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3.1 Properties of exponentials and logarithms

Core Preparatory Topics
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FEPS Mathematics Support Framework

3.1 Introduction

The aim of this unit is to assist you in consolidating and developing your knowledge and skills in working with exponentials and logarithms and graphs of these functions.

While studying these slides you should attempt the ‘Your Turn’ questions in the slides.

After studying the slides, you should attempt the Consolidation Questions.

3.1 Learning checklist

Learning resource	Notes	Tick when complete
Slides		
Your turn questions		
Consolidation questions		

3.1 Learning objectives

After completing this unit you should be able to

3.1.1 Sketch graphs of exponential functions (including e^x)

3.1.2 Sketch the graph of a natural logarithmic function

3.1.3 Write an expression in index form or logarithmic form

3.1.4 Apply the laws of logarithms

3.1.5 Solve exponential equations

What is the connection between exponentials and logarithms?

Exponential and logarithmic functions are closely related as one is the inverse of the other!

We will also see that when we write numbers in logarithmic form it’s not really that different to using index notation!

First, lets refresh our memories about the laws of indices…

The laws of indices are analogous to the laws of logarithms

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For any two real numbers a & b and indices m & n:

1. $a^0 = 1$, this is a convention in mathematics

2. $a^m a^n = a^{m+n}$

3. $a^m / a^n = a^{m-n}$

4. $a^{-n} = \frac{1}{a^n}$

5. $\sqrt{a} = a^{1/2}$, and in general $\sqrt[n]{a} = a^{1/n}$ and $\sqrt[n]{a^m} = a^{m/n}$

6. $(ab)^n = a^n b^n$

7. $(a/b)^n = a^n/b^n$

We will see some of the similarities of the laws of indices to the laws of logs a bit later in the lecture…

Your turn!

Simplify	Evaluate
$x^7 \times x^{-5}$	$64^{-\frac{2}{3}}$
$\left(y^{\frac{3}{5}}\right)^{-10}$	$\left(\frac{64}{81}\right)^{-\frac{1}{2}}$

Answers

x^2	$\frac{1}{16}$
y^{-6}	$\frac{9}{8}$

Exponential Functions

$$y = a^x$$

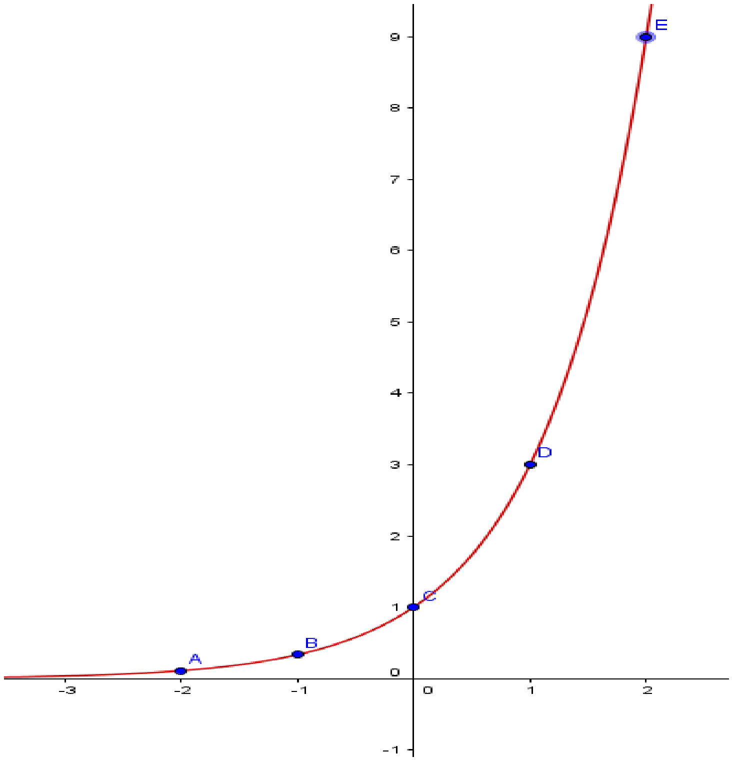
Exponent or index

Base

In the next few slides we discuss the behaviour of this function for different values of a and x.

3.1.1. Sketch graphs of exponential functions (including e^x)

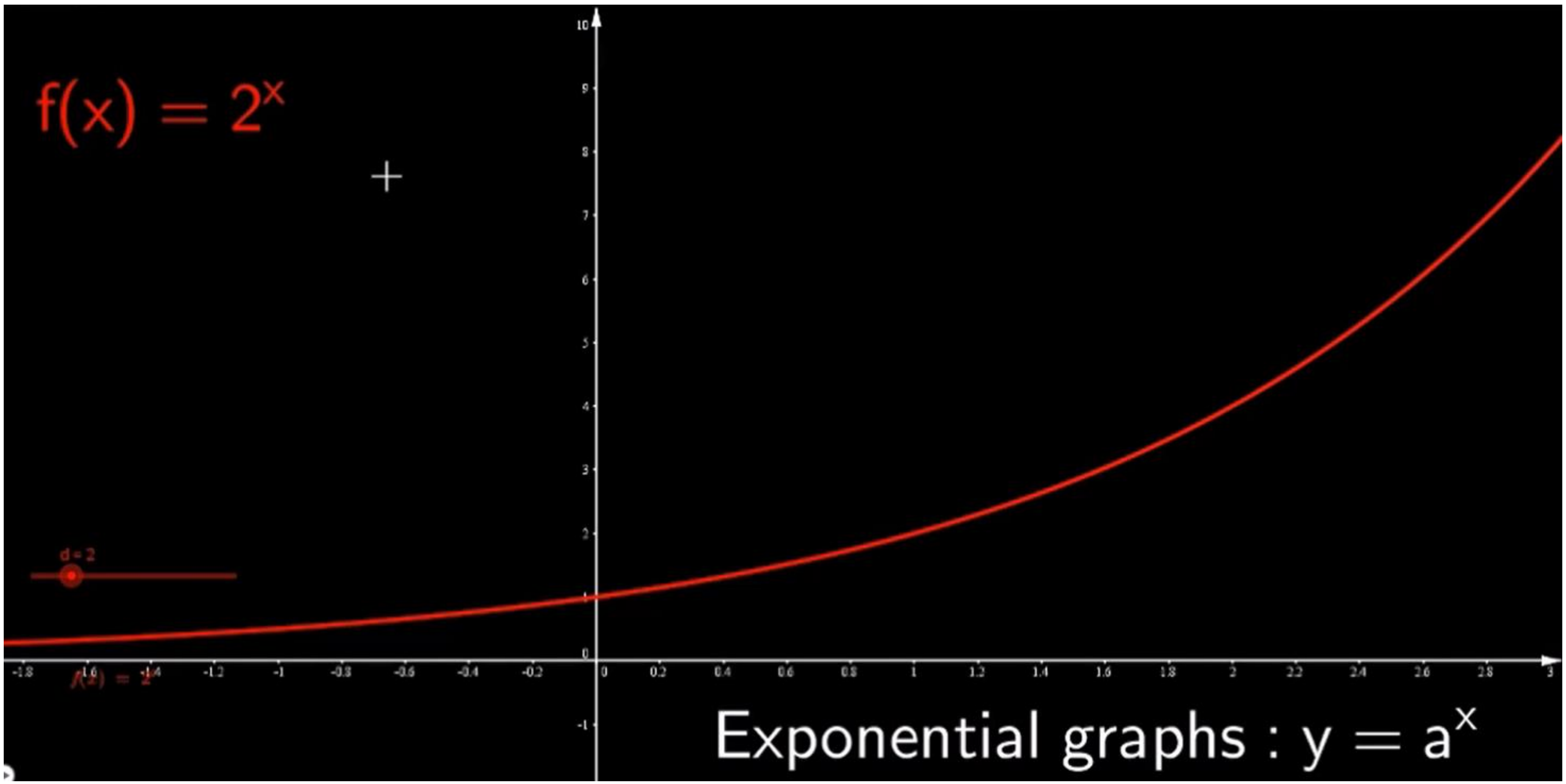
As we already know, an asymptote is a line that a curve gets closer and closer to, but never meets.



All graphs of exponential functions and logarithmic functions have asymptotes.

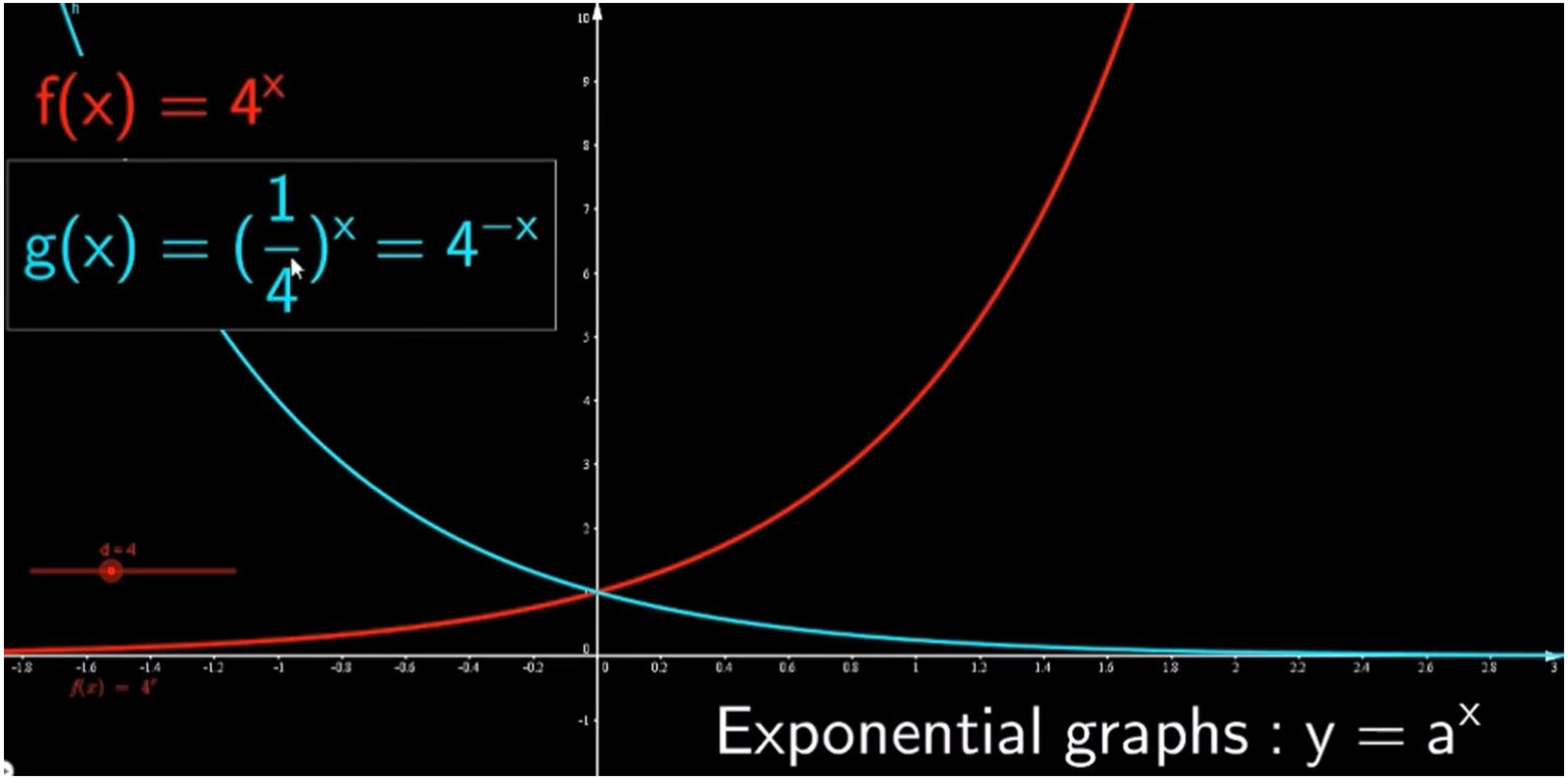
The graph of $y = a^x$ will always pass through the point $(0, 1)$

Graphs of exponential functions, a > 0



$y = a^x;\ 0 < a < 1$

The graph of $y = a^{-x}$ is a reflection of the graph for $y = a^x$ in the y-axis



Introducing the exponential function, $y = e^x$

$y = e^x$ where $e \approx 2.7$ is a common form of exponential function which has many applications in calculus and engineering. This is a special case of the general form of $y = a^x$.

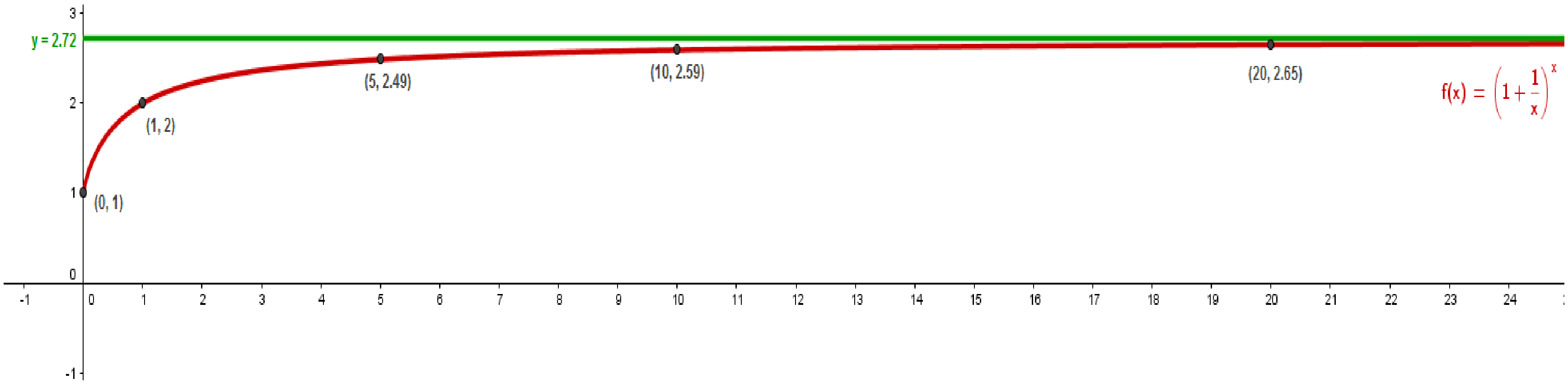
What is e?

It is a very well known **irrational number** which is called the Euler number. Its value is approximately 2.7, this number is also the base of the natural logarithm which was invented by Napier.

How can we compute e?

In mathematics there are many ways to find e :

- One approach is to use $\boldsymbol{e} = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$



$\boldsymbol{n}$	1	10	100	1000	10000	100000	1000000
$(1+\frac{1}{n})^n$	2	2.59	2.7	2.72	2.72	2.72	2.72

Another method is to use a power series: $\boldsymbol{e} = \sum_{n=0}^{\infty} \frac{1}{n!}$

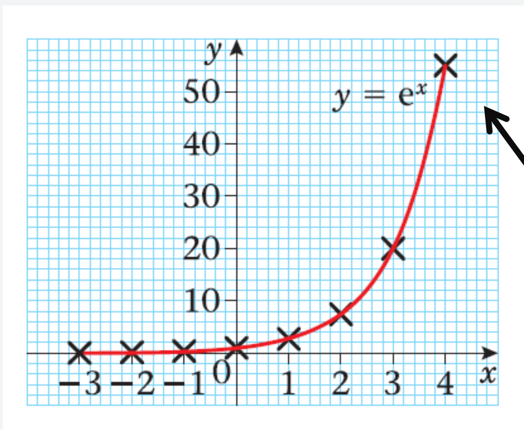
n	$n!$	$\sum_{k=0}^{n} \frac{1}{k!}$
0	1	$\sum_{k=0}^{0} \frac{1}{k!} = 1$
1	1	$\sum_{k=0}^{1} \frac{1}{k!} = 1 + 1 = 2$
2	2	$\sum_{k=0}^{2} \frac{1}{k!} = 1 + 1 + \frac{1}{2} = 2.5$
3	6	$\sum_{k=0}^{3} \frac{1}{k!} = 1 + 1 + \frac{1}{2} + \frac{1}{6} = 2.666667$
4	24	$\sum_{k=0}^{4} \frac{1}{k!} = 1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} = 2.666667$

Graphing $y = e^x$

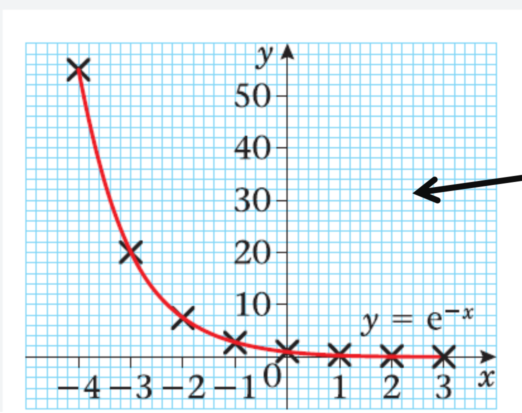
Draw the graphs of:

a $y = e^x$ **b** $y = e^{-x}$

a



b



A table of values will show you how rapidly this curve grows.

x	−2	−1	0	1	2	3	4	5
y	0.14	0.37	1	2.7	7.4	20	55	148

With these curves it is worth keeping in mind:

- as $x \to \infty$, $e^x \to \infty$ (it grows very rapidly)
- when $x = 0$, $e^0 = 1$ [(0, 1) lies on the curve]
- as $x \to -\infty$, $e^x \to 0$ (it approaches but never reaches the x-axis).

Exponential growth

Exponential decay

This curve is similar to the one in part **a** except that its value at $x = 2$ is e^{-2} and its value at $x = -2$ is e^2.

Hence it is a reflection of the curve of part **a** in the y-axis.

Inverses of exponential functions

The inverse of a^x is $\log_a x$

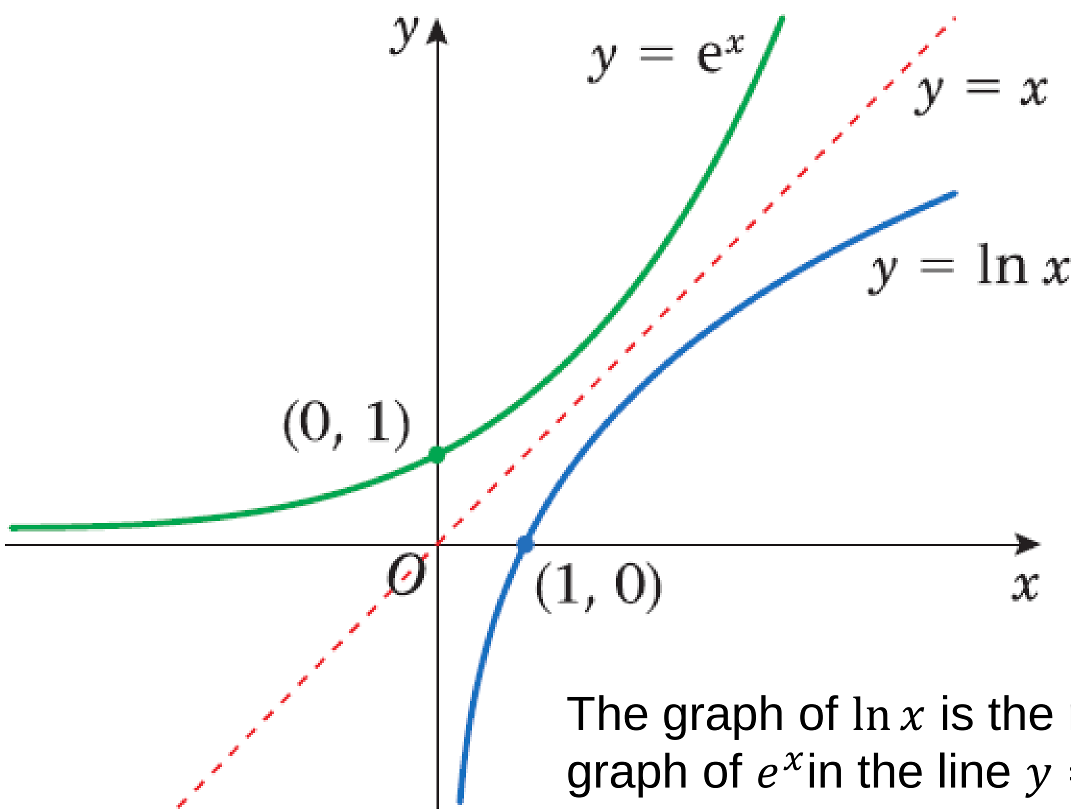
These are called logarithmic functions

The inverse of e^x is $\log_e x = \ln x$

Where $\ln x$ is the **Natural logarithmic function**.

As the two functions are inverses, the graph of $\ln x$ is a reflection of the graph for e^x in the line $y = x$

3.1.2. Sketch the graph of the natural logarithmic function



The graph of $\ln x$ is the reflection of the graph of e^x in the line $y = x$.

Your turn!

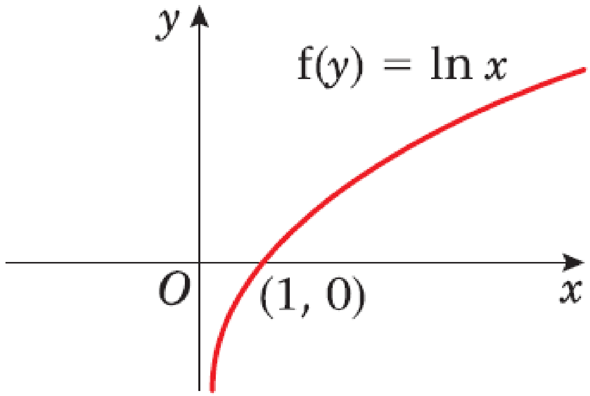
Sketch the graph of $y = -2\ln(x-3)$

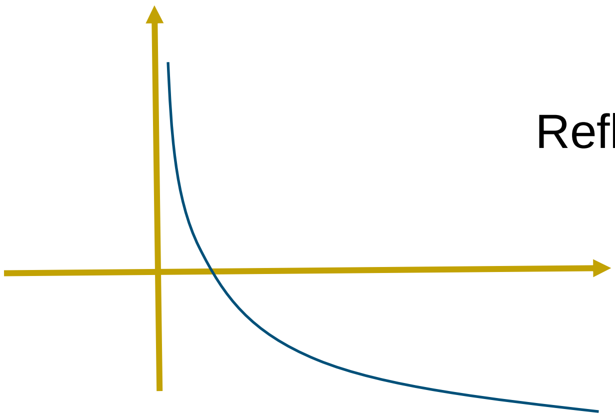
$-$ Sign is a reflection in the x-axis

2 is the vertical scaling

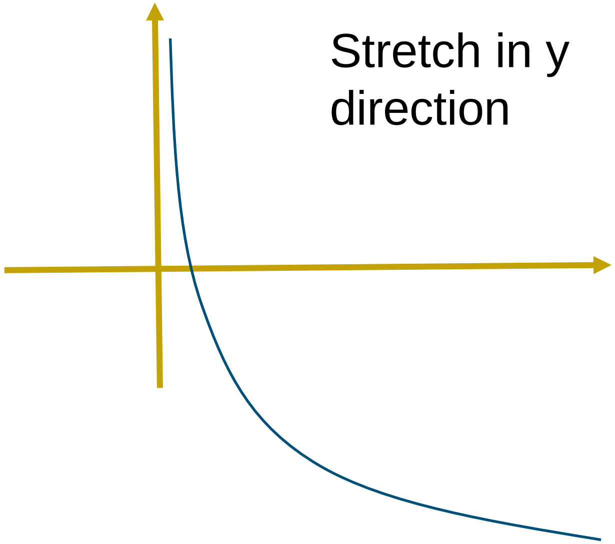
$(x-3)$ shifts the graph 3 units in the **positive** x $-$direction

Solution



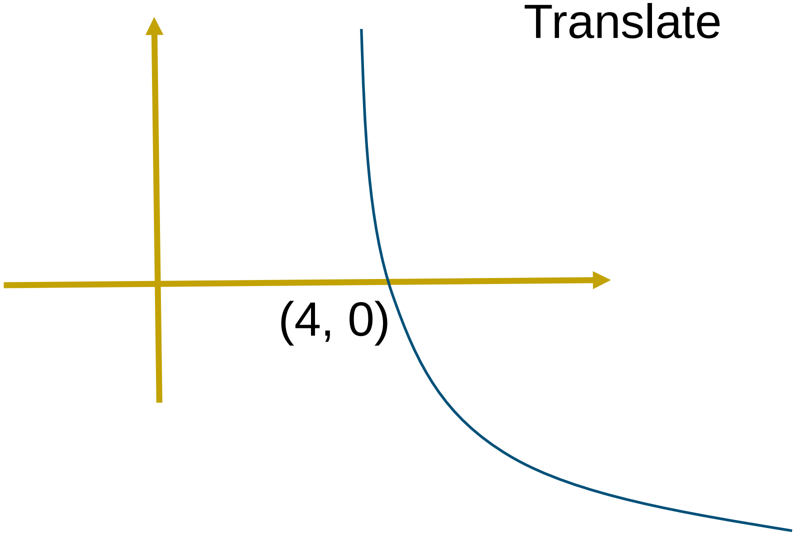


Reflect

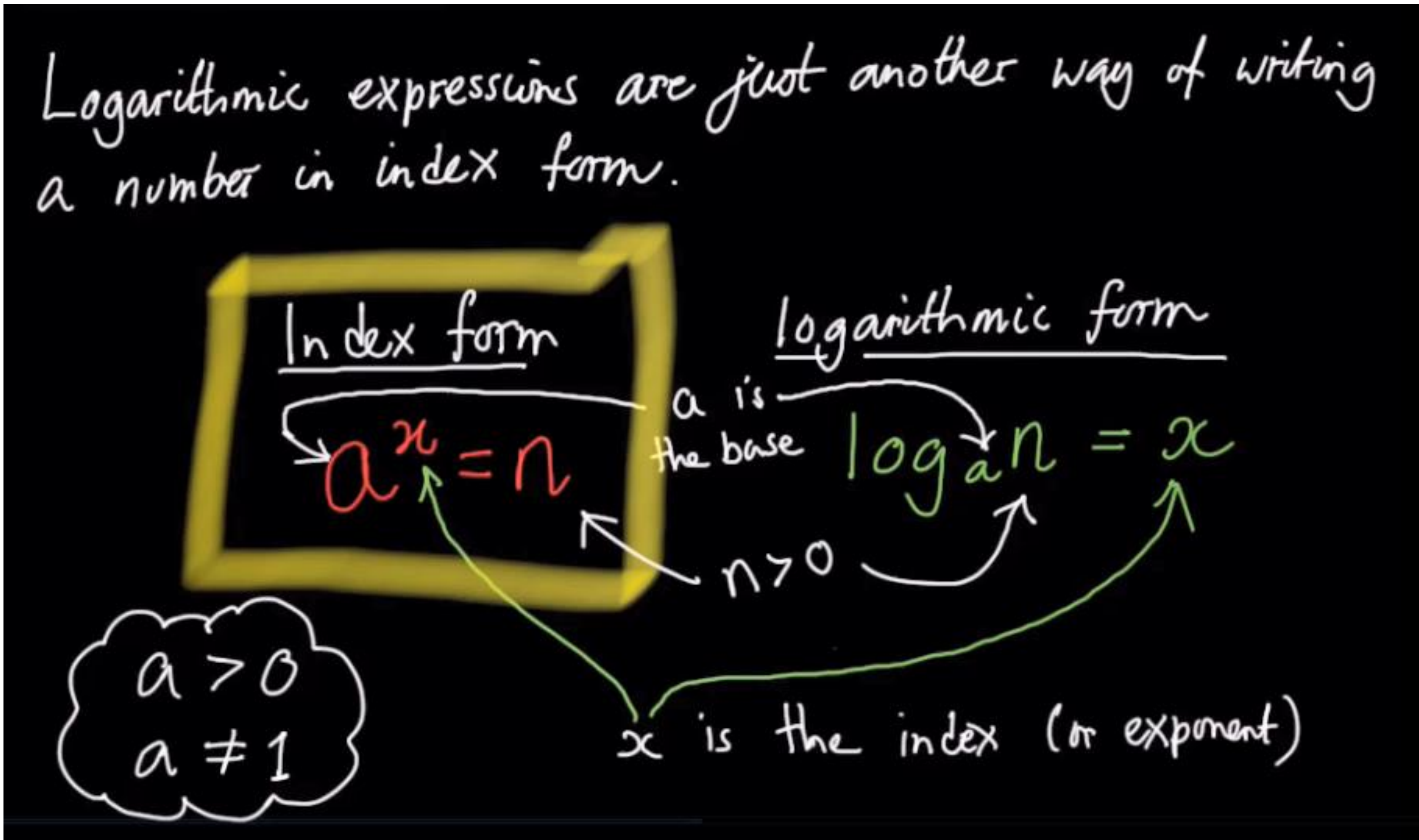


Stretch in y direction

Translate



3.1.3. Write an expression in index form or logarithmic form



Examples

$a^x = n$ $\log_a n = x$

Write as logarithms: (i) $2^4 = 16$

(ii) $4^{-2} = 1/16$

(iii) $11^0 = 1$

Write in index form: (i) $\log_3 81 = 4$

(ii) $\log_{16} 4 = 1/2$

(iii) $\log_5\left(\frac{1}{5}\right) = -1$

Answers

$$a^x = n \qquad \log_a n = x$$

Write as logarithms:

(i) $2^4 = 16$ $\quad \log_2 16 = 4$

(ii) $4^{-2} = 1/16$ $\quad \log_4 \left(\frac{1}{16}\right) = -2$

(iii) $11^0 = 1$ $\quad \log_{11} 1 = 0$

Write in index form

(i) $\log_3 81 = 4$ $\quad 3^4 = 81$

(ii) $\log_{16} 4 = 1/2$ $\quad 16^{1/2} = 4$

(iii) $\log_5 \left(\frac{1}{5}\right) = -1$ $\quad 5^{-1} = 1/5$

3.1.4. Apply the laws of logarithms

If $N = a^x$ then $\log_a N = x$

$$N = a^{\log_a N}$$

$$\log(pq) = \log p + \log q$$

$$\log\left(\frac{p}{q}\right) = \log p - \log q$$

$$\log p^n = n \log p$$

Examples

1) Write $\log_2 80$ in the form $k + \log_2 5$

$$\log_2 80 = \log_2 16 + \log_2 5$$
$$= \underline{4 + \log_2 5}$$

2) Write $\log\left(\frac{xy^3}{z^4}\right)$ as a sum of logarithms

$$= \log(xy^3) - \log(z^4)$$
$$= \log x + \log y^3 - \log z^4$$
$$= \log x + 3\log y - 4\log z$$

Examples

3) Write $2.5^{\log_{2.5} 4}$ as a number

$$2.5^{\log_{2.5} 4} = 4$$

4) Write $\log_2\left(\frac{8}{\sqrt[3]{7}}\right)$ in the form $k + a\log_2 7$

$$= \log_2 8 - \log_2 \sqrt[3]{7}$$

$$\sqrt[3]{7} = 7^{1/3}$$

$$= 3 - \frac{1}{3}\log_2 7$$

Your turn !

Write the following expression as a single logarithm,

$$2\log_{10} 20 - (\log_{10} 5 + \log_{10} 8)$$

Solution

$$2\log_{10} 20 - (\log_{10} 5 + \log_{10} 8)$$

$$= \log_{10} 400 - (\log_{10} 40)$$

$$= \log_{10}\left(\frac{400}{40}\right)$$

$$= \log_{10} 10 = 1$$

3.1.5. Solve exponential equations

Example 1

Solve $2^{x-1} = 5$

$$\log\left(2^{x-1}\right) = \log 5$$

$$(x-1)\log 2 = \log 5$$

$$x\log 2 - \log 2 = \log 5$$

$$x\log 2 = \log 5 + \log 2 = \log 10$$

$$x = \frac{1}{\log 2} = \underline{\underline{3.32}}$$

Another example

Example 2

Solve $4^{x+1} = \frac{1}{3^{x-2}}$

$$4^{x+1} = 3^{-(x-2)}$$

$$\log\left(4^{x+1}\right) = \log\left(3^{(2-x)}\right)$$

$$(x+1)\log 4 = (2-x)\log 3$$

$$x\log 4 + \log 4 = 2\log 3 - x\log 3$$

$$x\log 3 + x\log 4 = 2\log 3 - \log 4$$

$$x = \frac{2\log 3 - \log 4}{\log 3 + \log 4}$$

$$= \frac{\log(9/4)}{\log 12}$$

$$= \underline{\underline{0.33}}$$

Quadratic form – use a substitution

Example 3

Solve $3^{4x} + 6(3^{2x}) - 6 = 0$

$y = 3^{2x}$

$\therefore \quad y^2 + 6y - 6 = 0$

$y = \frac{-6 \pm \sqrt{60}}{2}, \quad y = \frac{-6 + \sqrt{60}}{2}$

$3^{2x} = 0.8729833462$

$2x \log 3 = \log(\quad) \quad \therefore x = \frac{\log(0.8729833462)}{2\log 3}$

$= -0.06$

Your turn!

Solve the following exponential equations

$$2^{2x} - 6(2^x) + 5 = 0$$

$$5^{2x} - 6(5^x) - 7 = 0$$

Solutions

1. $2^{2x} - 6(2^x) + 5 = 0$ subst $y = 2^x$

$y^2 - 6y + 5 = 0$

$(y-5)(y-1) = 0 \Rightarrow 2^x = 5$ or $2^x = 1$

Solve using, e.g $x \ln 2 = \ln 5$

$x \ln 2 = \ln 1$

$\therefore x = \frac{\ln 5}{\ln 2} = \underline{\underline{2.32}}$ $x = \frac{\ln 1}{\ln 2} = \underline{\underline{0}}$

2. Same procedure as (1)

Changing the base of a logarithm

In general

$$\boxed{\log_a x = \frac{\log_b x}{\log_b a}}$$

$$\boxed{\log_a b = \frac{1}{\log_b a}}$$

$\log_b b = 1$

special case can be written as a reciprocal

Example – solving an equation involving a base change

Example 1 Solve $\log_2 x = 8 + 9\boxed{\log_x 2}$

$$\log_2 x = 8 + 9\left(\frac{1}{\log_2 x}\right) \qquad y = \log_2 x$$

$$y = 8 + \frac{9}{y}$$

$$y^2 = 8y + 9 \Rightarrow y^2 - 8y - 9 = 0$$

$$(y-9)(y+1) = 0$$

(1) $\log_2 x = 9 \therefore 2^9 = \underline{\underline{512}}$

(2) $\log_2 x = -1 \therefore 2^{-1} = \underline{\underline{1/2}}$

Solving simultaneous equations involving logs

Example: Solve the simultaneous equations

$$8^y = 4^{2x+3} \qquad (1)$$

$$\log_2 y = \log_2 x + 4 \qquad (2)$$

$$\log_2(8^y) = \log_2(4^{2x+3})$$

$$y \log_2 8 = (2x+3)\log_2 4$$

$$3y = 4x + 6$$

$$y = \frac{4x+6}{3}$$

$$y = \frac{4x+6}{3}$$

$$8^y = 4^{2x+3} \quad (1)$$

$$\log_2 y = \log_2 x + 4 \quad (2)$$

$$\log_2\left(\frac{4x+6}{3}\right) = \log_2 x + 4$$

$$\log_2\left(\frac{4x+6}{3}\right) = \log_2 x + \log_2 16$$

$$\therefore \log_2\left(\frac{4x+6}{3}\right) = \log_2 16x$$

$$\therefore \frac{4x+6}{3} = 16x \Rightarrow x = \frac{3}{22}$$

$$y = 24/11$$

3.1 Summary

You should now be able to do the following

3.1.1 Sketch graphs of exponential functions (including e^x)

3.1.2 Sketch the graph of a natural logarithmic function

3.1.3 Write an expression in index form or logarithmic form

3.1.4 Apply the laws of logarithms

3.1.5 Solve exponential equations

Figure references

Some of the figures in the slide numbers listed in the table

17	19	21

have been reproduced from the following text book series:

Attwood, G., Macpherson, A., Moran, B., Petran, J., Pledger, K., Staley, G. and Wilkins, D. (2008), Edexcel AS and A Level Modular Mathematics series C1-C4, Pearson, Harlow, UK.

You may wish to refer to these text books for further information, examples and practice questions.

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