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5.3 Further trigonometric identities, equations and graphs

Core Preparatory Topics
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FEPS Mathematics Support Framework

5.3 Introduction

The aim of this unit is to assist you in consolidating and developing your knowledge and skills in working with some trigonometric identities, solving trigonometric equations and sketching/interpreting graphs of trigonometric functions .

While studying these slides you should attempt the ‘Your Turn’ questions in the slides.

After studying the slides, you should attempt the Consolidation Questions.

5.3 Learning checklist

Learning resource	Notes	Tick when complete
Slides		
Your turn questions		
Consolidation questions		

5.3 Learning objectives

After completing this unit you should be able to

5.3.1 Recall and use functions $\sec x$, $\operatorname{cosec} x$, $\cot x$

5.3.2 Recall the graphs of $\sec x$, $\operatorname{cosec} x$, $\cot x$

5.3.3 Solve problems involving $\sec x$, $\operatorname{cosec} x$, $\cot x$

5.3.4 Use the Pythagorean identities.

5.3.5 Use the inverse functions $\arcsin x$, $\arccos x$ and $\arctan x$ and their graphs.

5.3.1 Functions secθ, cosecθ, cotθ

Cosecant	Secant	Cotangent
$\mathrm{cosec}(x) = \dfrac{1}{\sin(x)}$	$\sec(x) = \dfrac{1}{\cos(x)}$	$\cot(x) = \dfrac{1}{\tan(x)}$

Provided $\sin(x) \neq 0$, $\cos(x) \neq 0$ and $\tan(x) \neq 0$

Third letter rule

$$\mathrm{cosec}(x) = \frac{1}{\sin(x)} \qquad \sec(x) = \frac{1}{\cos(x)} \qquad \cot(x) = \frac{1}{\tan(x)}$$

Examples

$$\sec(2\pi/3) = \frac{1}{\cos(2\pi/3)} = -\frac{1}{\cos(\pi/3)} = -2$$

$$\text{cosec}(\pi/6) = \frac{1}{\sin(\pi/6)} = 2$$

$$\cot(\pi/6) = \frac{1}{\tan(\pi/6)} = \frac{1}{1/\sqrt{3}} = \sqrt{3}$$

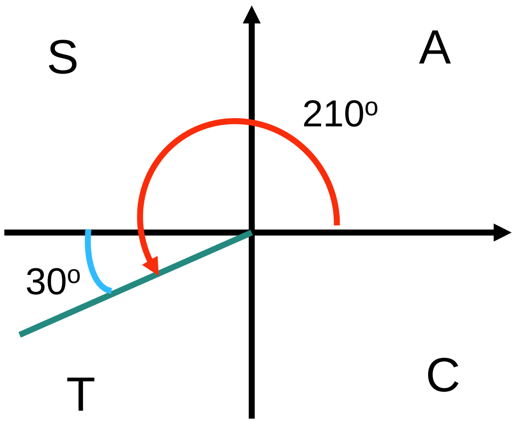
Another example

Work out the exact value of sec 210º

$$\sec 210^0 = \frac{1}{\cos 210^0}$$

so we need to find $\cos 210^0$

The angle 210^0 lies in the third quadrant where cos is negative



This means:

$$\cos 210^0 = -\cos 30^0 = -\frac{\sqrt{3}}{2}$$

$$\sec 210^0 = \frac{1}{-\sqrt{3}/_2} = -\frac{2}{\sqrt{3}} = -\frac{2\sqrt{3}}{3}$$

Your turn! (1)

Given that $\sin(A) = 4/5$, where A is obtuse, and $\cos(B) = \sqrt{3}/2$ where B is acute, find the exact values of:

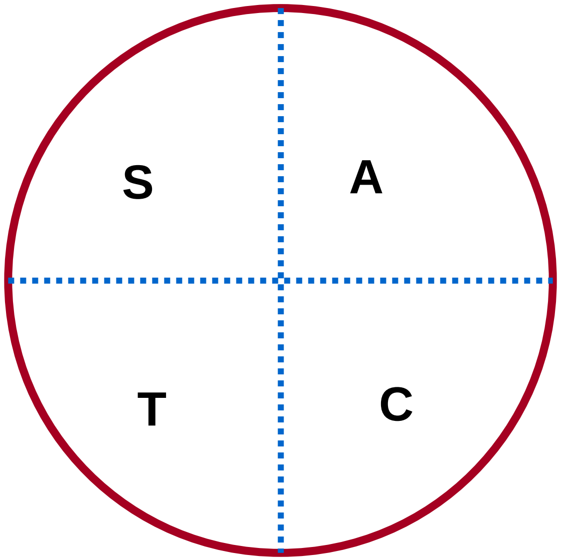
$$\text{(i) } \sec(A), \quad \text{(ii) } \operatorname{cosec}(B), \quad \text{(iii) } \cot(A)$$

Solutions

$$\text{(i) } \sec(A) = \frac{1}{\cos(A)} = -\frac{1}{3/5} = -\frac{5}{3}$$

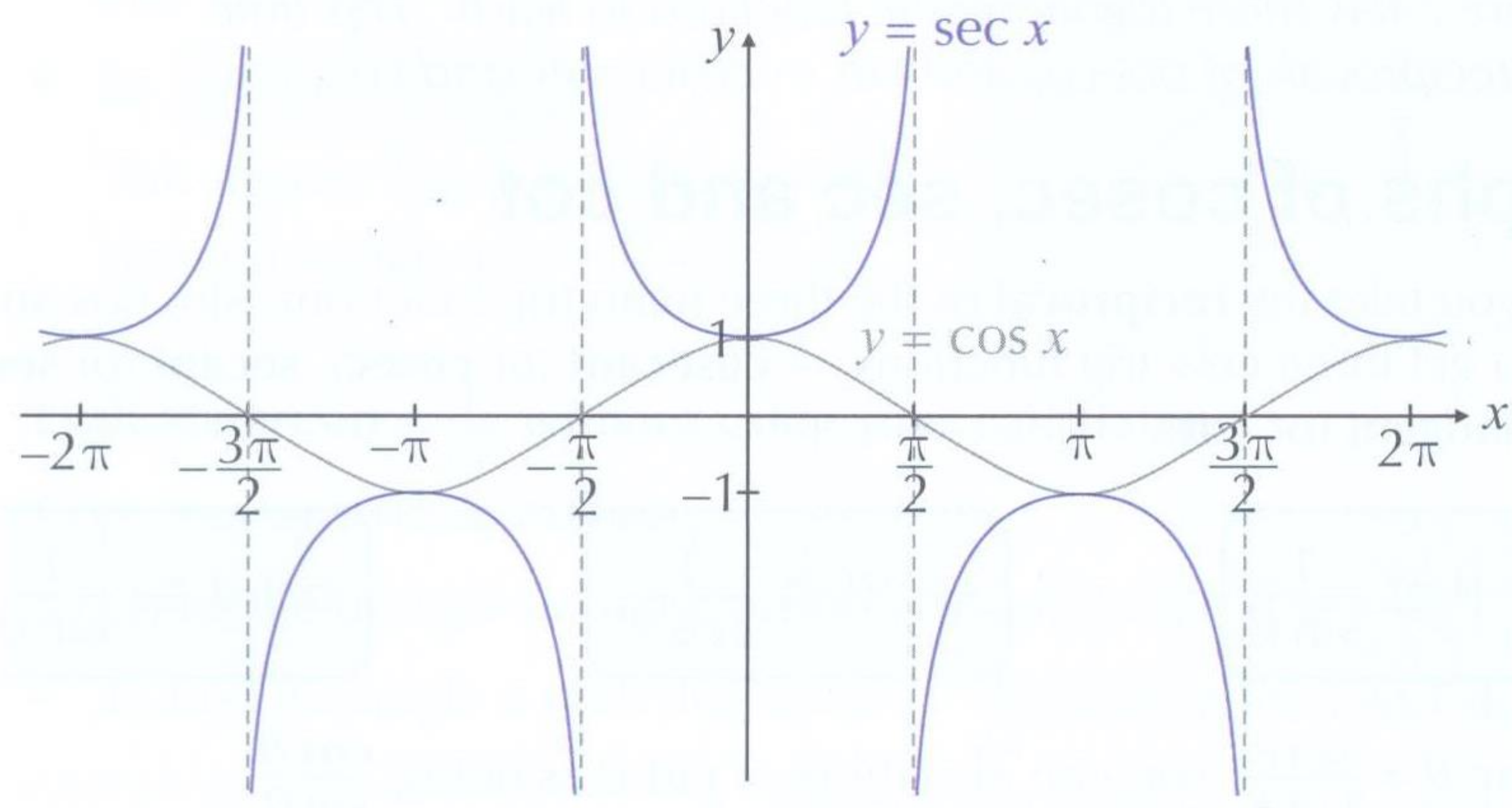
$$\text{(ii) } \operatorname{cosec}(B) = \frac{1}{\sin(B)} = \frac{1}{1/2} = 2$$

$$\text{(iii) } \cot(A) = \frac{1}{\tan(A)} = -\frac{1}{4/3} = -\frac{3}{4}$$



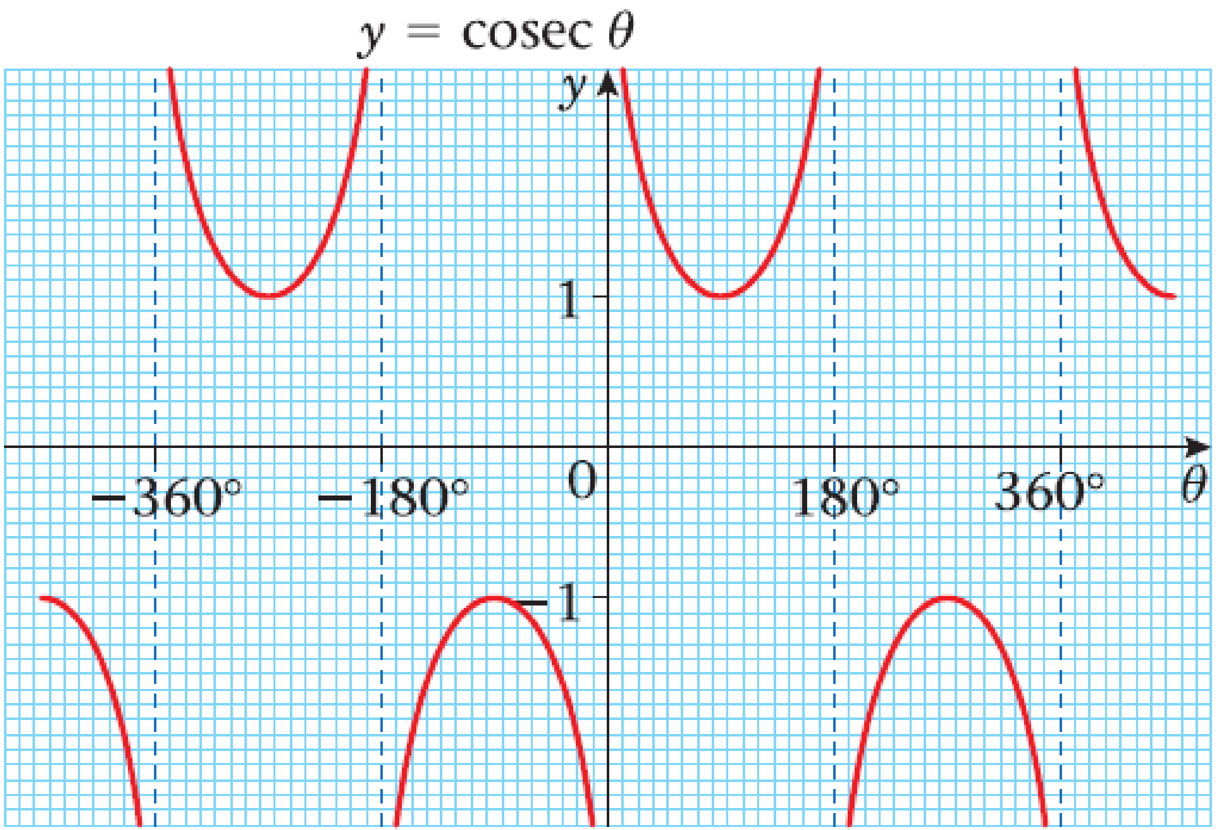
5.3.2 Graphs of secθ, cosecθ, cotθ

The graph of $y = \sec\theta, \theta \in \mathbb{R},$ has symmetry in the y-axis and repeats itself every 360^0
It has vertical asymptotes at all the values of θ for which $\cos\theta = 0,$ i.e. at $\theta = (90 + 180n)^0$



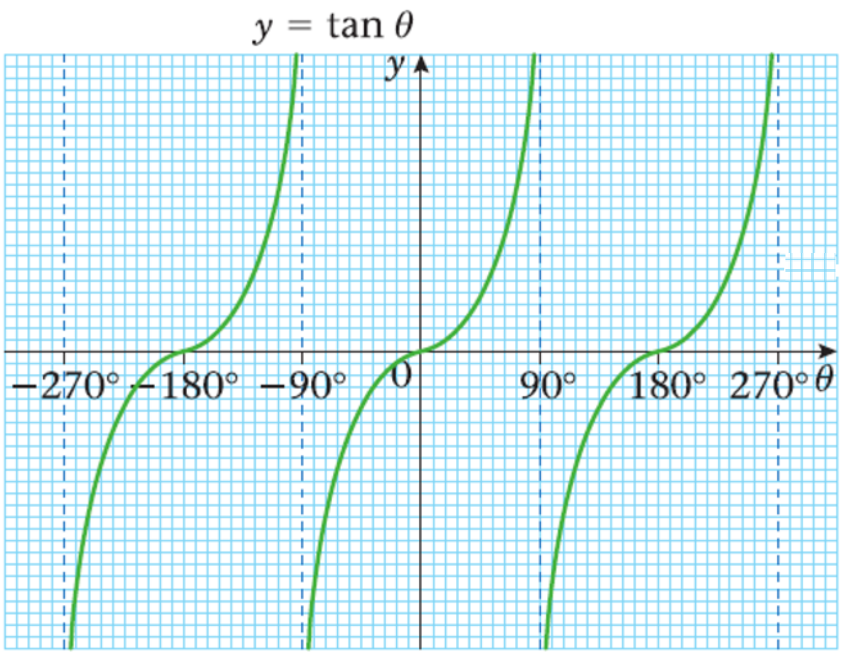
The graph of cosecθ

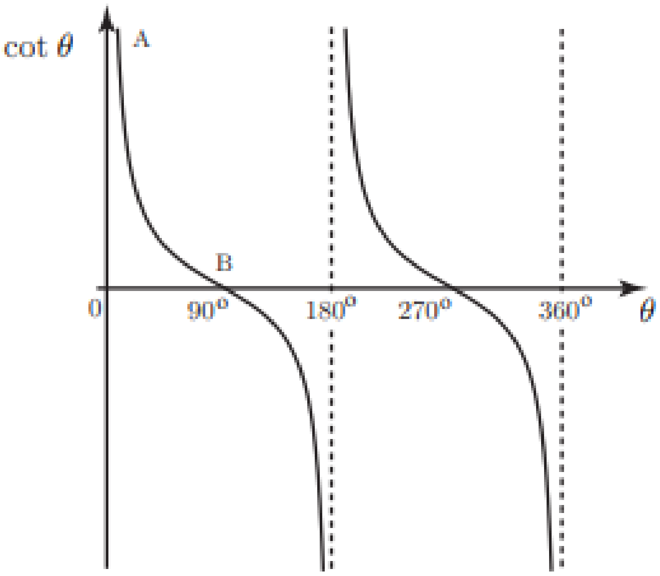
The graph of $y = cosec\,\theta, \theta \in \mathbb{R}$, has vertical asymptotes at all the values of θ for which $\sin\theta = 0$, i.e. at $\theta = 180n^0, n \in \mathbb{Z}$ and repeats itself every 360^0



The graph of $\cot\theta$

The graph of $y = cot\,\theta, \theta \in \mathbb{R}$, has vertical asymptotes at all the values of θ for which $\sin\theta = 0$, i.e. at $\theta = 180n^0, n \in \mathbb{Z}$ and repeats itself every 180^0





5.3.3 Solve problems involving sec(x), cosec(x), and cot(x)

Example: Prove the following identity

$$\sin(A)\cos(A)\left[\operatorname{cosec}(A)+\sec(A)\right]=\cos(A)+\sin(A)$$

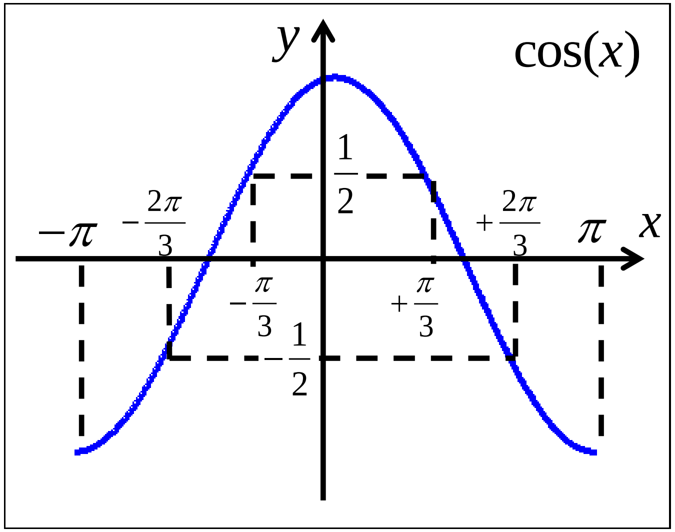
$$\sin(A)\cos(A)\left[\operatorname{cosec}(A)+\sec(A)\right]=\sin(A)\cos(A)\left[\frac{1}{\sin(A)}+\frac{1}{\cos(A)}\right]$$

$$=\frac{\sin(A)\cos(A)}{\sin(A)}+\frac{\sin(A)\cos(A)}{\cos(A)}=\cos(A)+\sin(A)$$

Example

Solve $\sec^2 x = 4, \ -\pi < x < \pi$

$$\Rightarrow \cos(x) = \pm\frac{1}{2} \Rightarrow x = \pm\frac{\pi}{3} \text{ or } x = \pm\frac{2\pi}{3}$$



Example

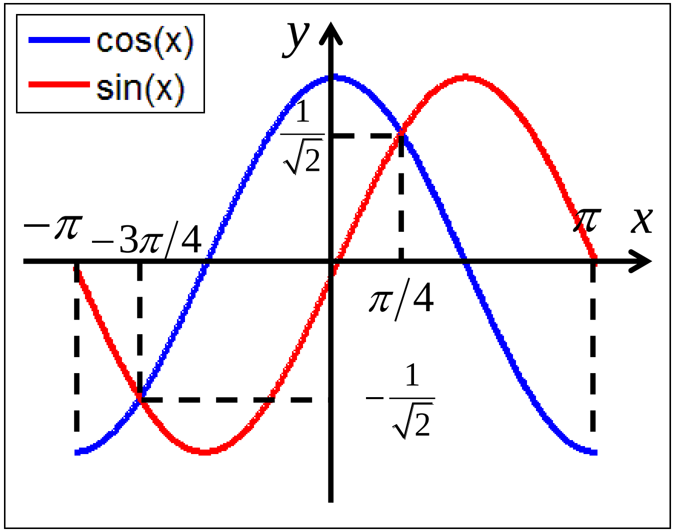
Solve $\cot(2x) = 1$ for $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$

Change the domain

$$-\frac{\pi}{2} \le x \le \frac{\pi}{2} \Rightarrow -\pi \le 2x \le \pi$$

$$\cot(2x) = 1 \Rightarrow \cos(2x) = \sin(2x)$$

$$\Rightarrow 2x = \frac{\pi}{4} \text{ or } 2x = -\frac{3\pi}{4} \Rightarrow x = \frac{\pi}{8} \text{ or } x = -\frac{3\pi}{8}$$



Another example

Solve the equation $4\text{cosec}^2\theta - 9 = \cot\theta$, in the interval $0 \le \theta \le 360^0$

Substituting

$1 + \cot^2\theta = \text{cosec}^2\theta$ gives:

$4(1 + \cot^2\theta) - 9 = \cot\theta$

$4 + 4\cot^2\theta - 9 = \cot\theta$

$4\cot^2\theta - \cot\theta - 9 = 0$

$(4\cot\theta - 5)(\cot\theta + 1) = 0$

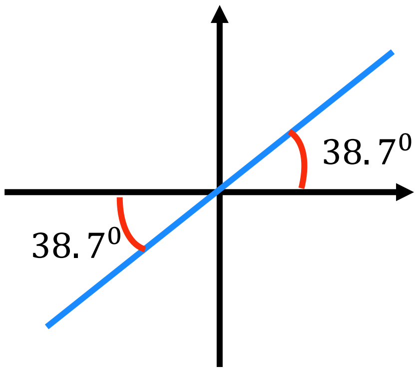
$\cot\theta = \frac{5}{4}$ or $\cot\theta = -1$

$\tan\theta = \frac{4}{5}$ or $\tan\theta = -1$

$\tan\theta = \frac{4}{5}$

$\theta = 38.7^0, 219^0$ (3 s.f.)

as tan is positive in quadrants 1 & 3

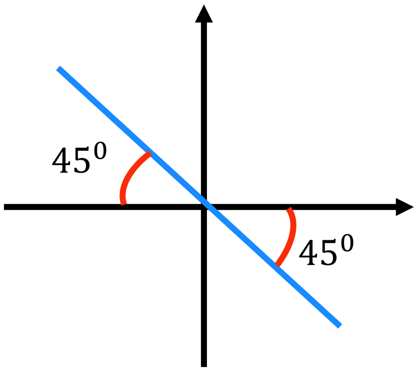


$\tan\theta = -1$

$\tan^{-1}(-1) = -45^0$

$\theta = 135^0, 315^0$

as tan is negative in quadrants 2 & 4



Your turn! (2)

Solve $2\cos x = \cot x$ for $0 \leq x \leq 2\pi$

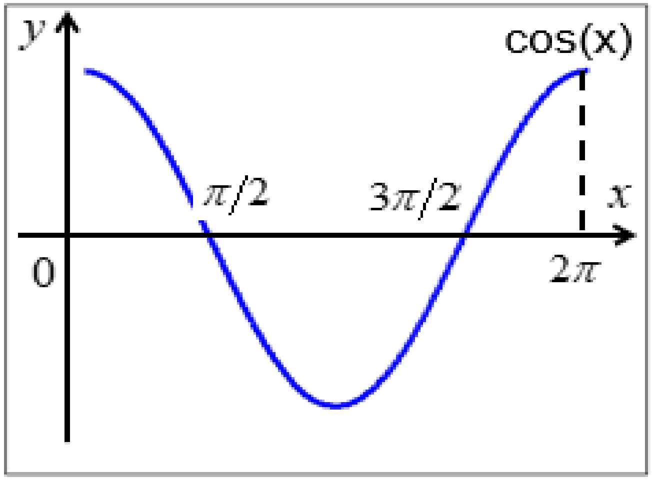
Solution

$2\cos(x) = \cot(x)$ for $0 \le x \le 2\pi$

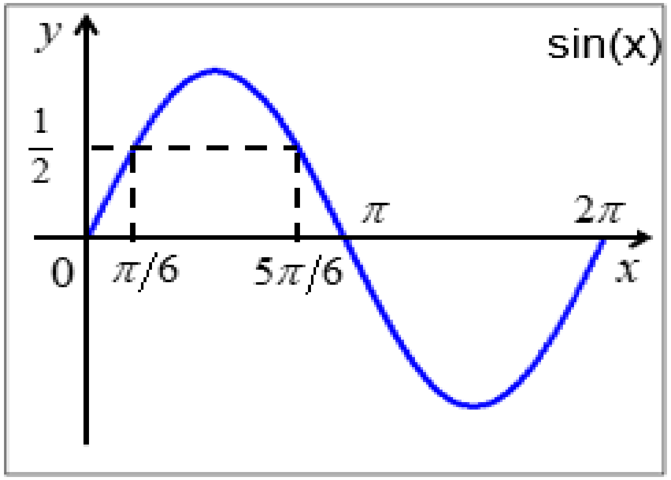
$\Rightarrow \; 2\cos(x) = \cos(x) / \sin(x)$

$\Rightarrow \; 2\cos(x)\sin(x) - \cos(x) = 0$

$\Rightarrow \; \cos(x)\,[2\sin(x) - 1] = 0$



$\cos(x) = 0$ ***or*** $\sin(x) = ½$



$\cos(x) = 0 \Rightarrow x = \pi/2$ or $x = 3\pi/2$

$\sin(x) = ½ \Rightarrow x = \pi/6$ or $x = 5\pi/6$

5.3.4 Solve problems using the Pythagorean identity

Example: prove $1+\tan^2(x) \equiv \sec^2(x)$

$$1+\tan^2(x) = \frac{\cos^2(x)}{\cos^2(x)} + \frac{\sin^2(x)}{\cos^2(x)} = \frac{\cos^2(x)+\sin^2(x)}{\cos^2(x)} = \sec^2(x)$$

Example

Prove $$1+\cot^2(x) \equiv \text{cosec}^2(x)$$

$$1+\cot^2(x) = \frac{\sin^2(x)}{\sin^2(x)} + \frac{\cos^2(x)}{\sin^2(x)} = \frac{\cos^2(x)+\sin^2(x)}{\sin^2(x)} = \text{cosec}^2(x)$$

Example: Prove $$\frac{\text{cosec}(A)\cot(A)}{\text{cosec}^2(A)+\sec^2(A)} \equiv \cos^3(A)$$

$$\frac{\text{cosec}(A)\cot(A)}{\text{cosec}^2(A)+\sec^2(A)} = \frac{\dfrac{\cos(A)}{\sin^2(A)}}{\dfrac{\sin^2(A)+\cos^2(A)}{\sin^2(A)\cos^2(A)}} = \frac{\cos(A)}{\sin^2(A)}\sin^2(A)\cos^2(A) = \cos^3(A)$$

Your turn! (3)

Prove

$$[1 - \cos(A)][1 + \sec(A)] \equiv \sin(A)\tan(A)$$

Solution

$$[1 - \cos(A)][1 + \sec(A)] = 1 + \sec(A) - \cos(A) - \cos(A)\sec(A)$$

$$= 1 + \sec(A) - \cos(A) - 1$$

$$= \sec A - \cos A$$

$$= \frac{1}{\cos(A)} - \cos(A)$$

$$= \frac{1 - \cos^2(A)}{\cos(A)}$$

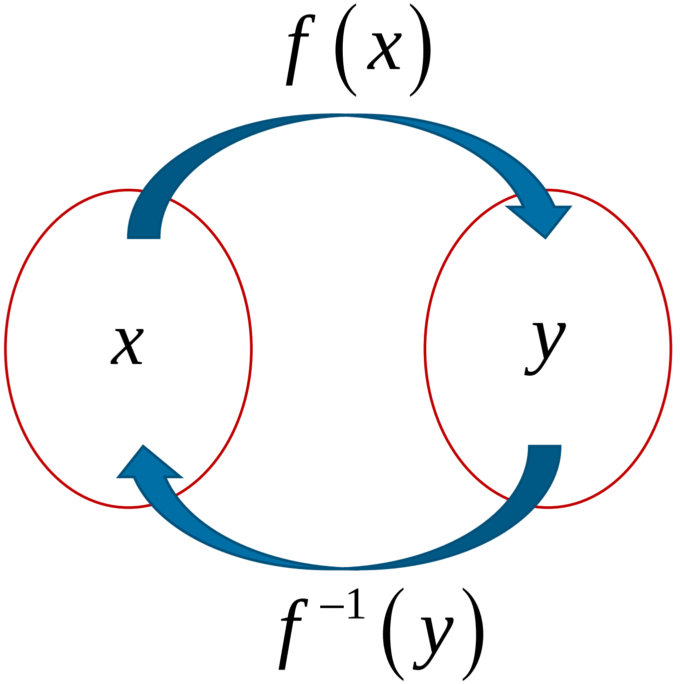
$$= \frac{\sin^2(A)}{\cos(A)}$$

$$= \sin(A)\tan(A)$$

5.3.5 Use the inverse functions arcsin(x), arccos(x) and arctan(x) and their graphs

The inverse function performs the opposite operation to the function. It takes elements of the range and maps them back into elements of the domain. Inverse functions only exist for one-to-one functions.

$$y = f(x) \Leftrightarrow x = f^{-1}(y)$$

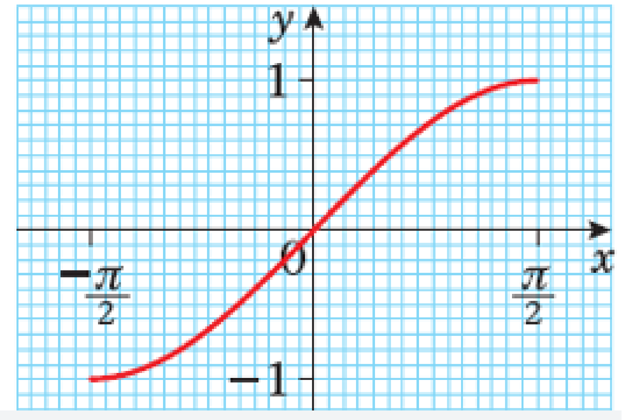


Notation used for inverse trig functions

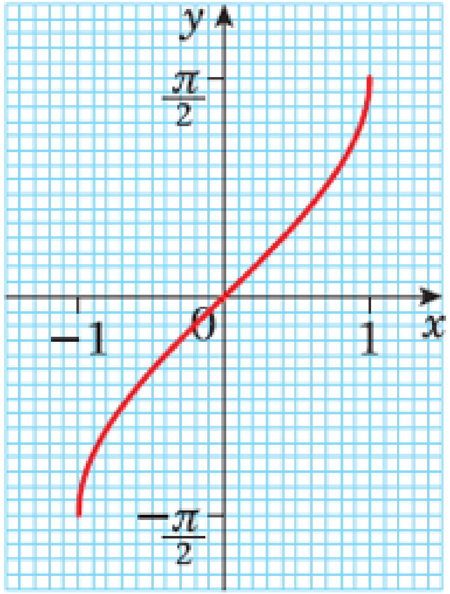
Function	Inverse	Alternative notation
$y = \sin x$	$y = \sin^{-1} x$	$y = \arcsin x$
$y = \cos x$	$y = \cos^{-1} x$	$y = \arccos x$
$y = \tan x$	$y = \tan^{-1} x$	$y = \arctan x$

$y = \sin^{-1}x$

Step 1: Draw the graph of $y = \sin x$, with the restricted domain $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$
This is a **one-to-one** function, taking all values in the range $-1 \leq \sin \mathrm{x} \leq 1$



$y = \arcsin x$



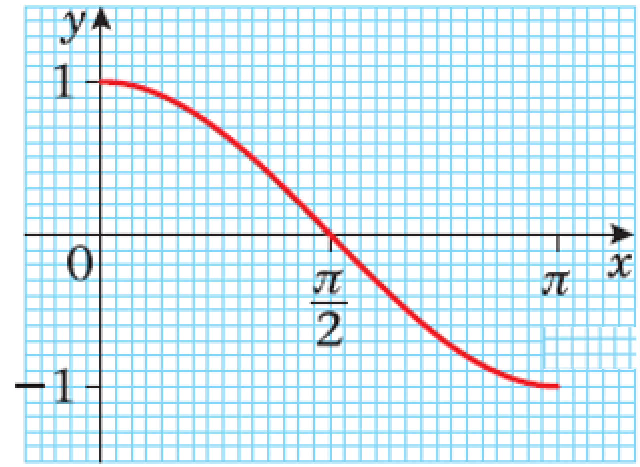
Step 2: Reflect the graph of $y = \sin x$ in the line $y = x$

The domain of $\arcsin x$ is $-1 \leq x \leq 1$
and the range is $-\frac{\pi}{2} \leq \arcsin x \leq \frac{\pi}{2}$

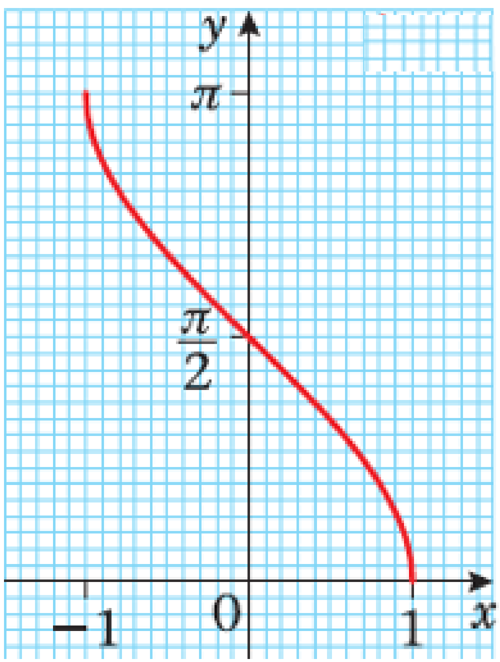
Remember that the x and y coordinates of the points interchange when reflecting in $y = x$

$y = \cos^{-1}x$

Step 1: Draw the graph of $y = \cos x$, with the restricted domain , $0 \leq x \leq \pi$
Tis is a **one-to-one** function taking all values in the range $-\mathbf{1} \leq \mathbf{cos}\,\boldsymbol{x} \leq \mathbf{1}$



$\mathrm{y} = \arccos x$



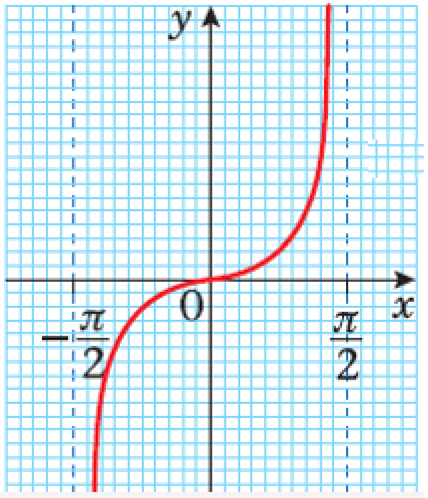
Step 2: Reflect the graph of $y = \cos x$ in the line $y = x$

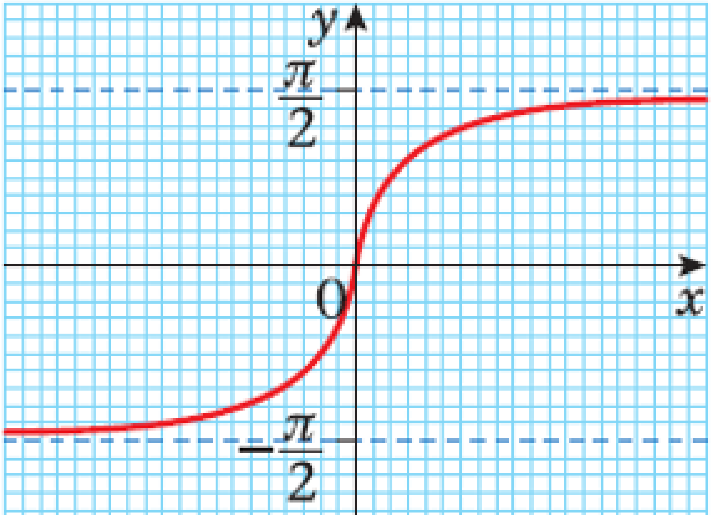
The domain of $\arccos x$ is $-1 \leq x \leq 1$
and the range is $0 \leq \arccos x \leq \pi$

Remember that the x and y coordinates of the points interchange when reflecting in $y = x$

$y = \tan^{-1} x$

Step 1: Draw the graph of $y = \tan x,\ -\frac{\pi}{2} \le x \le \frac{\pi}{2}$





Step 2: Reflect the graph of $y = \tan x$ in the line $y = x$

The domain of $\arcsin x$ is $-1 \le x \le 1$ and the range is $-\frac{\pi}{2} < \arctan x < \frac{\pi}{2}$

Remember that the x and y coordinates of the points interchange when reflecting in $y = x$

Examples

Work out in radians the values of:

a) $\arcsin\left(-\frac{\sqrt{2}}{2}\right)$

b) $\arccos(-1)$

c) $\arctan\left(\sqrt{3}\right)$

Solution

a) $\arcsin\left(-\frac{\sqrt{2}}{2}\right) = -\frac{\pi}{4} = -0.785$ (3 s.f.) domain $-\frac{\pi}{2} \leq \arcsin x \leq \frac{\pi}{2}$

b) $\arccos(-1) = \pi = 3.14$ (3 s.f.) domain $0 \leq \arccos x \leq \pi$

Your turn! (4)

$$\arctan(\sqrt{3}) = ?$$

Solution

c) $\arctan(\sqrt{3}) = \frac{\pi}{3} = 1.05\ \ (3\text{ s.f.})$ domain $-\frac{\pi}{2} < \arctan x < \frac{\pi}{2}$

5.3 Summary

You should now be able to

5.3.1 Recall and use functions $\sec x$, $\operatorname{cosec} x$, $\cot x$

5.3.2 Recall the graphs of $\sec x$, $\operatorname{cosec} x$, $\cot x$

5.3.3 Solve problems involving $\sec x$, $\operatorname{cosec} x$, $\cot x$

5.3.4 Use the Pythagorean identities.

5.3.5 Use the inverse functions $\arcsin x$, $\arccos x$ and $\arctan x$ and their graphs.

Figure references

Some of the figures in the slides for this unit, as listed in the table below

10	11	12	14	15	18	25	26	27

have been reproduced from the following text book series:

Attwood, G., Macpherson, A., Moran, B., Petran, J., Pledger, K., Staley, G. and Wilkins, D. (2008), Edexcel AS and A Level Modular Mathematics series C1-C4, Pearson, Harlow, UK.

You may wish to refer to these text books for further information, examples and practice questions.

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