5.1 Solutions to consolidation questions

Key words: angle, arc length, degree, radian, circumference, diameter, centre, circle, perimeter, area of a sector, area of a (circular) segment, Pythagoras theorem, adjacent leg, opposite leg, hypotenuse, quadrant, clockwise/anticlockwise, sine, cosine, tangent, signs of trigonometric functions, graphs of trigonometry functions, periodic, vertical and horizontal stretch, vertical and horizontal translation, reflection.

Trigonometric ratios: $\sin\theta = \dfrac{\text{opposite leg}}{\text{hypotenuse}}$ $\cos\theta = \dfrac{\text{adjacent leg}}{\text{hypotenuse}}$ $\tan\theta = \dfrac{\text{opposite leg}}{\text{adjacent leg}}$

Area formulae:

	arc length	area of a sector	*area of a (circular) segment*
	$l = r\theta$	$A = \frac{1}{2}r^2\theta = \frac{1}{2}lr$	$\text{area}(\Delta ABC) = \frac{1}{2}ac\sin B$

You need to remember these formulae. It will not be given to you in your assessments.

1. Convert the following angles in radians to degrees:

a. $\dfrac{\pi}{2} \rightarrow 90^0$

b. $\dfrac{7\pi}{9} \rightarrow 140^0$

Since, $\pi \leftrightarrow 180^0$, $\dfrac{7\pi}{9} \leftrightarrow x$ $\Leftrightarrow x = \dfrac{180^0 \times \frac{7\pi}{9}}{\pi} \Leftrightarrow x = 140^0$

c. $\dfrac{7\pi}{6} \rightarrow 210^0$

Since, $\pi \leftrightarrow 180^0$, $\dfrac{7\pi}{6} \leftrightarrow x$ $\Leftrightarrow x = \dfrac{180^0 \times \frac{7\pi}{6}}{\pi} \Leftrightarrow x = 210^0$

2. Use calculator to convert the following angles to degrees, giving your answer to the nearest $0{,}1^0$:

a. $2{,}5^c \rightarrow \approx 143{,}2$

Since, $1^c \leftrightarrow \dfrac{180^0}{\pi}$, $2{,}5^c \leftrightarrow x$ $\Leftrightarrow x = \dfrac{\frac{180^0}{\pi} \times 2{,}5^c}{1^c} \Leftrightarrow x = 143{,}2394\ldots \approx 143{,}2^0$

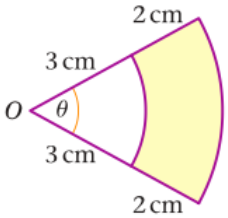
b. $3{,}14^c$

Since, $1^c \leftrightarrow \dfrac{180^0}{\pi}$, $3{,}14^c \leftrightarrow x$ $\Leftrightarrow x = \dfrac{\frac{180^0}{\pi} \times 3{,}14^c}{1^c} \Leftrightarrow x = 179{,}9087\ldots \approx 179{,}9^0$

c. $3{,}49^c$

Since, $\begin{array}{ccc} 1^c & \leftrightarrow & \dfrac{180^0}{\pi} \\ 3{,}49^c & \leftrightarrow & x \end{array} \Leftrightarrow x = \dfrac{\dfrac{180^0}{\pi} \times 3{,}49^c}{1^c} \Leftrightarrow x = 199{,}9622... \approx 200^0$

3. Referring to the diagram, find



a. The perimeter of the shaded region when $\theta = 0{,}8$ radians.
We have that length, l, of the arc of a circle with radius r and angle q is given by, $l = r\theta$, from which we can conclude that

i) the length, l_1, of the arc of the circle with radius 3 is

$$l_1 = 3 \times 0{,}8 = 2{,}4 \text{ cm}$$

ii) the length, l_2, of the arc with radius 5 is

$$l_2 = 5 \times 0{,}8 = 4 \text{ cm}$$

Thus, the perimeter, P, of the shaded region is equal to

$$P = 2{,}4 + 4 + 2 + 2 = 10{,}4 \text{ cm}$$

b. The value of q when the perimeter of the shaded region is 14 cm.
Using the same formula, $l = r\theta$, we have that

$$P = 14 \Leftrightarrow 3\theta + 5\theta + 2 + 2 = 14 \Leftrightarrow 8\theta = 10 \Leftrightarrow \theta = \frac{5}{4} \text{ radians}$$

4. In the diagram, BC is the arc of a circle, centre O and radius 8 cm. The points A and D are such that $OA = OD = 5$ cm. Given that $\angle BOC = 1{,}6$ radians, calculate the area of the shaded region.

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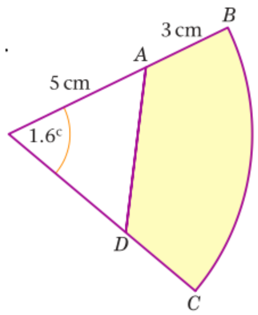


The area of the sector of the circle with centre O and arc BC, is given by the formula $A_C = \frac{1}{2} r^2 \theta$,

while the area of the triangle defined by points O, A and D is obtained using $A_T = \frac{1}{2} r^2 \sin\theta$.

Thus, the shaded area is:

$$A = A_C - A_T = \frac{1}{2} r^2 \theta - \frac{1}{2} r^2 \sin\theta = \frac{1}{2} \times 8^2 \times 1{,}6^c - \frac{1}{2} \times 5^2 \times \sin 1{,}6^c \approx 38{,}7 \text{ cm}^2$$

5. In the diagram, AD and BC are arcs of circles with centre O, such that $OA = OD = r$ cm, $AB = DC = 8$ cm and $\angle BOC = \theta$ radians.

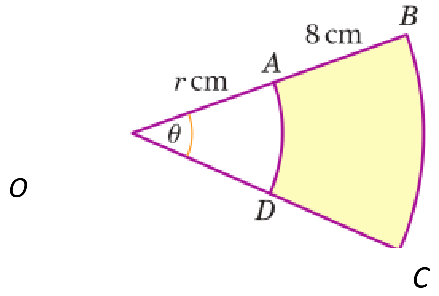
a. Given that the area of the shaded region is $48\,\text{cm}^2$, show that $r = \frac{6}{\theta} - 4$

Using the formula, $A = \frac{1}{2}r^2\theta$, we have that the area of the shaded region can be obtained in the following way:

$$48 = \frac{1}{2}(r+8)^2\theta - \frac{1}{2}r^2\theta \iff 48 = \frac{1}{2}(r^2+64+16r)\theta - \frac{1}{2}r^2\theta \iff 48 = \frac{1}{2}(64+16r)\theta$$

$$\iff \frac{96}{\theta} = 64+16r \iff r = \frac{96}{16\theta} - \frac{64}{16} = \frac{6}{\theta} - 4$$

b. Given also that $r = 10\theta$, calculate the perimeter of the shaded region.



We have, using a., that

$$10\theta = \frac{6}{\theta} - 4 \iff 10\theta^2 + 4\theta - 6 = 0 \iff 5\theta^2 + 2\theta - 3 = 0$$

$$\iff \theta = \frac{-1 \pm \sqrt{(-1)^2 - 5\times(-3)}}{5} \iff \theta = \frac{3}{5} \text{ or } \theta = -1$$

Since q can not be negative, we can conclude that $\theta = \frac{3}{5} = 0{,}6$ radians, and therefore $r = 10\theta = 10\times 0{,}6 = 6$ cm.

Thus, the perimeter, P, of the shaded region is equal to

$$P = 8+8+r\theta+(r+8)\theta = 16+6\times 0{,}6+(6+8)\times 0{,}6 = 28\,\text{cm}.$$

6. Write down the values of:

a. $\sin(-90)^0 = -1$

b. $\sin(450)^0 = \sin\left((360)^0 + (90)^0\right) = \sin(90)^0 = 1$

c. $\cos(-180)^0 = -1$

d. $\cos(-270)^0 = 0$

e. $\tan 360^0 = \tan 0^0 = 0$

f. $\tan(-180)^0 = 0$

7. Write down the values of the following, where the angles are in radians:

a. $\sin\left(-\frac{\pi}{2}\right)=-1$

b. $\cos\pi=-1$

c. $\tan\left(-2\pi\right)=0$

8. By drawing diagrams, express the following in terms of trigonometric ratios of acute angles:

a. $\cos\left(-\frac{3\pi}{4}\right)=-\cos\frac{\pi}{4}$

b. $\tan\frac{7\pi}{5}=\tan\left(\frac{2\pi}{5}+\pi\right)=\tan\frac{2\pi}{5}$

c. $\sin\left(-\frac{6\pi}{7}\right)=-\sin\frac{\pi}{7}$

d. $\tan\frac{15\pi}{8}=\tan\left(\frac{7\pi}{8}+\pi\right)=\tan\frac{7\pi}{8}=-\tan\frac{\pi}{8}$

9. Given that q is an acute angle measured in degrees, express in terms of $\sin\theta$:

a. $\sin\left(-180^0+\theta\right)=\sin\left(-180^0+\theta+360^0\right)=\sin\left(180^0+\theta\right)=-\sin\theta$

b. $\sin\left(-360^0+\theta\right)=\sin\theta$

c. $\sin\left(720^0-\theta\right)=\sin\left(-\theta\right)=-\sin\theta$

d. $\sin\left(720^0+\theta\right)=\sin\theta$

10. Given that q is an acute angle measured in degrees, express in terms of $\cos\theta$ or $\tan\theta$:

a. $\cos\left(-\left(180^0-\theta\right)\right)=\cos\left(-180^0+\theta\right)=\cos\left(-180^0+\theta+360^0\right)=\cos\left(180^0+\theta\right)=-\cos\theta$

b. $\cos\left(\theta-360^0\right)=\cos\theta$

c. $\cos\left(\theta-540^0\right)=\cos\left(\theta-540^0+360^0\right)=\cos\left(\theta-180^0\right)=\cos\left(180^0-\theta\right)=-\cos\theta$

d. $\tan\left(-\theta\right)=-\tan\theta$

e. $\tan\left(180^0-\theta\right)=\tan\left(-\theta\right)=-\tan\theta$

f. $\tan\left(180^0+\theta\right)=\tan\theta$

11. Express the following as trigonometric ratios of either 30^0, 45^0 or 60^0, and hence find their exact values.

a. $\cos300^0=\cos\left(300^0-360^0\right)=\cos\left(-60\right)^0=\cos60^0=\frac{1}{2}$

b. $\cos225^0=\cos\left(180^0+45^0\right)=-\cos45^0=-\frac{\sqrt{2}}{2}$

c. $\tan\left(-225^0\right)=\tan\left(-225^0+180^0\right)=\tan\left(-45^0\right)=-\tan45^0=-1$

d. $\tan 210^0 = \tan\left(30^0 + 180^0\right) = \tan 30^0 = \dfrac{\sqrt{3}}{3}$

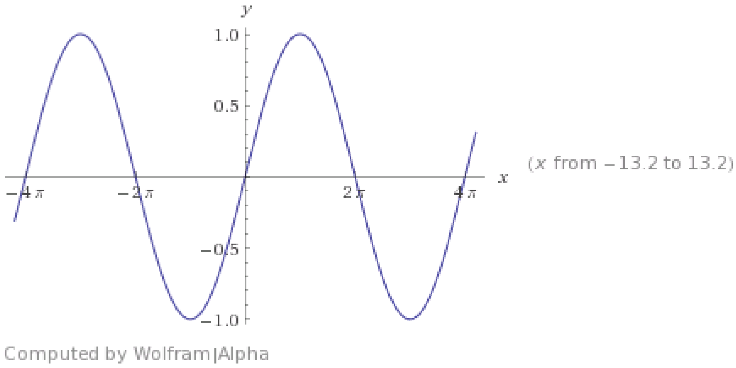
12. In this question q is measured in radians. Sketch, on separate axes, the graphs of the following in the interval $-2\pi \le \theta \le 2\pi$. In each case give the periodicity of the function.

a. $y = \sin\left(\dfrac{1}{2}\theta\right)$

This is the graph of the function

$$y = \sin\theta$$

stretched horizontally by a factor of 2.



To find the period, let us assume that

$$\sin\left(\frac{1}{2}\theta\right) = \sin\left(\frac{1}{2}(\theta + k)\right) = \sin\left(\frac{1}{2}\theta + \frac{k}{2}\right)$$

and the smaller positive value of the real number k when this happens is:

$$2\pi$$

Thus,

$$\frac{k}{2} = 2\pi \quad \Leftrightarrow \quad k = 4\pi$$

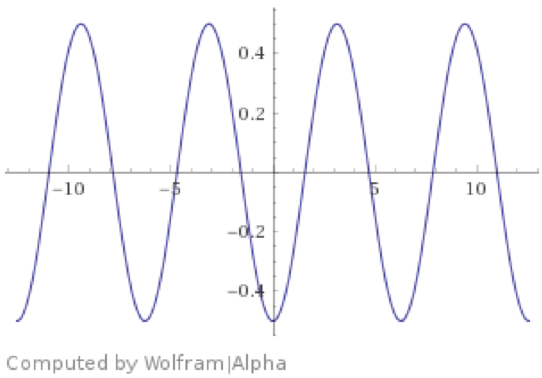
which is the period of the function.

b. $y = -\dfrac{1}{2}\cos\theta$

This is the graph of the function

$$y = \cos\theta$$

stretched vertically by a factor of 1/2 and reflected in the horizontal axes.

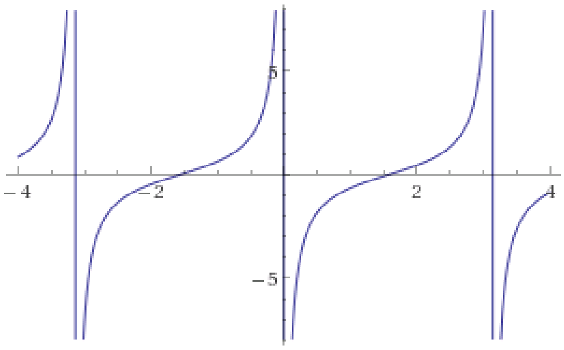


The period of this function is

2π

c. $y = \tan\left(\theta - \frac{\pi}{2}\right)$

This is the graph of the function $y = \tan\theta$ with a horizontal translation of $\frac{\pi}{2}$ to the right.

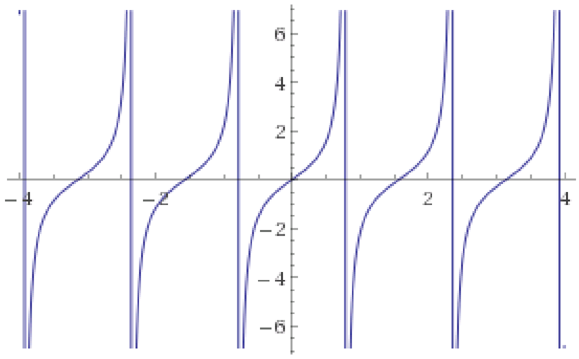


Computed by Wolfram|Alpha

Its period is p.

d. $y = \tan(2\theta)$

This is the graph of the function $y = \tan\theta$ stretched by scale factor $\frac{1}{2}$ horizontally.



Computed by Wolfram|Alpha

To find its period we proceed like in a., where k is a real number:

$$\tan(2\theta) = \tan(2(\theta + k)) = \tan(2\theta + 2k)$$

and the smaller positive value of k for which the previous equality is true is when $2k = \pi$, that is the period is equal to $\frac{\pi}{2}$.

13. Describe geometrically the transformations which map:

a. The graph of $y = \tan x$ onto the graph of $y = \tan\left(\frac{1}{2}x\right)$.

This is a horizontal stretch by a scale factor 2.

b. The graph of $y = \tan\left(\frac{1}{2}x\right)$ onto the graph of $y = \tan\left(\frac{1}{2}x\right) + 3$.

This is an upper vertical translation of three units.

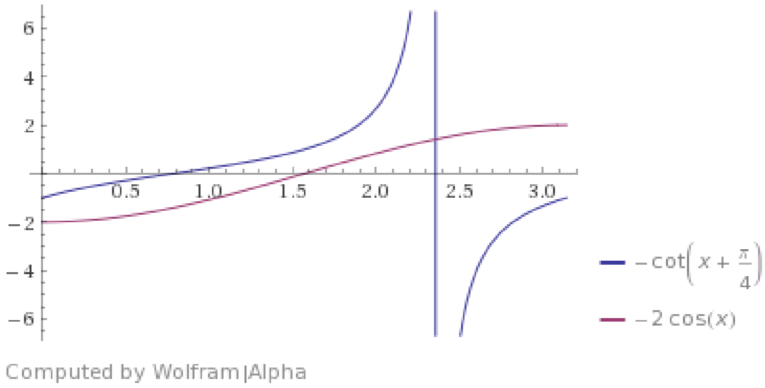
c. The graph of $y = \cos x$ onto the graph of $y = -\cos x$.

This is a reflection over the x-axes.

d. The graph of $y = \sin\left(x - \frac{\pi}{18}\right)$ onto the graph of $y = \sin\left(x + \frac{\pi}{18}\right)$.

This defines a horizontal translation of $\frac{2\pi}{18}$ units to the left.

14. a. Sketch on the same set of axes, in the interval $0 \le x \le \pi$, the graphs of $y = \tan\left(x - \frac{\pi}{4}\right)$ and $y = -2\cos x$, showing the coordinates of points of intersection with the axes.



Intersection of $y = -2\cos x$ with the x-axes:

$$-2\cos x = 0 \quad \Leftrightarrow \quad \cos x = 0 \underset{0 \le x \le \pi}{\Leftrightarrow} x = \frac{\pi}{2}$$

Intersection of $y = -2\cos x$ with the y-axes:

$$x = 0 \quad \Rightarrow \quad y = -2\cos 0 = -1$$

Intersection of $y = \tan\left(x - \frac{\pi}{4}\right)$ with the x-axes:

$$\tan\left(x - \frac{\pi}{4}\right) = 0 \underset{-\frac{\pi}{4} \le x - \frac{\pi}{4} \le \frac{3\pi}{4}}{\Leftrightarrow} x - \frac{\pi}{4} = 0 \quad \Leftrightarrow \quad x = \frac{\pi}{4}$$

Intersection of $y = \tan\left(x - \frac{\pi}{4}\right)$ with the y-axes:

$$x = 0 \quad \Rightarrow \quad y = \tan\left(0 - \frac{\pi}{4}\right) = -\tan\frac{\pi}{4} = -1$$

b. Deduce the number of solutions of the equation $\tan\left(x - \frac{\pi}{4}\right) + 2\cos x = 0$, in the interval $0 \le x \le \pi$

We have that

$$\tan\left(x - \frac{\pi}{4}\right) + 2\cos x = 0 \quad \Leftrightarrow \quad \tan\left(x - \frac{\pi}{4}\right) = -2\cos x$$

and therefore to find the solutions of this equation in the interval $0 \le x \le \pi$, is equivalent to find the intersection of the functions in the graph presented in a.

But, as we can immediately see they do not intersect in the given interval.

Thus, the number of solutions in the given interval is zero.

Additional question

1. Consider the model of the form:

$$y = A\sin\left(Bx + C\right) + D,$$

where A, B, C and D are real constants with $A > 0$.
This is a model that helps to illustrate the behaviour of event and phenomenon that are sinusoidal, that is, regularly fluctuating between a maximum and a minimum.
We call ***critical points*** as those where a function attains a maximum or minimum value.
For example, for many weather patterns we have a period of 12 months and the regularity mentioned, works.
For this model:

a. Find an expression for the period, P, of the function, in terms of the given constants.
We have a repetition of the values of y, when k is a real number for which

$$A\sin\left(Bx + C\right) + D = A\sin\left(B\left(x + k\right) + C\right) + D \quad \Leftrightarrow \quad \sin\left(Bx + C\right) = \sin\left(Bx + Bk + C\right)$$

To find the smaller value of k, for which this equality holds for every x, we need that

$$Bk = 2\pi \quad \Leftrightarrow \quad k = \frac{2\pi}{B}$$

(2π is the period of the sine function).
Therefore,

$$P = \frac{2\pi}{B}$$

b. Apply a. to find the value of *B* in weather models.
In weather models, we know that the period is 12. Thus,

$$12 = \frac{2\pi}{B} \quad \Leftrightarrow \quad B = \frac{\pi}{6}$$

c. Assuming that M is the maximum value and m is the minimum value of $y = A\sin\left(Bx + C\right) + D$, find expressions for A, B, C and D.

The maximum, M, of $y = A\sin\left(Bx + C\right) + D$ occurs when

$$\sin\left(Bx + C\right) = 1$$

Likewise, the minimum, m, occurs when

$$\sin\left(Bx + C\right) = -1.$$

$$\begin{cases} M = A\cdot 1 + D \\ m = A\cdot(-1) + D \end{cases} \Leftrightarrow \begin{cases} A + D = M \\ -A + D = m \end{cases} \underset{E_1+E_2}{\Leftrightarrow} \begin{cases} A + D = M \\ 2D = M + m \end{cases}$$

$$\Leftrightarrow \begin{cases} A + D = M \\ D = \dfrac{M+m}{2} \end{cases} \Leftrightarrow \begin{cases} A + \dfrac{M+m}{2} = M \\ D = \dfrac{M+m}{2} \end{cases}$$

$$\Leftrightarrow \begin{cases} A = \dfrac{M-m}{2} \\ D = \dfrac{M+m}{2} \end{cases}$$

We know, from a., that

$B = \dfrac{2\pi}{P}$, where P is the period of function y.

To find an expression for constant C, let us assume that, for example, m occurs at $x = x_1$. This means that

$$m = A\sin(Bx_1 + C) + D \Leftrightarrow \sin(Bx_1 + C) = \frac{m-D}{A}$$

Since, the minimum value of sine function is -1, we have, where P is the period, that

$$\sin(Bx_1 + C) = -1 \Rightarrow Bx_1 + C = \frac{3\pi}{2} \underset{B=\frac{2\pi}{P}}{\Rightarrow} C = \frac{3\pi}{2} - \frac{2\pi}{P}x_1$$

We can, therefore, conclude that the values A, B, C and D such that

$$y = A\sin(Bx + C) + D$$

adjust data, when M is the maximum (attained in x_2) and m is the minimum (attained in x_1) are given by:

$$A = \frac{M-m}{2}$$

$$B = \frac{2\pi}{P}$$

$$C = \frac{3\pi}{2} - \frac{2\pi}{P}x_1$$

$$D = \frac{M+m}{2}$$

where P is the period.

Note that an alternative expression for C, can be:

$$C = \frac{\pi}{2} - \frac{2\pi}{P}x_2$$

References:

Some of the questions on this worksheet were reproduced from the following sources;

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