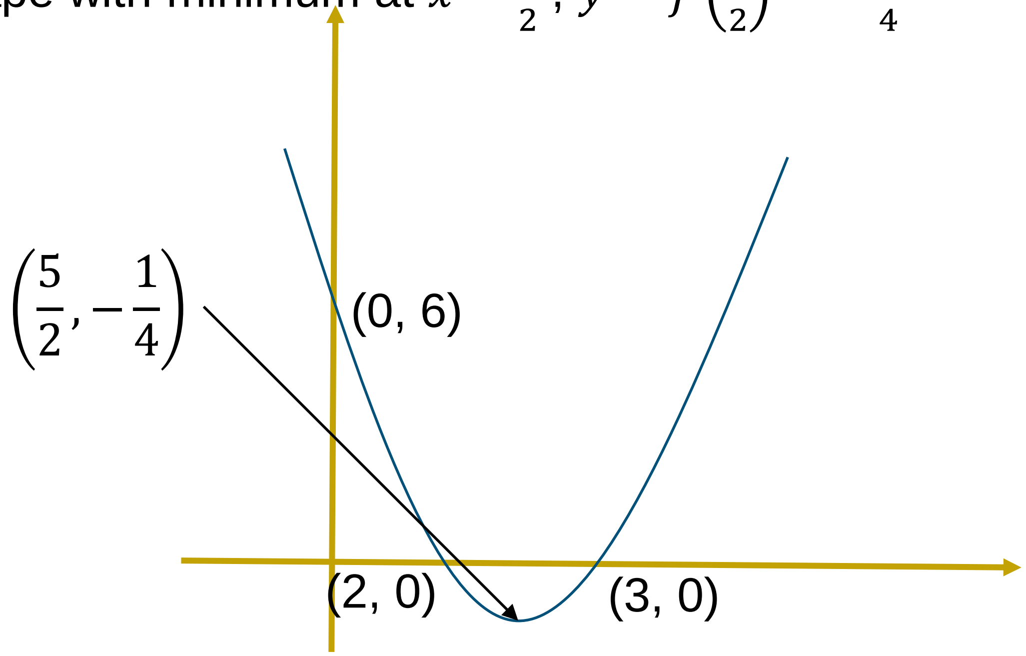
$$y = x^2 - 5x + 6$$

Intercept on y-axis: $f(0) = 6$

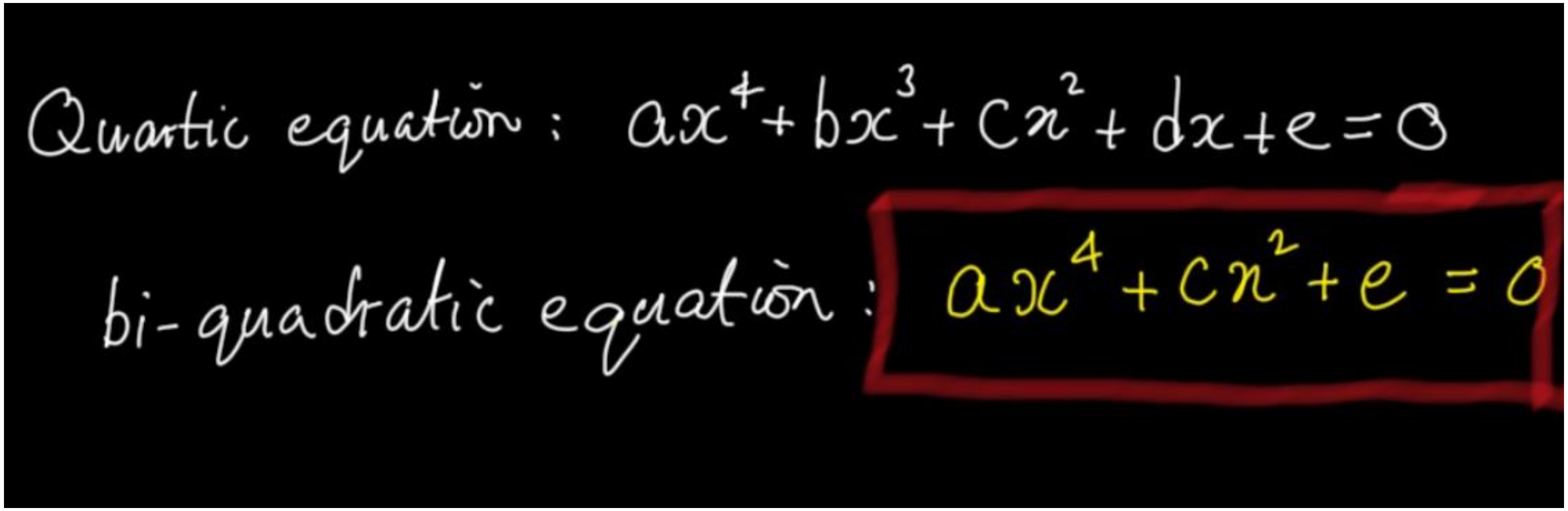
Roots (x-intercepts): $(x-3)(x-2) = 0 \Rightarrow x = 2 \; or \; x = 3$

Axis of symmetry: $-\frac{b}{2a} = \frac{5}{2}$

$a > 0 \therefore$ valley shape with minimum at $x = \frac{5}{2}$, $y = f\left(\frac{5}{2}\right) = -\frac{1}{4}$



2.2.6. Solve a bi-quadratic equation



Examples: bi-quadratic equation

E.g. 1 solve $x^4 - 3x^2 + 2 = 0$

subst. $t = x^2$

$$t^2 - 3t + 2 = 0$$

$$(t-2)(t-1) = 0$$

$$x^2 = 2 \qquad x^2 = 1$$

$$x = \pm\sqrt{2} \qquad x = \pm 1$$

$$x \in \{1, -1, \sqrt{2}, -\sqrt{2}\}$$

Examples: bi-quadratic equation

E.g. 2 Solve $4x^4 - 16x^2 + 15 = 0$

subst $t = 2x^2$

$$t^2 - 8t + 15 = 0$$

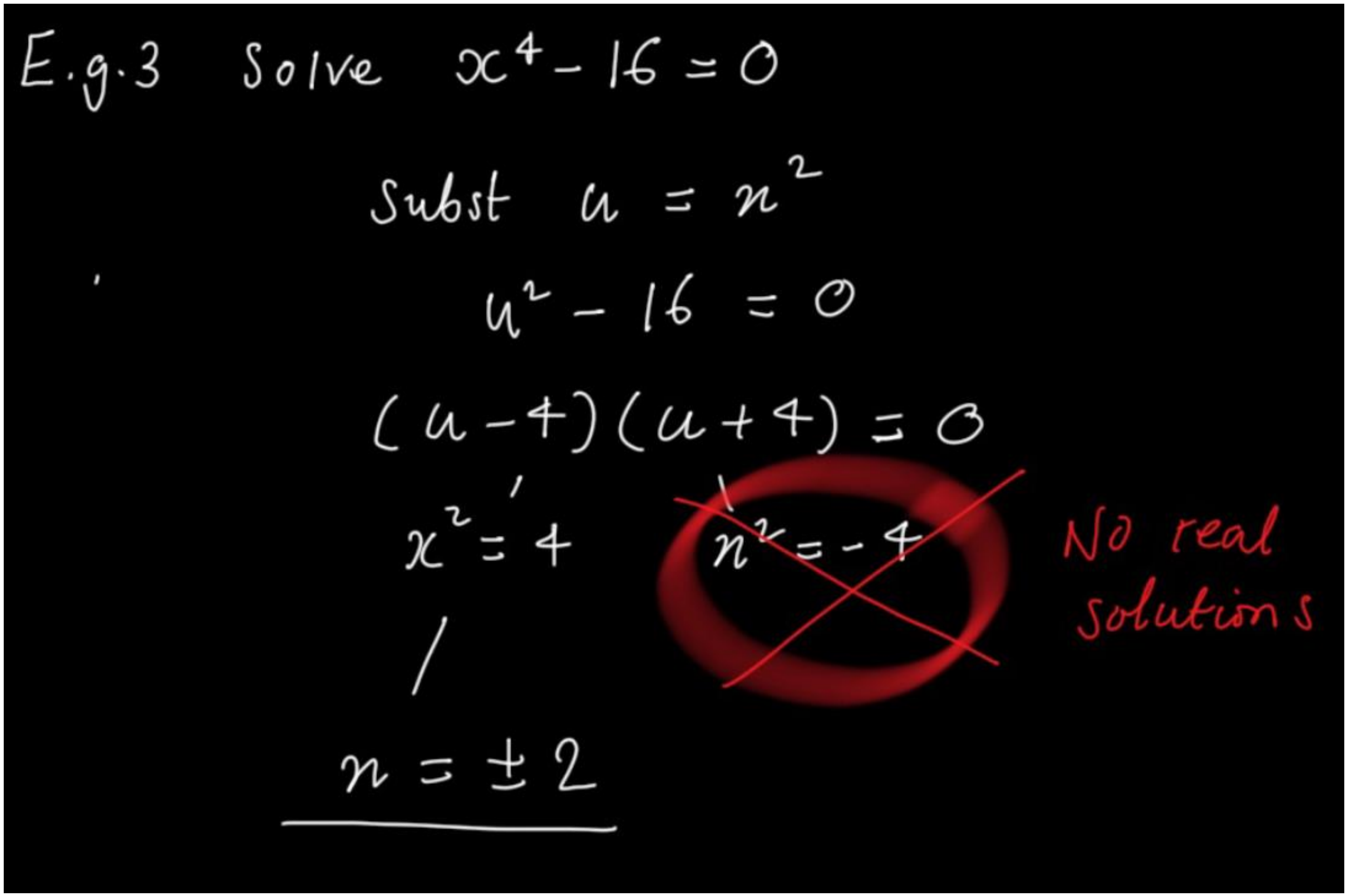
$$(t-5)(t-3) = 0$$

$$2x^2 = 5 \qquad 2x^2 = 3$$

$$x^2 = 5/2 \Rightarrow x = \pm\sqrt{5/2}$$

$$x^2 = 3/2 \Rightarrow x = \pm\sqrt{3/2}$$

Examples: bi-quadratic equation



Your turn!

Solve the biquadratic equation $4x^4 - 8x^2 + 3 = 0$

Solution

$$4x^4 - 8x^2 + 3 = 0$$

$$t^2 - 4t + 3$$

$$(t-1)(t-3) = 0$$

$$x^2 - 1 = 0$$

$$x = \pm 1$$

$$x^2 - 3 = 0$$

$$x = \pm\sqrt{3}$$

Substitute

$t = 2x^2$

2.2.7. Interpret graphically the nature of the roots of a polynomial equation

Polynomial: the general form of a polynomial expression is the sum of terms,

$$a_n x^n + a_{n-1} x^{n-1} + \dots + a_0.$$

The highest power, n is the degree or order of the polynomial (it is also the dominant term). The powers in each term are non-negative integers. a_n is called the leading coefficient and a_0 is a constant.

Behaviour of polynomial graphs

Polynomials do not have any horizontal or vertical asymptotes. We can consider how they behave at infinity though.

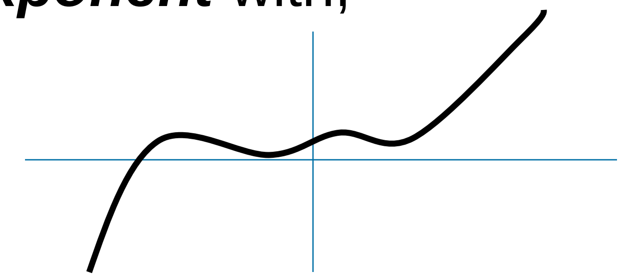
For example, $f(x) = x^3 - 2x^2 + 5$ tends to positive infinity as $x \to +\infty$ and $f(x)$ tends to negative infinity as $x \to -\infty$.

We need only consider the behaviour of the ***highest order*** (dominant) term and the sign of its coefficient. It should be clear from the above example, if it were "$-x^3$", then $f(x)$ would tend to positive infinity as $x \to -\infty$ and to negative infinity as $x \to +\infty$.

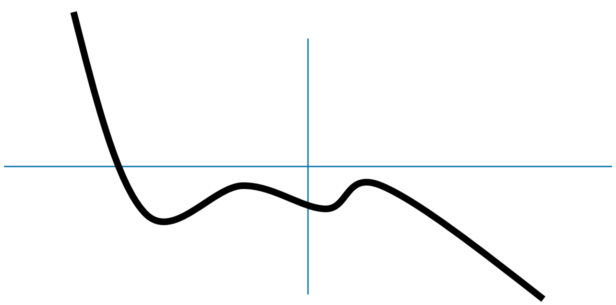
When the highest order term has a ***even exponent***, e.g "x^4", then $f(x)$ will tend to positive infinity as $x \to \pm\infty$. Clearly, if it were "$-x^4$", then $f(x)$ will tend to negative infinity as $x \to \pm\infty$.

Illustration of polynomial graphs

Dominant term has an ***odd exponent*** with,

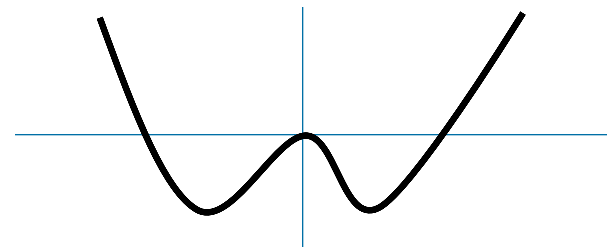


a positive coefficient e.g. $2x^7$

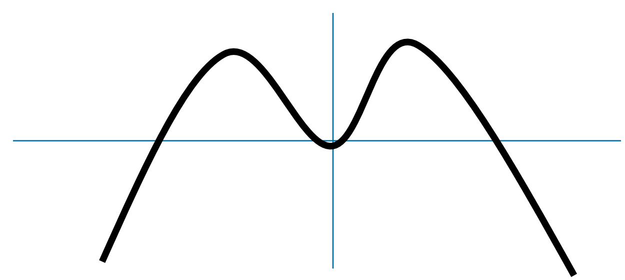


negative coefficient e.g. $-2x^7$

Dominant term has an ***even exponent*** with,



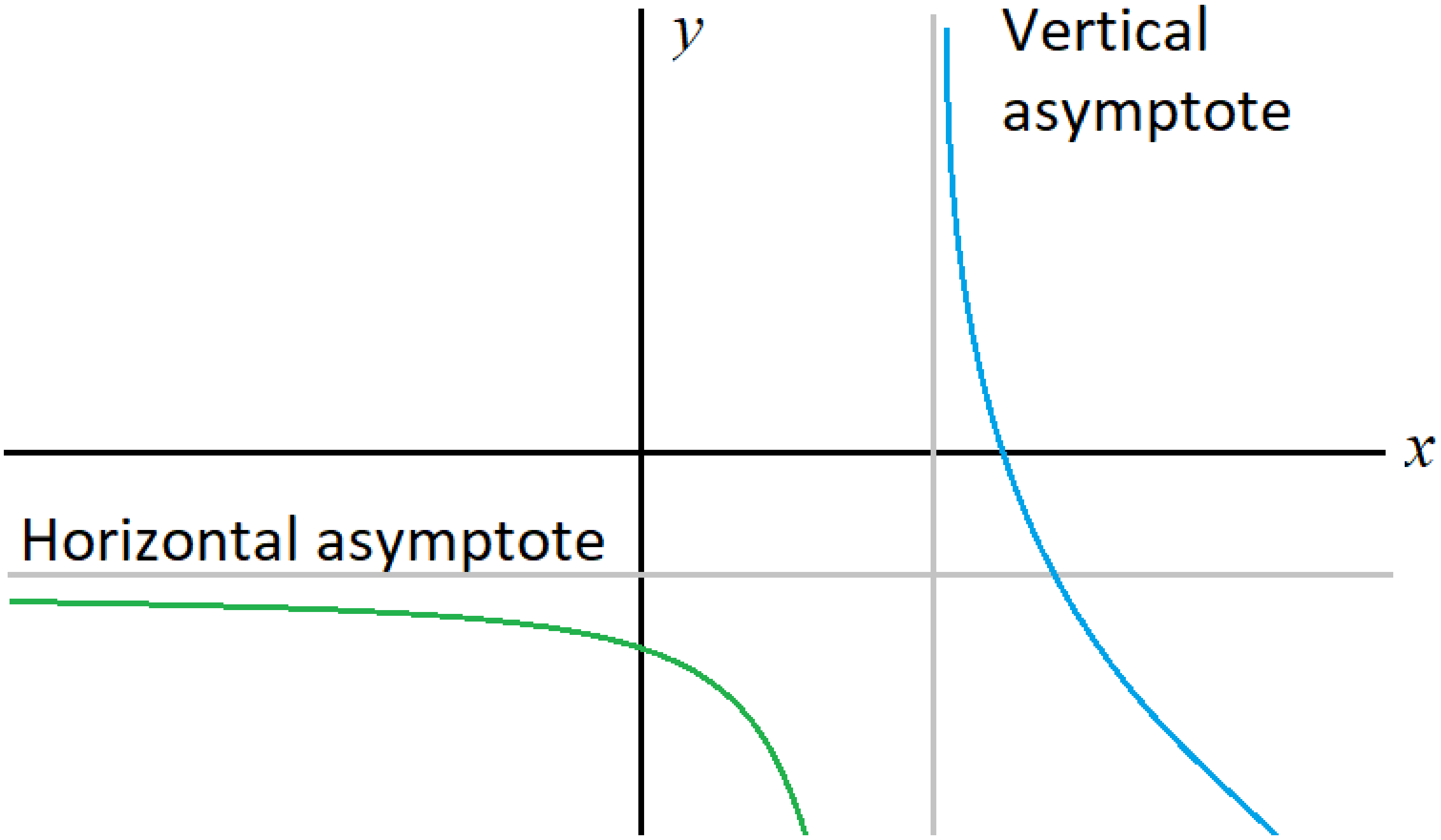
a positive coefficient e.g. $3x^4$



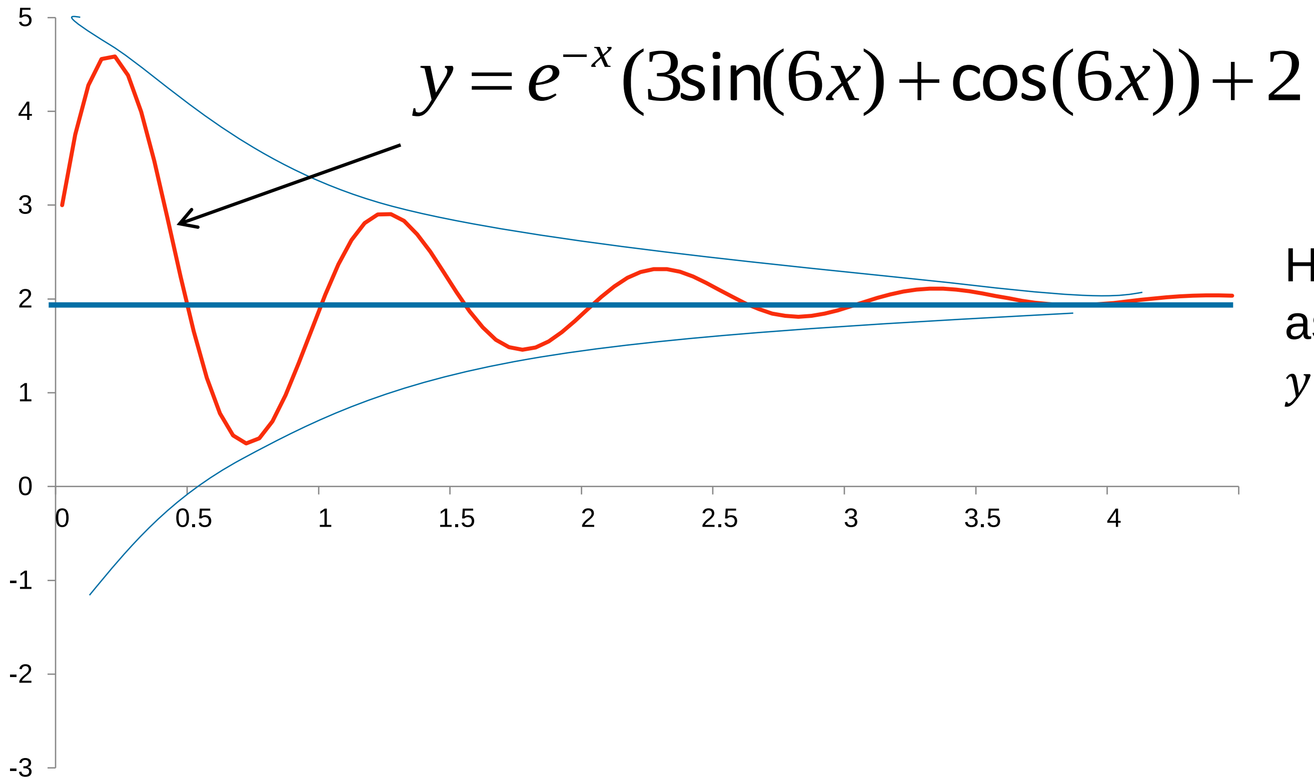
negative coefficient e.g $-3x^4$

Asymptotes and behavior “at infinity”

Asymptotes can be horizontal or vertical:



Certain types of function can cross over their horizontal asymptotes.



Horizontal asymptote $y = 2$

Roots of polynomials

A polynomial of degree n has n roots.

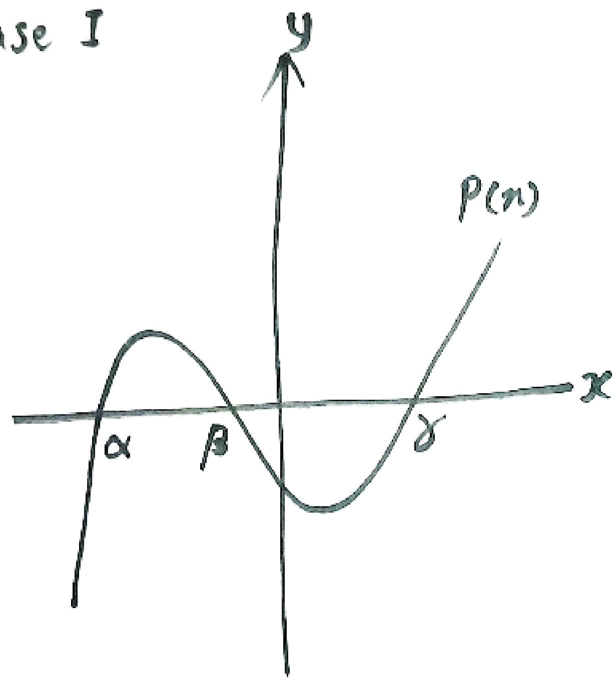
Some of the roots may be complex and these always occur in complex conjugate pairs, the product of a complex conjugate pair (when expressed as two linear factors of the polynomial) is an irreducible quadratic (see complex conjugate root theorem).

A polynomial can be factored into all real factors such that the factors are a combination of linear factors (and their powers, in the case of repeated roots) and irreducible quadratics (more about that in semester 2).

The cubic polynomial

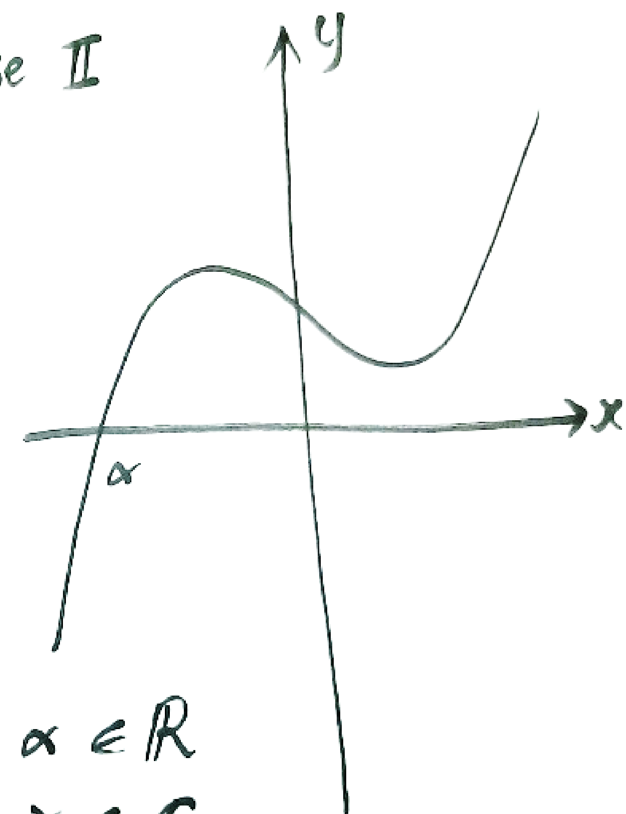
Cubic polynomial, $p(x) = a(x-\alpha)(x-\beta)(x-\gamma)$

Case I



$\alpha, \beta, \gamma \in \mathbb{R}$

Case II

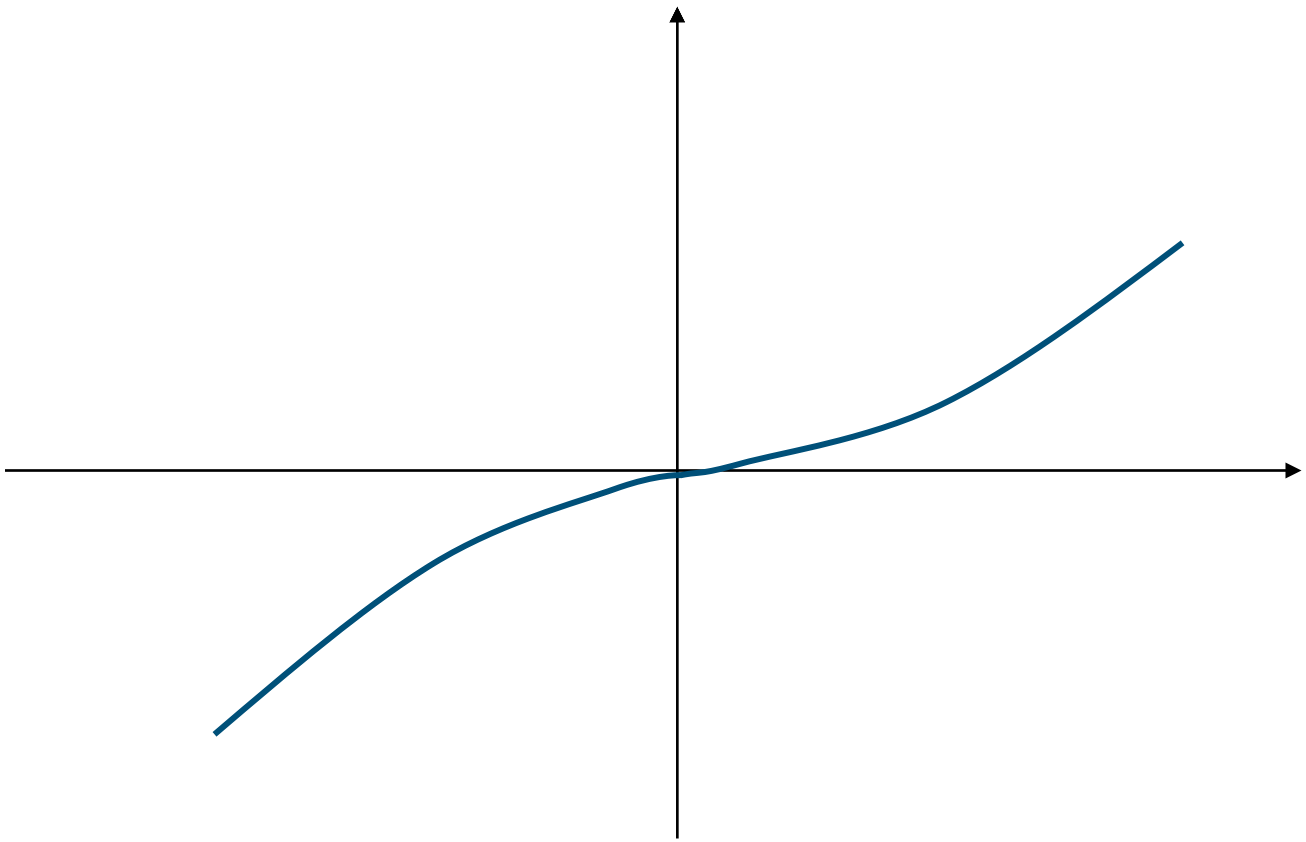


$\alpha \in \mathbb{R}$

$\beta, \gamma \in \mathbb{C}$

$\beta^* = \gamma$

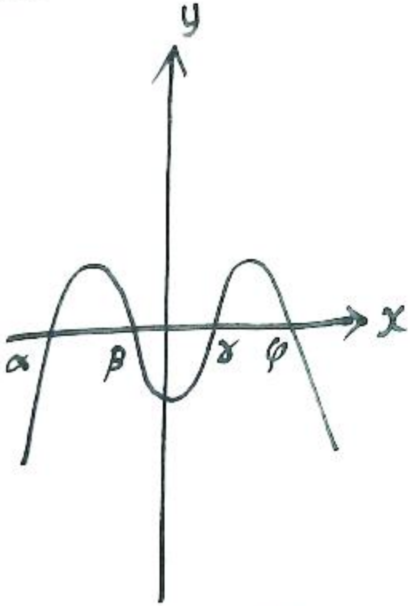
Cubic functions can look like this too…



The quartic polynomial (4th order)

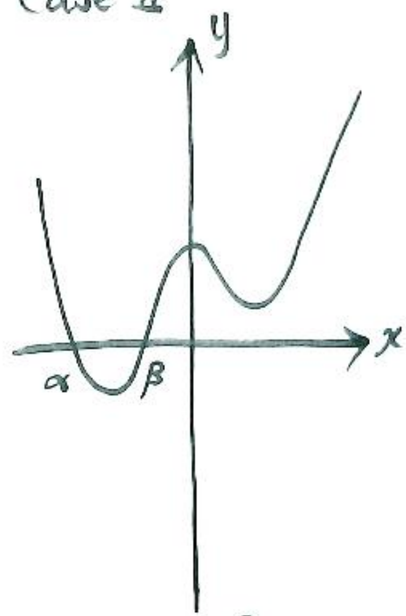
Quartic polynomial, $p(x) = a(x-\alpha)(x-\beta)(x-\gamma)(x-\varphi)$

Case I



$\alpha, \beta, \gamma, \varphi \in \mathbb{R}$

Case II

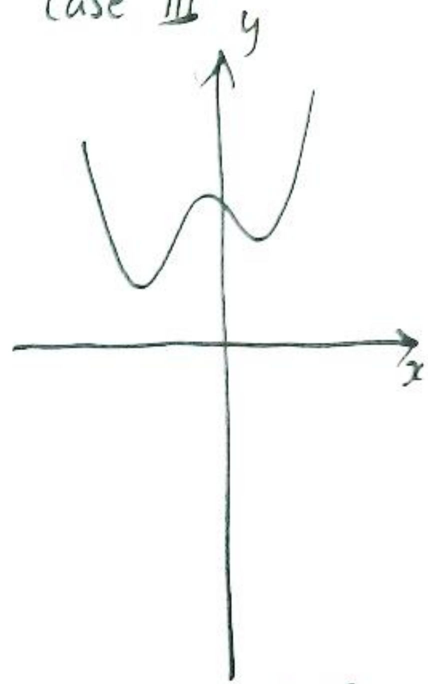


$\alpha, \beta \in \mathbb{R}$

$\gamma, \varphi \in \mathbb{C}$

$\gamma^* = \varphi$

Case III



$\alpha, \beta, \gamma, \varphi \in \mathbb{C}$

$\alpha^* = \beta$

$\gamma^* = \varphi$

Your turn!

A 7th order polynomial equation has the form

$$x^7 + gx^6 + fx^5 + ex^4 + dx^3 + cx^2 + bx + a$$

Write down the possible combinations of real and complex roots

Solution

Real	Complex
5	2
3	4
1	6
7	0

2.2 Summary

You should now be able to

2.2.1. Solve a quadratic equation by factorisation

2.2.2. Complete the square on a quadratic function

2.2.3. Solve a quadratic equation using the quadratic formula

2.2.4. Apply the quadratic discriminant in problem solving

2.2.5. Sketch the graph of a quadratic function

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