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5.2 Trigonometric equations

Core Preparatory Topics
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FEPS Mathematics Support Framework

5.2 Introduction

The aim of this unit is to assist you in consolidating and developing your knowledge and skills in working with some trigonometric identities and solving trigonometric equations.

While studying these slides you should attempt the ‘Your Turn’ questions in the slides.

After studying the slides, you should attempt the Consolidation Questions.

5.2 Learning checklist

Learning resource	Notes	Tick when complete
Slides		
Your turn questions		
Consolidation questions		

5.2 Learning objectives

After completing this unit you should be able to

5.2.1 Apply the identity $\cos^2\theta + \sin^2\theta \equiv 1$

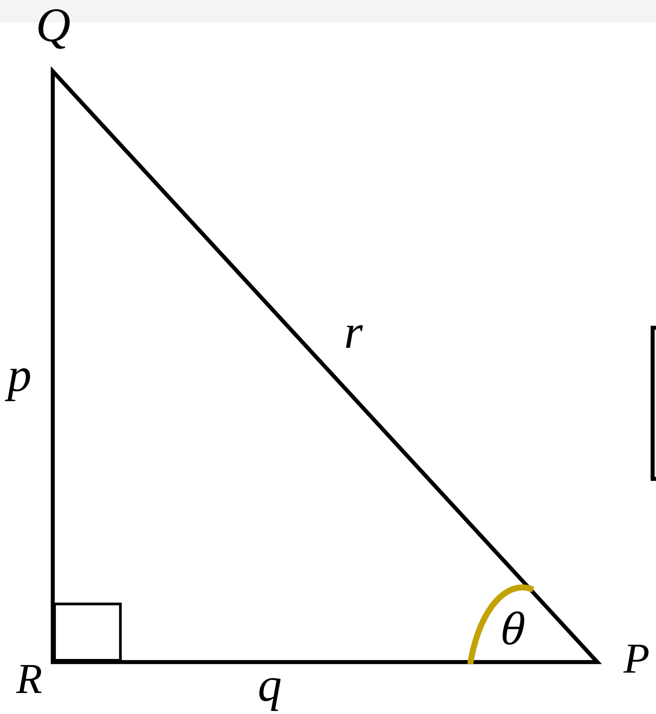
5.2.2 Apply the identity $\tan\theta = \sin\theta / \cos\theta$

5.2.3 Solve simple equations of the form $\sin\theta = k$

5.2.4 Solve equations of the form $\sin(n\theta \pm \alpha) = k$

5.2.5 Solve quadratic equations involving trigonometric functions

5.2.1. Apply the identities $\cos^2\theta + \sin^2\theta \equiv 1$



$$\sin\theta = \frac{p}{r} \qquad \cos\theta = \frac{q}{r} \qquad \tan\theta = \frac{p}{q}$$

$$\sin^2\theta + \cos^2\theta = \frac{y^2}{r^2} + \frac{x^2}{r^2} = 1$$

using $x^2 + y^2 = r^2$

$$\boxed{\sin^2\theta + \cos^2\theta = 1}$$

Example 1

Simplify: $\sin^2 3\theta + \cos^2 3\theta$

Solution:

Using the identity $\sin^2 \theta + \cos^2 \theta = 1$ with θ replaced by 3θ,

$$\sin^2 3\theta + \cos^2 3\theta = 1$$

In general, $\sin^2 n\theta + \cos^2 n\theta = 1$

Your turn! (1)

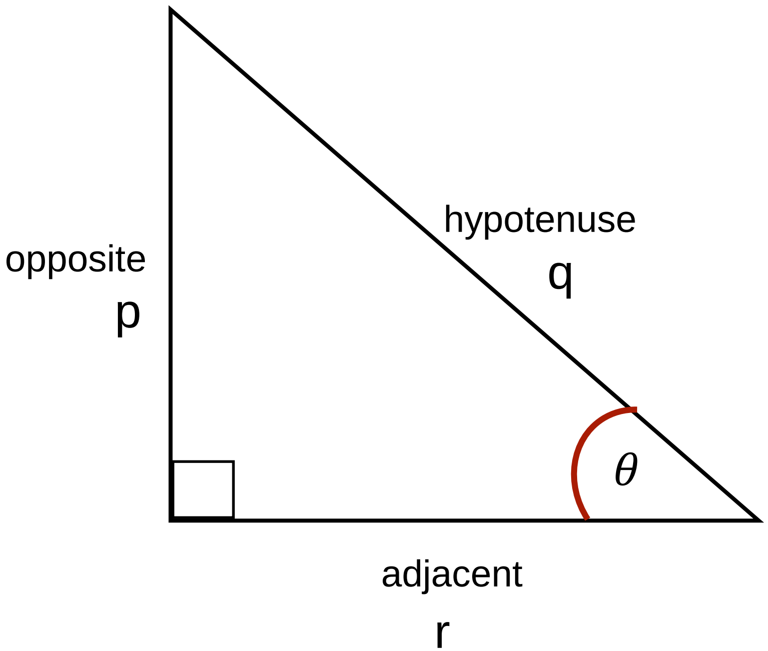
Simplify : $5 - 5\sin^2\theta$

Solution

$$5 - 5\sin^2\theta \;=\; 5\,(1 - \sin^2\theta)$$

$$= \; 5\cos^2\,\theta$$

5.2.2 Apply the identity $\tan\theta = \sin\theta / \cos\theta$



$$\frac{\sin\theta}{\cos\theta} = \frac{p}{q} \cdot \frac{r}{x} = \frac{p}{r} = \tan\theta$$

$$\boxed{\tan\theta = \frac{\sin\theta}{\cos\theta}}$$

Example 2

Simplify : $\dfrac{\sin 2\theta}{\sqrt{1-\sin^2 2\theta}}$ $\quad(\cos 2\theta > 0)$

$$\frac{\sin 2\theta}{\sqrt{1-\sin^2 2\theta}} = \frac{\sin 2\theta}{\sqrt{\cos^2 2\theta}}$$

$$= \frac{\sin 2\theta}{\cos 2\theta}$$

$$= \tan 2\theta$$

Note: $\sqrt{\cos^2 2\theta} = \pm\cos 2\theta$ depending on the sign of $\cos 2\theta$

Your turn! (2)

Show that:

$$\frac{\cos^4\theta - \sin^4\theta}{\cos^2\theta} \equiv 1 - \tan^2\theta$$

Solution

Starting with LHS

$$\frac{\cos^4\theta - \sin^4\theta}{\cos^2\theta} = \frac{(\cos^2\theta - \sin^2\theta)(\cos^2\theta + \sin^2\theta)}{\cos^2\theta}$$

$$= \frac{\cos^2\theta - \sin^2\theta}{\cos^2\theta}$$

$$= \frac{\cos^2\theta}{\cos^2\theta} - \frac{\sin^2\theta}{\cos^2\theta}$$

$$= 1 - \tan^2\theta$$

LHS = RHS shown as required

Example 3

Given that $\cos\theta = -\frac{3}{5}$ and that θ is reflex, find the value of $\sin\theta$ and $\tan\theta$.

Method 1 (using identities)

We have:

$$\sin^2\theta + \cos^2\theta = 1$$

$$\sin^2\theta + \left(-\frac{3}{5}\right)^2 = 1$$

$$\sin^2\theta = 1 - \frac{9}{25}$$

$$= \frac{16}{25}$$

$$\sin\theta = \sqrt{\frac{16}{25}} = \pm\frac{4}{5}$$

$$\sin\theta = -\frac{4}{5}$$

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{-{}^4/_5}{-{}^3/_5} = \frac{4}{3}$$

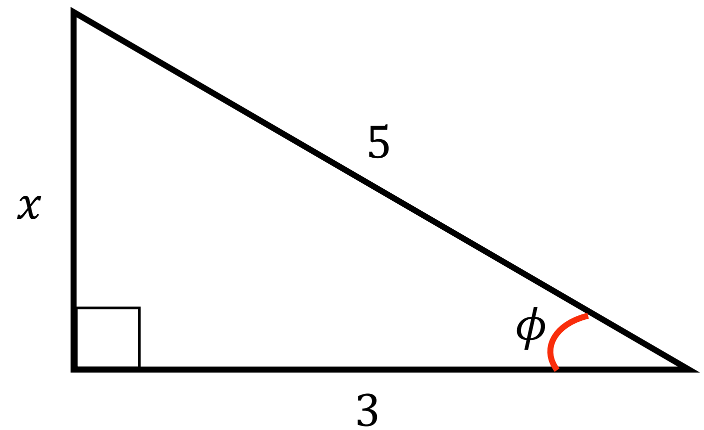
(as θ is reflex it lies in the 3rd or 4th quadrant. For $\cos$ and $\sin$ to be both negative it has to be in the 3rd quadrant)

Method 2 (using a triangle)

Using a right angled triangle with the acute angle ϕ, where $\cos\phi = \frac{3}{5}$

Using Pythagoras' theorem

$$x = 4$$



So: $\sin\phi = \frac{4}{5}$ and as $\sin\theta = -\sin\theta$

$$\sin\theta = -\frac{4}{5}$$

Also we can see: $\tan\phi = \frac{4}{3}$

As: $\tan\theta = +\tan\phi$

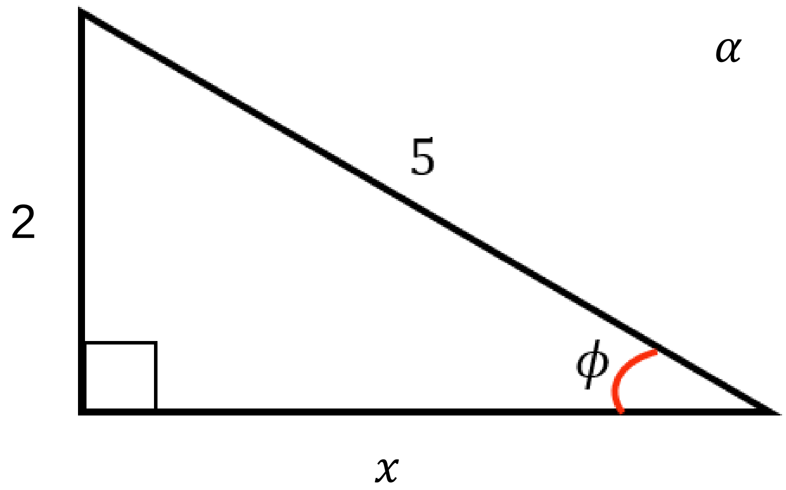
$$\tan\theta = +\frac{4}{3}$$

Your turn! (3)

Given that $\sin\alpha = \frac{2}{5}$ and that α is obtuse, find the exact value of $\cos\alpha$.

Solution

Given that $\sin\alpha = \frac{2}{5}$ and that α is obtuse, find the exact value of $\cos\alpha$.



Using Pythagoras' theorem:

$$x = \sqrt{5^2 - 2^2} = \sqrt{21}$$

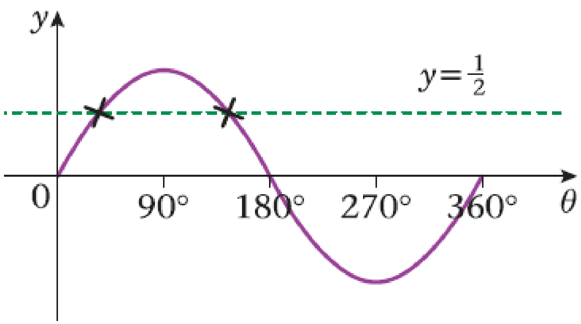
$$\cos\phi = \frac{\sqrt{21}}{5}$$

as α is obtuse, α lies in the 2nd quadrant, so it is negative.

$$\cos\phi = -\frac{\sqrt{21}}{5}$$

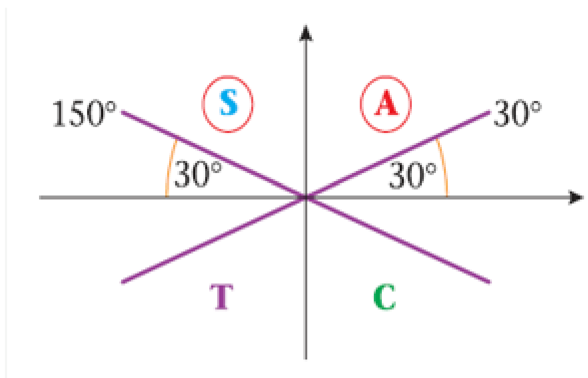
5.2.3 Solve equations of the form $\sin\theta = k$

You need to be able to solve simple trigonometric equations of the form $\sin\theta = k, \cos\theta = k,$ (where $-1 \leq k \leq 1$ and $\tan\theta = p \ (p \in \mathbb{R})$



$$\sin\theta = \frac{1}{2}$$

Using your calculator: $\sin^{-1}\left(\frac{1}{2}\right) = 30^0$



As the angle has to be positive we can only use quadrants 1and 2.

So $\theta = \ 30^0$ and 150^0

Some rules to follow

- For the equation $\sin x = k$ your first solution, using your calculator is $x = \sin^{-1} k$.
  A second solution is $(180^0 - x)$ or $(\pi - x)$ if you are working in radians.
  Other solutions can be found by adding or subtracting multiples of 360^0 or 2π radians

- For the equation $\cos x = k$ your first solution, using your calculator is $x = \cos^{-1} k$.
  A second solution is $(360^0 - x)$ or $(2\pi - x)$ if you are working in radians.
  Other solutions can be found by adding or subtracting multiples of 360^0 or 2π radians

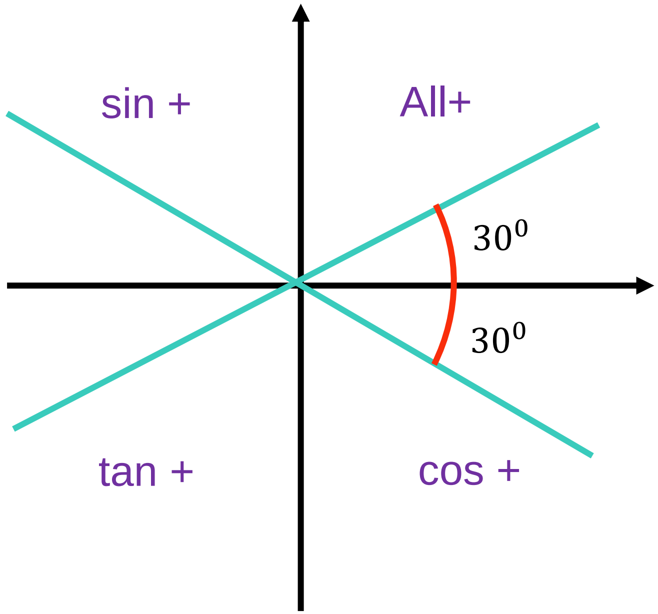
- For the equation $\tan x = k$ your first solution, using your calculator is $x = \tan^{-1} k$.
  A second solution is $(180^0 + x)$ or $(\pi + x)$ if you are working in radians.
  Other solutions can be found by adding or subtracting multiples of 360^0 or 2π radians

You should know how to find an angle without using a calculator, if the angle is a variant of 0^o,30^o, 45^o,60^o, or 90^o.

Example

Solve $\cos x = \frac{\sqrt{3}}{2}$ in the interval $0^o < x \leq 360^o$.

Solution:
Find one solution in the first quadrant and use it to find the other solution in the fourth quadrant.



$x = 30^0$

$x = 330^0$

Your turn! (4)

Solve , in the interval $0 < \theta \leq 2\pi$, the equation
$\sin\theta = \sqrt{3}\cos\theta$.

Solution

Solve , in the interval $0 < \theta \leq 2\pi$, the equation $\sin\theta = \sqrt{3}\cos\theta$.

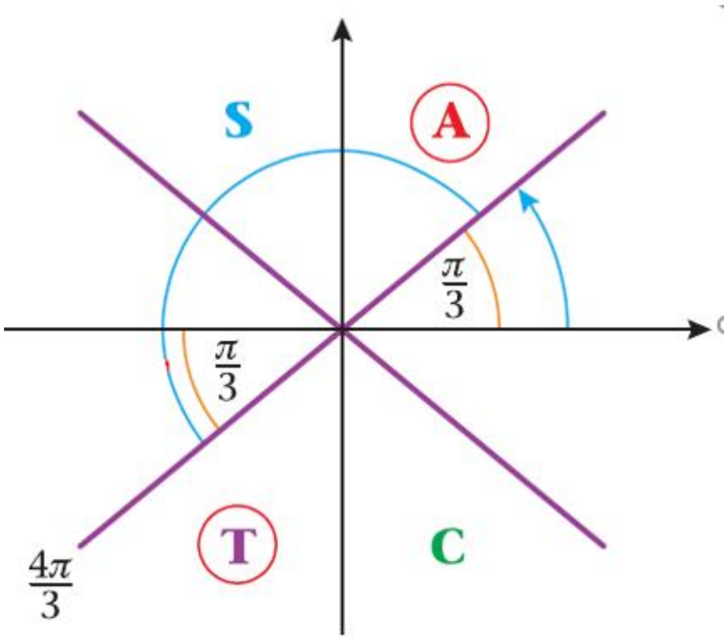
$$\sin\theta = \sqrt{3}\cos\theta$$

So $$\frac{\sin\theta}{\cos\theta} = \sqrt{3}$$

$$\tan\theta = \sqrt{3}$$

So one solution is $\theta = \frac{\pi}{3}$

$$\theta = \frac{\pi}{3} = \frac{4\pi}{3}$$



Your turn! (5)

Solve, in the intervals indicated, the following equations for θ, where θ is measured in radians.

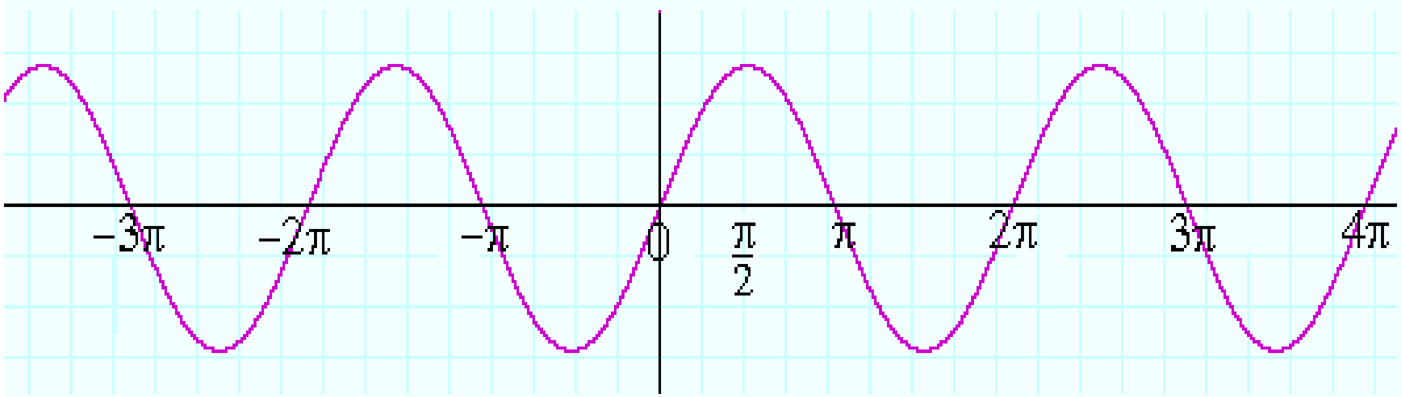
Give your answer in terms of π or 2 decimal places.

a) $\sin\theta = 0, \quad -2\pi < \theta \leq 2\pi$

b) $\cos\theta = -\frac{1}{2}, -2\pi < \theta \leq \pi$

Solution 1

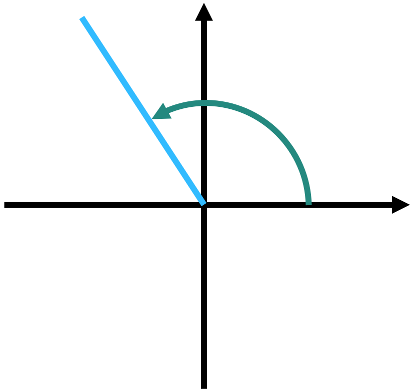
Here is the graph of $y = \sin x$:



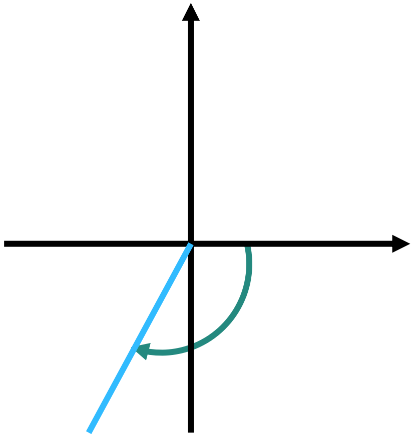
For $\sin\theta = 0, \quad -2\pi < \theta \leq 2\pi \qquad \theta = -\pi,\ 0,\ \pi,\ 2\pi$

Solution 2

a) $\cos\theta = -½ \qquad -2\pi < \theta \leq \pi$



$\theta = 120^0 = \frac{2\pi}{3}$ in quadrant 2



$\theta = -\frac{2\pi}{3}$ in quadrant 3

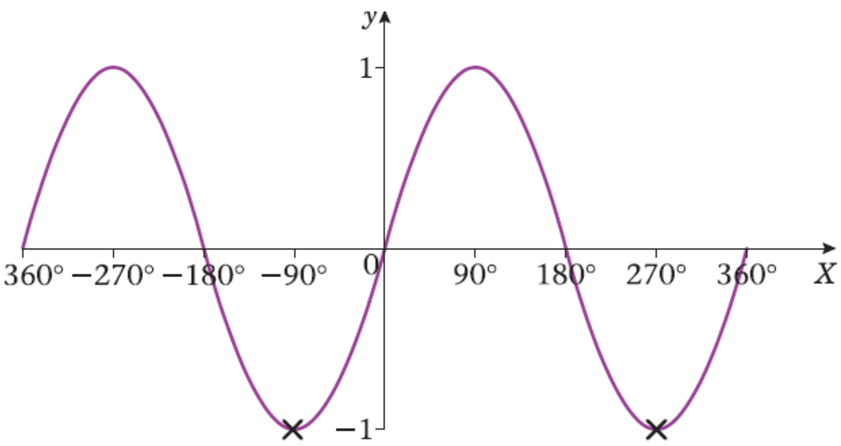
5.2.4 Solve equations of the form $\sin(n\,\theta \pm \alpha) = k$

You need to be able to solve equations of the form $\sin(n\theta + \alpha) = k$
$\cos(n\theta + \alpha) = k$ and $\tan(n\theta + \alpha) = k$

Example

Solve the equation $\sin(2\theta - 35)^0 = -1$

Looking at the graph $\sin^{-1}(-1) = -90^0,\ 270^0$



$$2\theta - 35 = -90^0$$
$$\theta = -27.5^0$$

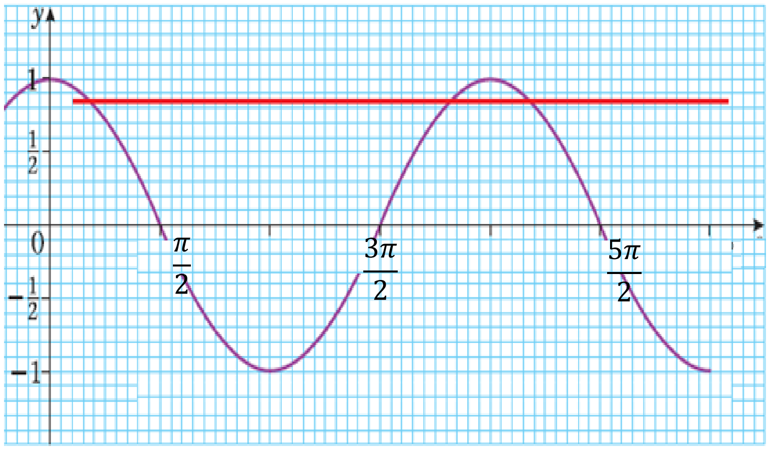
$$2\theta - 35 = 270^0$$
$$\theta = 152.5^0$$

Your turn! (6)

Solve the equation $cos\left(3\theta + \frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$, in the interval $0 \leq \theta \leq \pi$.

Solution

Solve the equation $\cos\left(3\theta + \frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$, in the interval $0 \leq \theta \leq \pi$.



$$\cos^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{6}, \frac{11\pi}{6} \text{ and } \frac{13\pi}{6}$$

So $\quad 3\theta + \frac{\pi}{6} = \frac{\pi}{6} \qquad \theta = 0$

$$3\theta + \frac{\pi}{6} = \frac{11\pi}{6} \qquad \theta = \frac{5\pi}{9}$$

$$3\theta + \frac{\pi}{6} = \frac{13\pi}{6} \qquad \theta = \frac{2\pi}{3}.$$

5.2.5 Solve quadratic equations involving trigonometric functions

You need to be able to solve quadratic equations in $\sin\theta, \cos\theta$ or $\tan\theta$

An equation like $\sin^2\theta + 2\sin\theta - 3 = 0$ can be solved in the same way as $x^2 + 2x - 3 = 0$ with $\sin\theta$ replacing x.

In general, follow these steps.

Step 1 : use trigonometric identities so that only one of $\sin\theta$, $\cos\theta$, or $\tan\theta$ appears in the given equation.

Step 2 : Factorise the equation.

Step 3 : Solve two simple equations.

Example

Solve for θ, in the interval $0^o \le \theta \le 360^o$, the equation $2\cos^2\theta - \cos\theta - 1 = 0$.

Solution:

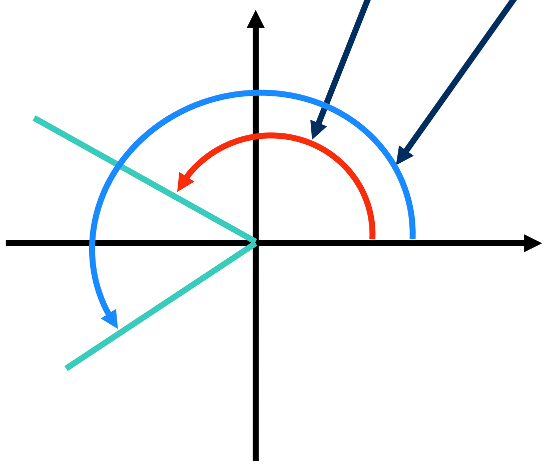
$$2\cos^2\theta - \cos\theta - 1 = 0$$

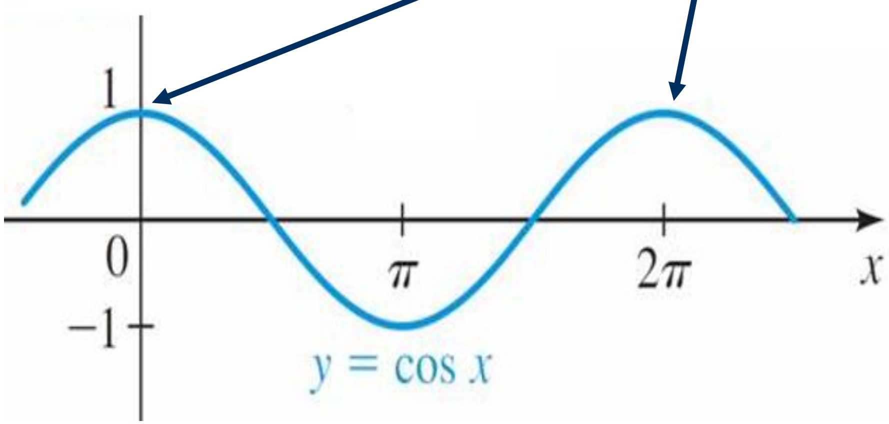
$$(2\cos\theta + 1)(\cos\theta - 1) = 0$$

So $\cos\theta = -\frac{1}{2}$ or $\cos\theta = 1$

$\cos\theta = -\frac{1}{2} \quad \theta = 120^0$ or 240^0

$\cos\theta = 1 \quad \theta = 0^0$ or 360^0





Your turn! (7)

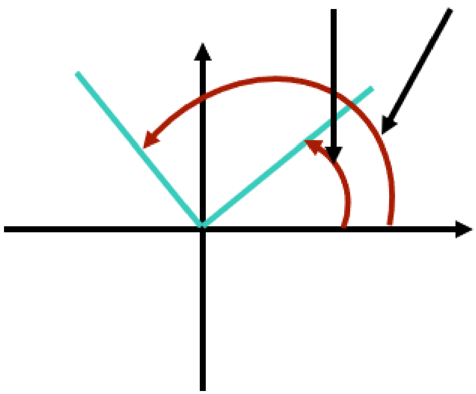
Solve for θ, in the interval $0^{\mathrm{o}} \leq \theta \leq 360^{\mathrm{o}}$, the equation $\sin^2(\theta - 30^o) = \frac{1}{2}$.

Solution

Solve for θ, in the interval $0^{\mathrm{o}} \leq \theta \leq 360^{\mathrm{o}}$, the equation $\sin^2(\theta - 30^o) = \frac{1}{2}$.

$$\sin(\theta - 30^o) = \pm\sqrt{\frac{1}{2}}$$

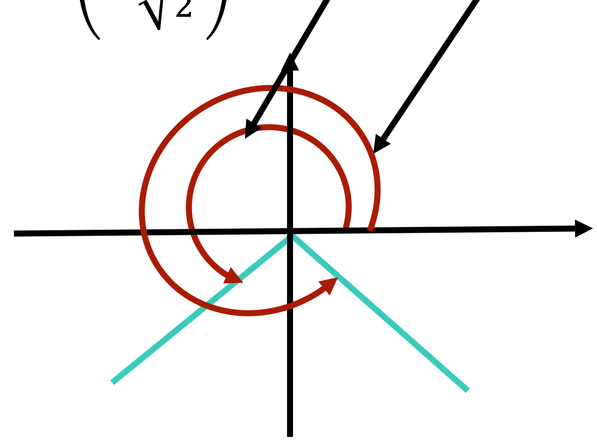
$$\sin^{-1}\left(\sqrt{\frac{1}{2}}\right) = 45^0, 135^0$$



$$\theta - 30^0 = 45^0, 135^0$$

$$\theta = 75^0, 165^0$$

$$\sin^{-1}\left(-\sqrt{\frac{1}{2}}\right) = 225^0, 315^0$$



$$\theta - 30^0 = 225^0, 315^0$$

$$\theta = 255^0, 345^0$$

Example

Solve for x, in the interval $-\pi \leq x \leq \pi$, $2\cos^2 x + 9\sin x = 3\sin^2 x$.

$$2\cos^2 x + 9\sin x = 3\sin^2 x$$

$$2(1 - \sin^2 x) + 9\sin x = 3\sin^2 x$$

$$2 - 2\sin^2 x + 9\sin x = 3\sin^2 x$$

$$5\sin^2 x - 9\sin x - 2 = 0$$

$$(5\sin x + 1)(\sin x - 2) = 0$$

$\sin x = -\dfrac{1}{5}$

$x = -2.940 \text{ or } -0.201$

(answers to 3 d.p.)

$\sin x = 2$

reject as $-1 \leq \sin x \leq 1$

Your turn! (8)

Solve for x, in the interval $-\pi \leq x \leq \pi$,

$$2\cos^2 x - \sin x - 1 = 0.$$

Solution

Solve for x, in the interval $-\pi \leq x \leq \pi$, the equation
$2\cos^2 x - \sin x - 1 = 0$.

Step 1

$2(1 - \sin^2 x) - \sin x - 1 = 0$ using $\sin^2 x + \cos^2 x = 1$

$2\sin^2 x + \sin x - 1 = 0$

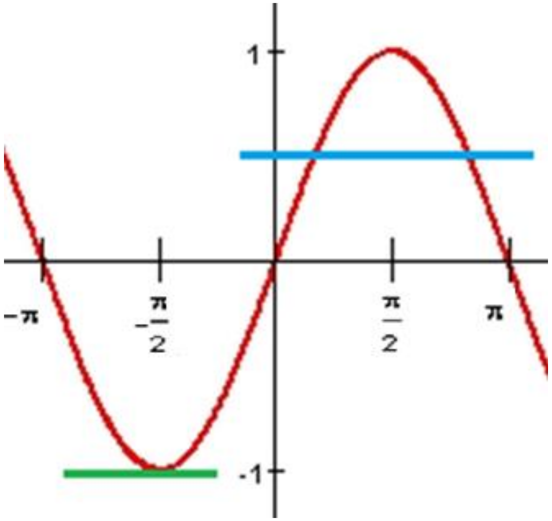
Step 2

$(\sin x + 1)(2\sin x - 1) = 0$

Step 3

Solve two equations : $\sin x = -1$ and $\sin x = \frac{1}{2}$.

Solution (continued)



$\sin x = -1$ if $x = -\frac{\pi}{2}$

$\sin x = \frac{1}{2}$ if $x = \frac{\pi}{6}$ or $\frac{5\pi}{6}$

The solutions are $x = -\frac{\pi}{2}$ or $x = \frac{\pi}{6}$ or $x = \frac{5\pi}{6}$.

Additional questions

1. Solve $\tan\left(x+\frac{\pi}{6}\right)=-1$ on the interval $-\pi<x\leq\pi$.

2. Solve $\tan^2 x+\tan x-2=0$

   on the interval $0\leq x<2\pi$.

   (Use $\tan 1.11=2$ and $\pi=3.14$)

Answers

1. Solve $\tan\left(x+\frac{\pi}{6}\right)=-1$ on the interval $-\pi < x \leq \pi$.

   Answer : $x=\frac{7\pi}{12}$ or $x=-\frac{5\pi}{12}$

2. Solve $\tan^2 x+\tan x-2=0$

   on the interval $0 \leq x < 2\pi$.

   Use $\tan 1.11 = 2$ and $\pi = 3.14$.

   Answer : $x \approx 2.03$ or $x 5.17$ or $x=\frac{\pi}{4}$ or $x=\frac{5\pi}{4}$

5.2 Summary

You should now be able to

5.2.1 Apply the identity $\cos^2\theta + \sin^2\theta \equiv 1$

5.2.2 Apply the identity $\tan\theta = \sin\theta / \cos\theta$

5.2.3 Solve simple equations of the form $\sin\theta = k$

5.2.4 Solve equations of the form $\sin(n\theta \pm \alpha) = k$

5.2.5 Solve quadratic equations involving trigonometric functions

Figure references

Some of the figures in the slides for this unit, as listed in the table below

18	20	22	24	27	29	31	37

have been reproduced from the following text book series:

Attwood, G., Macpherson, A., Moran, B., Petran, J., Pledger, K., Staley, G. and Wilkins, D. (2008), Edexcel AS and A Level Modular Mathematics series C1-C4, Pearson, Harlow, UK.

You may wish to refer to these text books for further information, examples and practice questions.

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