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2.3 Factor and remainder theorems

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FEPS Mathematics Support Framework

2.3 Introduction

The aim of this unit is to assist you in consolidating and developing your knowledge and skills in working with the factor and remainder theorems. It will also refresh your skills in algebraic manipulation and in solving two linear equations simultaneously.

While studying these slides you should attempt the 'Your Turn' questions in the slides.

After studying the slides, you should attempt the Consolidation Questions.

2.3 Learning checklist

Learning resource	Notes	Tick when complete
Slides		
Your turn questions		
Consolidation questions		

2.3 Learning objectives

After completing this unit you should be able to

2.3.1 Factorise polynomial expressions

2.3.2 Divide a polynomial by a linear or quadratic factor

2.3.3 Apply the remainder theorem

2.3.4 Apply the factor theorem

Some terminology (1)

Identity: a statement that two mathematical expressions are equal for all values of their variables (symbol $\equiv$).

Irreducible quadratic: a quadratic expression which cannot be written as a product of two linear factors using the set of real numbers, R, e.g., $x^2 + 1 = 0$. In other words, it is a quadratic with a negative discriminant ($b^2 - 4ac < 0$.)

Linear: a function or expression containing a variable with a first degree power, e.g., $x + 1, 2x + y$.

Some terminology (2)

Polynomial: the general form of a polynomial expression is the sum of terms,

$$a_n x^n + a_{n-1} x^{n-1} + \dots + a_0$$

The highest power, n is the degree or order of the polynomial (it is also the dominant term). The powers in each term are non-negative. a_n is called the leading coefficient and a_0 is a constant.

Quotient: the result of dividing one number (or polynomial) by another.

2.3.1. Factorise polynomial expressions

Factorisation is the process of rewriting an algebraic (often a polynomial) expression as a product of simpler, **irreducible** factors. The original expression is divisible by its factors.

Eg 1, a quadratic expression can sometimes be expressed as a product of two linear factors:

$$x^2 - 2x - 6 \equiv (x - 4)(x + 2)$$

Eg 2, an irreducible quadratic is a quadratic expression that cannot be factorised, $x^2 + 1$, for instance.

$$x^3 + x^2 + x + 1 \equiv (x^2 + 1)(x + 1)$$

Cannot be simplified further in the real number system

Note

Any polynomial with real coefficients can be written as a product of linear factors and irreducible quadratic factors.

Most polynomials that arise in real world applications cannot be factorised easily. Usually, though it’s not our goal to factorise but to find the roots of the polynomial. This is done numerically with software.

Clearly, if we can factorise the polynomial manually, we can also write down the roots. For now, we are concerned only with real roots.

There are general algebraic solutions to cubic and quartic polynomial equations (analogous to the quadratic formula).

Some useful identities

$$a^2 - b^2 \equiv (a-b)(a+b)$$

$$a^2 \pm 2ab + b^2 \equiv (a \pm b)^2$$

$$a^3 + 3a^2b + 3ab^2 + b^3 \equiv (a+b)^3$$

$$a^3 - 3a^2b + 3ab^2 - b^3 \equiv (a-b)^3$$

Using the greatest common factor (GCF) to factorise

1. Factorise $3x^3 - 9x^2$

Here, we can see that each term is a multiple of 3 and x^2. So, $3x^2$ is the GCF

$\therefore \quad 3x^3 - 9x^2 = 3x^2(x-3)$ ■

2. Factorise $2x^5 + 2x^4 - 4x^3$

In this case, $2x^3$ is the GCF

$\therefore \quad 2x^5 + 2x^4 - 4x^3 = 2x^3(x^2 + x - 2)$

$= 2x^3(x-1)(x+2)$ ■

Factorising by grouping

1. Factorise $x^5 - 2x^4 + x - 2$

The trick is to regroup terms that can be factorised:

Rewrite as: $(x^5 + x) - (2x^4 + 2)$

$$x(x^4+1) - 2(x^4+1)$$

$\therefore\ x^5 - 2x^4 + x - 2 = (x^4+1)(x-2)$

Factorisation using identities

Factorise $2x^4 - 6x^3y + 6x^2y^2 - 2xy^3$

first, take out the GCF, $2x$:

$$2x(x^3 - 3x^2y + 3xy^2 - y^3)$$

we recognise the expression in the brakets is an expansion of the form $(x-y)^3$, from the identity.

$$\therefore 2x^4 - 6x^3y + 6x^2y^2 - 2xy^3 = 2x(x-y)^3 \blacksquare$$

Using a substitution to factorise

Factorise $x^4 + x^2 - 20$

let $u = x^2$

$$
\begin{aligned}
x^4 + x^2 - 20 &= u^2 + u - 20 \\
&= (u+5)(u-4) \\
&= (x^2+5)(x^2-4) \\
&= (x^2+5)(x-2)(x+2) \quad \blacksquare
\end{aligned}
$$

difference between two squares

Factorisation summary

In general, only relatively simple expressions can be factorised easily. We can sometimes use GCF, grouping, identities or substitutions to help.

Trial and error may be involved!

There are other useful techniques available to us for finding any factors of polynomial expressions and solving polynomial equations. These will be considered next.

Your turn! (1)

Factorise the following,

$$x^4 - 625$$

$$x^4 - 3x^3 - x + 3$$

Solutions

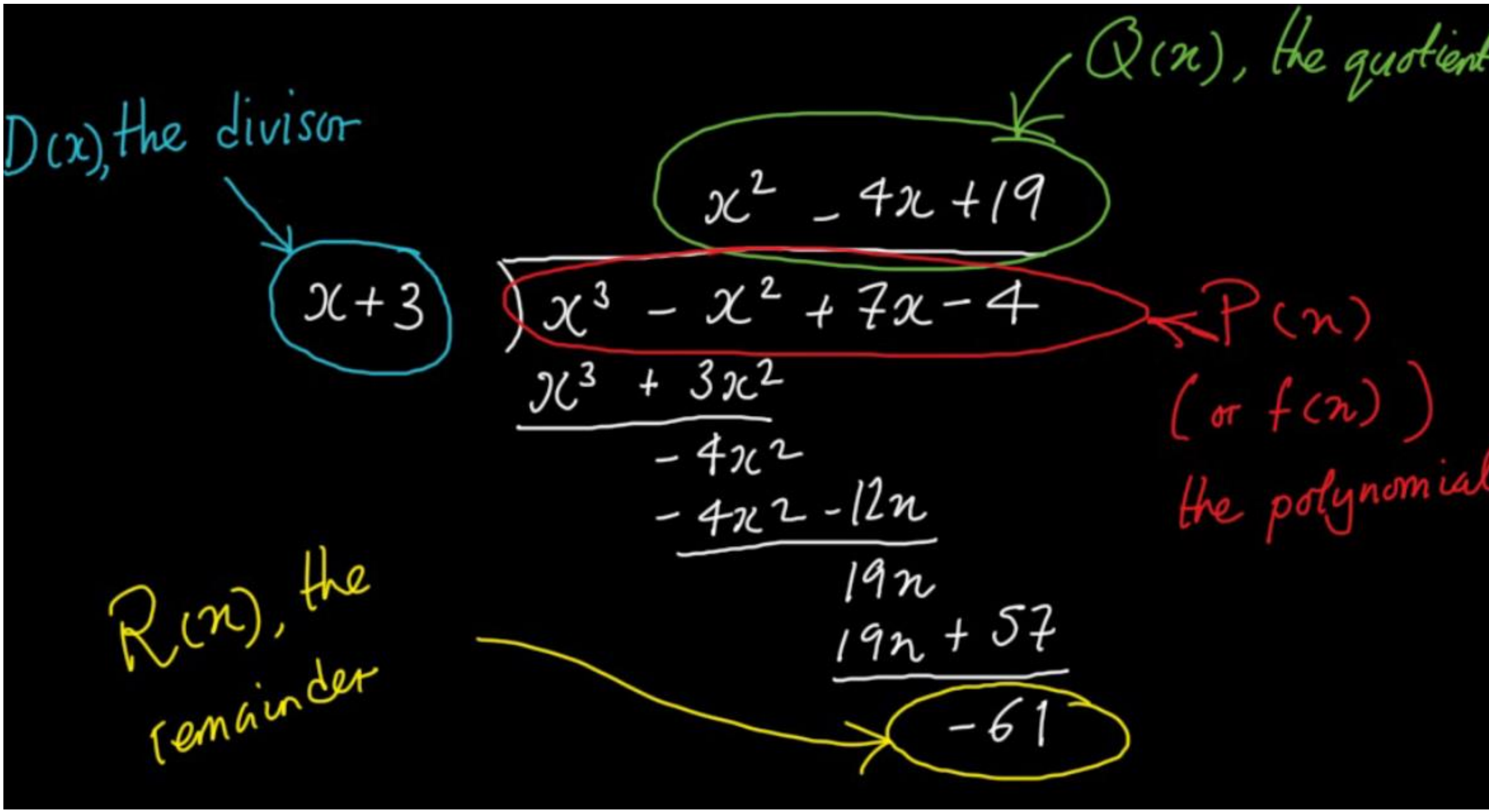
$$x^4 - 625 = (x^2 + 25)(x^2 - 25)$$

$$x^4 - 3x^3 - x + 3 = x^4 - x - 3x^3 + 3$$

$$= x(x^3 - 1) - 3(x^3 - 1)$$

$$= (x - 3)(x^3 - 1)$$

2.3.2. Divide a polynomial by a linear or quadratic factor



How can we use this to solve equations?

A polynomial $P(x)$ (or $f(x)$) can be written as:

$$\boxed{P(x) = D(x)Q(x) + R(x)}$$

Any value of x, such that $P(x) = 0$ is called a 'zero' of $P(x)$.

Example – long division

$\frac{x^3}{x} \rightarrow x^2 - 5x - 4 \leftarrow$ quotient

$x - 2 \;\big)\; x^3 - 7x^2 + 6x - 2$

$x^3 - 2x^2 \leftarrow$ multiply out $x^2(x-2)$

subtract

$-7x^2 - (-2x^2) \rightarrow -5x^2 + 6x - 2 \leftarrow$ bring down the next terms

$-5x^2 + 10x$

then repeat the division

$-4x - 2$

$-4x + 8$

$-10 \leftarrow$ remainder

$$\therefore\; x^3 - 7x^2 + 6x - 2 \equiv (x-2)(x^2 - 5x - 4) - 10$$

Example: long division

$$
\begin{array}{r|l}
 & x^2 + x - 6 \\
\hline
x+1 & x^3 + 2x^2 - 5x - 6 \\
 & \underline{x^3 + x^2} \\
 & x^2 - 5x - 6 \\
 & \underline{x^2 + x} \\
 & -6x - 6 \\
 & \underline{-6x - 6} \\
 & 0
\end{array}
$$

$$\therefore \quad x^3 + 2x^2 - 5x - 6 \equiv (x+1)(x^2 + x - 6)$$

$$\equiv (x+1)(x-2)(x+3)$$

The quotient factorises so we can write the polynomial as a product of three linear factors

Example: Long division

Divide $2x^4 - x^2 - 6x + 5$ by $2x^2 + 4x + 5$

$2x^4/2x^2$

$$
\begin{array}{r|l}
 & x^2 - 2x + 1 \\
\hline
2x^2 + 4x + 5 & 2x^4 + 0x^3 - x^2 - 6x + 5 \\
x^2(2x^2+4x+5) & \underline{-(2x^4 + 4x^3 + 5x^2)} \\
 & -4x^3 - 6x^2 - 6x \\
 & \underline{-(-4x^3 - 8x^2 - 10x)} \\
 & 2x^2 + 4x + 5 \\
 & \underline{-(2x^2 + 4x + 5)} \\
 & 0
\end{array}
$$

Subtract and bring down $-6x$

$\therefore\ 2x^4 - x^2 - 6x + 5 \div 2x^2 + 4x + 5 = x^2 - 2x + 1$ ■

$(x-1)^2$

Your turn! (2)

Divide $x^3 - 3x^2 + 4x - 12 \ by \ x - 3$

Solution

Quotient = $x^2 + 4$

Remainder = 0

2.3.3. Apply the remainder theorem

From the first two examples, we can see that,

1. $f(x) \equiv x^3 - 7x^2 + 6x - 2$

   When $f(x)$ was divided by $(x-2)$ the remainder was -10 and $f(2) = -10$

2. $f(x) \equiv x^3 + 2x^2 - 5x - 6$

   When $f(x)$ was divided by $(x+1)$ the remainder was 0 and $f(-1) = 0$

The remainder theorem

In general when a polynomial $f(x)$ is divided by $(x-a)$, there is a quotient, $Q(x)$ and a remainder, R.

We can write, $f(x) \equiv (x-a)Q(x) + R$

in the case $x = a$, then $f(a) = R$

This is known as the remainder theorem and can be stated as,

$$\boxed{\text{When } f(x) \text{ is divided by } (x-a) \text{ the remainder is } f(a)}$$

Example: Remainder theorem

1. Find the remainder when $5x^2 - 2x + 3$ is divided by $(3x+2)$

$$f(-2/3) = 5(-2/3)^2 - 2(-2/3) + 3$$

$$= 5(4/9) + 4/3 + 3$$

$$= \frac{20}{9} + \frac{12}{9} + \frac{27}{9} = 59/9$$

Example: Remainder theorem

2. Express $6x^2 + 2x - 3$ in the form $(2x-2)Q(x) + R$, where $Q(x)$ and R are to be found.

$$6x^2 + 2x - 3 \equiv (2x-2)Q(x) + R$$

$$\begin{array}{r}
3x + 4 \\
2x-2 \,\big)\overline{\,6x^2 + 2x - 3} \\
\underline{6x^2 - 6x \phantom{{}-3}} \\
8x - 3 \\
\underline{8x - 8} \\
5
\end{array}$$

$\therefore \; Q(x) = 3x + 4$

$R = 5$

$$6x^2 + 2x - 3 \equiv (2x-2)(3x+4) + 5 \; \blacksquare$$

Your turn! (3)

Find the remainder when $x^3 - 5x^2 - 2x - 5$ is divided by $(x + 3)$

Solution

$$f(-3) = (-3)^3 - 5(-3)^2 - 2(-3) - 5 = -71$$

2.3.4. Apply the factor theorem

The factor theorem concerns the case where the remainder is zero.

If $x-a$ is a factor of $f(x)$ there will be no remainder, ie. $f(a)=0$ and $R=0$.

$$\boxed{\begin{array}{l}\text{For a polynomial function } f(x) \\ \text{if } f(a)=0 \text{ then } (x-a) \text{ is a factor of } f(x)\end{array}}$$

Why the factor theorem is useful

We can use the factor theorem to help us factorise polynomials and to solve polynomial equations!

Knowing $(x - a)$ is a factor means that you also know a is a root and vice versa.

Example: Factor theorem

factorise $f(x) = x^4 - 3x^3 - 5x^2 + 3x + 4$

hence write down the solutions to $f(x) = 0$

$f(1) = 1 - 3 - 5 + 3 + 4 = 0 \quad \therefore \ (x-1)$ is a factor

$f(-1) = 1 - (-3) - 5 - 3 + 4 = 0 \quad \therefore \ (x+1)$ is a factor

$(x-1)(x+1) = x^2 - 1$

$$
\begin{array}{r l}
 & x^2 - 3x - 4 \\
\cline{2-2}
x^2 - 1 \, \big) & x^4 - 3x^3 - 5x^2 + 3x + 4 \\
 & x^4 \qquad\quad - x^2 \\
\cline{2-2}
 & \quad -3x^3 - 4x^2 + 3x + 4 \\
 & \quad -3x^3 \qquad\quad + 3x \\
\cline{2-2}
 & \qquad\quad -4x^2 \quad + 4 \\
 & \qquad\quad -4x^2 \quad + 4 \\
\cline{2-2}
 & \qquad\qquad\qquad 0
\end{array}
$$

$f(x) = (x-1)(x+1)(x^2 - 3x - 4)$

$\quad = (x-1)(x+1)(x-4)(x+1)$

$f(x) = 0 : \ x = 1, -1, 4, -1$ ■

Example: Factor theorem

One root of $x^2 - 3x + a = 0$ is 2. Find the other root.

$$f(x) = x^2 - 3x + a$$

$$f(2) = 4 - 6 + a = 0$$

$$\therefore \quad a = 2$$

$$f(x) = x^2 - 3x + 2$$

$$= (x-2)(x-1)$$

$\therefore$ the other root is $x = 1$ ■

Your turn! (4)

Find all the solutions to $x^3 + x^2 - 37x + 35 = 0$

Solution

Try $f(1)$ first

$f(1) = 1 + 1 - 37 + 35 = 0 \implies (x - 1)$ is a factor

Now try $f(5)$

$f(5) = 125 + 25 - 185 + 35 = 0 \implies (x - 5)$ is a factor

Finally try $f(-7)$

$f(5) = -343 + 49 + 259 + 35 = 0 \implies (x + 7)$ is a factor

Roots: $x \in \{-7, 1, 5\}$

Further examples: Factor and remainder theorem

1. $f_1(x) = x^3 + 2x^2 - 5ax - 7$

$f_1(-1) = R_1$

$f_2(x) = x^3 + ax^2 - 12x + 6$

$f_2(2) = R_2$

$$\boxed{2R_1 + R_2 = 6}$$

find a

Further examples: Factor and remainder theorem

Solution

$$f_1(x) = x^3 + 2x^2 - 5ax - 7$$
$$f_2(x) = x^3 + ax^2 - 12x + 6$$

$$f(-1) = -1 + 2 + 5a - 7 = R_1$$
$$5a - 6 = R_1$$
$$f_2(2) = 8 + 4a - 24 + 6 = R_2$$
$$4a - 10 = R_2$$

$$2R_1 + R_2 = 6$$

$$2(\overset{R_1}{5a - 6}) + (\overset{R_2}{4a - 10}) = 6$$
$$10a - 12 + 4a - 10 = 6$$
$$14a = 28 \Rightarrow a = 2$$

Further examples: Factor and remainder theorem

2\. $f(x) = 2x^3 + 5x^2 - 18x + 5$, given the condition $f(a) = f(-a)$ find the possible values of a

$$2a^3 + \cancel{5a^2} - 18a + \cancel{5} = -2a^3 + \cancel{5a^2} + 18a + \cancel{5}$$

$$4a^3 - 36a = 0$$

$$4a(a^2 - 9) = 0 \Rightarrow a = 0$$

$$(a+3)(a-3) = 0 \Rightarrow a = \pm 3$$

$$\boxed{a \in \{-3, 0, 3\}}$$

Further examples: Factor and remainder theorem

3. $g(x) = 3x^3 + 2x^2 - px + q$, given that $(x-1)$ is a factor of $g(x)$ and $g(-1) = 10$, find p and q.

$$g(1) = 3 + 2 - p + q = 0 \Rightarrow q - p = -5 \quad (1)$$

$$g(-1) = -3 + 2 + p + q = 10 \Rightarrow q + p = 11 \quad (2)$$

$$(1)+(2): \ 2q = 6 \Rightarrow q = 3$$

$$p = 11 - 3 = 8$$

$$\boxed{p = 8, \ q = 3}$$

2.3 Summary

You should now be able to,

2.3.1 Factorise polynomial expressions

2.3.2 Divide a polynomial by a linear or quadratic factor

2.3.3 Apply the remainder theorem

2.3.4 Apply the factor theorem

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