Pat Warner

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9.1 Introduction to vectors

Core Preparatory Topics
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2.1
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11.1
11.5

FEPS Mathematics Support Framework

9.1 Introduction

The aim of this unit is to assist you in consolidating and developing your knowledge and skills in working with basic vector concepts in 2 and 3 dimensions.

While studying these slides you should attempt the ‘Your Turn’ questions in the slides.

After studying the slides, you should attempt the Consolidation Questions.

9.1 Learning checklist

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Learning resource	Notes	Tick when complete
Slides		
Your turn questions		
Consolidation questions		

9.1 Learning objectives

After studying this unit you should be able to

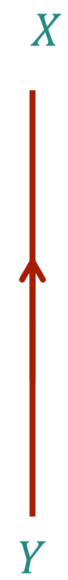
9.1.1 Recognise the difference between scalar and vector quantities
9.1.2 Construct a simple vector diagram and apply the triangle law
9.1.3 Find the magnitude (modulus) of a vector
9.1.4 Find a unit vector
9.1.5 Perform simple vector arithmetic
9.1.6 Solve simple geometric problems in vector diagrams
9.1.7 Use vectors to describe the location of a point in a 2 and 3 dimensional Cartesian framework
9.1.8 Use the Cartesian component form of a vector in 2 and 3 dimensions
9.1.9 Extend 2 dimensional operations such as modulus, to 3 dimensions

9.1.1 Scalar and vector quantities

A scalar quantity can be described by using a single number (*the magnitude* or *size*)

The distance from X to Y is 50 metres

Distance is a scalar



A vector quantity has both magnitude and direction

From X to Y you go 50 meters north:

This is the displacement of X from Y

Displacement is a vector

Scalars and vectors

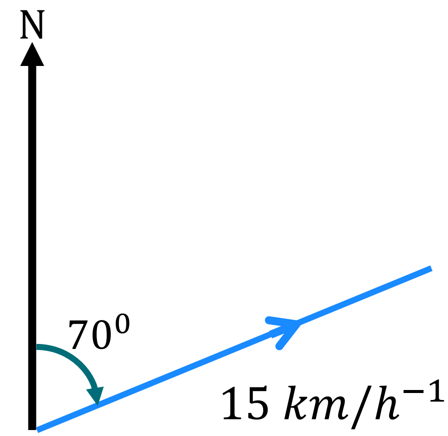
A ship is sailing at $15\ km/h^{-1}$

Speed is a scalar

A ship is sailing at $15\ km\ /h^{-1}$, on a bearing of 070^0.

This is called the velocity of the ship.

Velocity is a vector.



Example 1

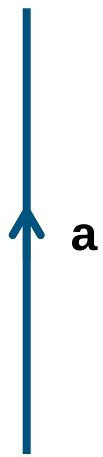
Show on a diagram the displacement vector from X to Y, where Y is 400m due east of Y

This is called a ‘directed line segment’,
The direction of the arrow shows the direction of the vector.

The vector is written as $\overrightarrow{XY}$

The length of the line segment $\overrightarrow{XY}$ represents the distance 400m

Another type of notation



Sometimes, instead of using the end points X and Y, a small (lower case) letter is used.

In print, the small letter will be in **bold type.** In writing, you should underline the small letter to show it is a vector: $\underline{a}$

Your turn!(1)

State whether each example described below refers to a **scalar quantity** or a **vector quantity**

a) A pilot flies due south for a distance of 200 kilometres.

b) The time taken to travel from London to Exeter is 3 hours.

Solution:

a) Vector quantity as we have direction and magnitude

b) Scalar as we only have magnitude.

9.1.2 Construct a vector diagram and apply the triangle law

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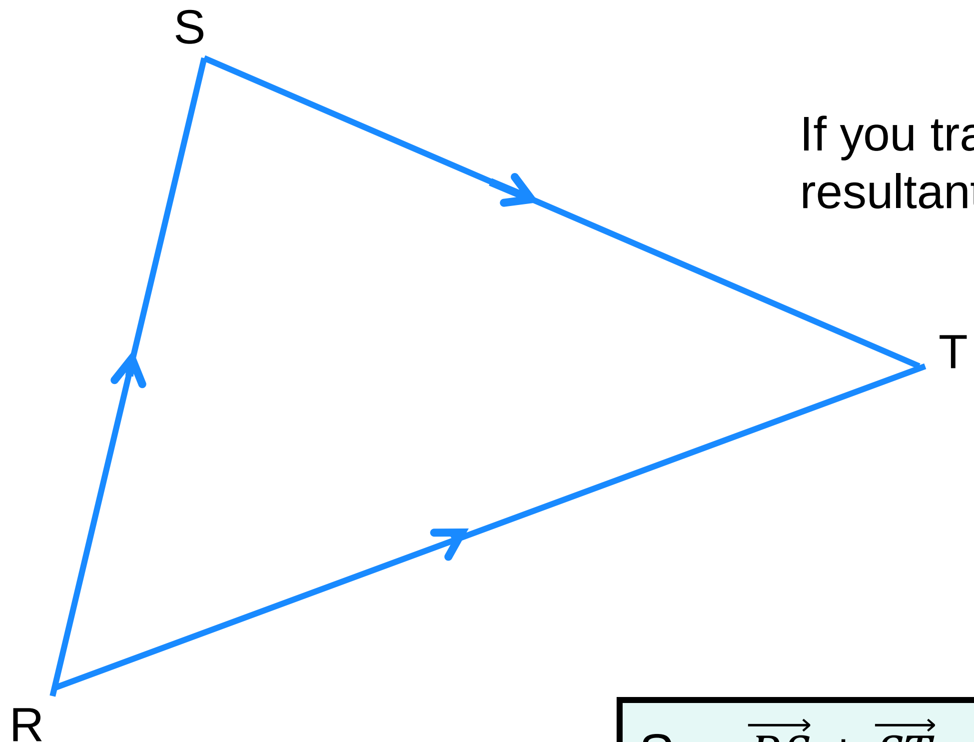
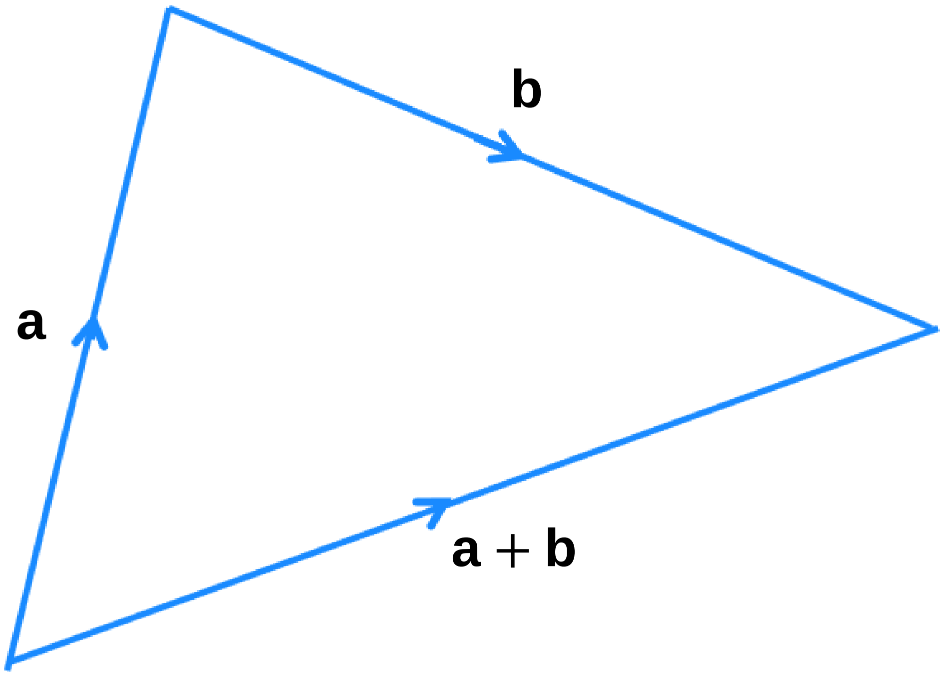


If you travel from R to S then S to T the resultant journey is R to T

So: $\overrightarrow{RS} + \overrightarrow{ST} = \overrightarrow{RT}$

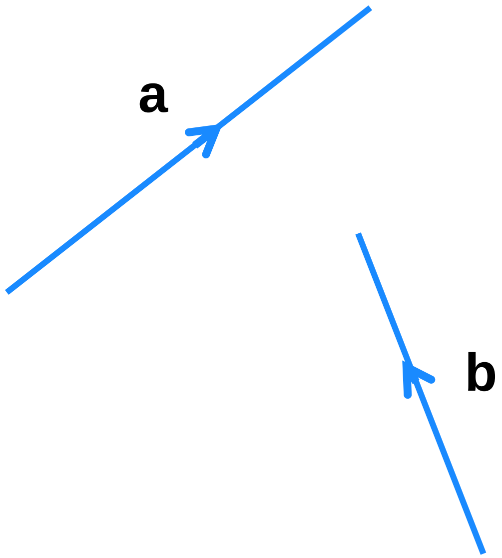
Vector addition: the "triangle law"

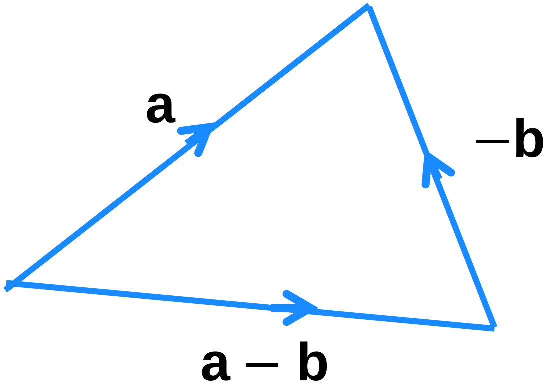


When you add the vectors **a** and **b,** the resultant vector **a** + **b** goes from 'the start of **a** to the finish of **b**'

Vector subtraction

Subtracting a vector is equivalent to ‘adding a negative vector’, so $\mathbf{a} - \mathbf{b}$ is defined as $\mathbf{a} + (-\mathbf{b})$





The zero vector

Adding the vectors $\overrightarrow{QR}$ and $\overrightarrow{RQ}$ gives the zero vector $\mathbf{0}$.

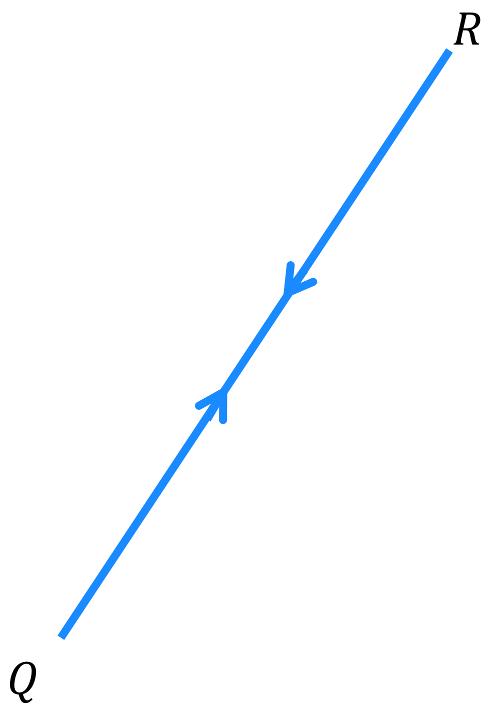
$$\overrightarrow{QR} + \overrightarrow{RQ} = \mathbf{0}$$

The zero displacement vector is $\mathbf{0}$.
It is printed in bold type, or underlined in written work.

You can also write:

$\overrightarrow{RQ}$ as $-\overrightarrow{QR}$

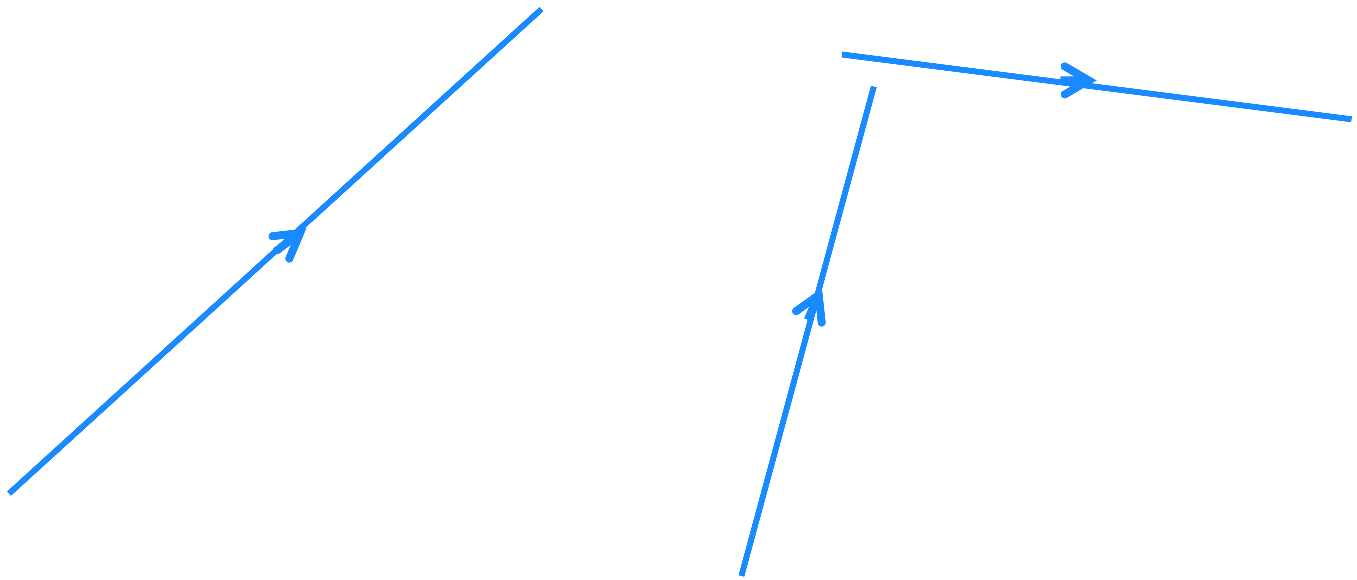
So $\overrightarrow{QR} + \overrightarrow{RQ} = \mathbf{0}$ or $\overrightarrow{QR} - \overrightarrow{QR} = \mathbf{0}$



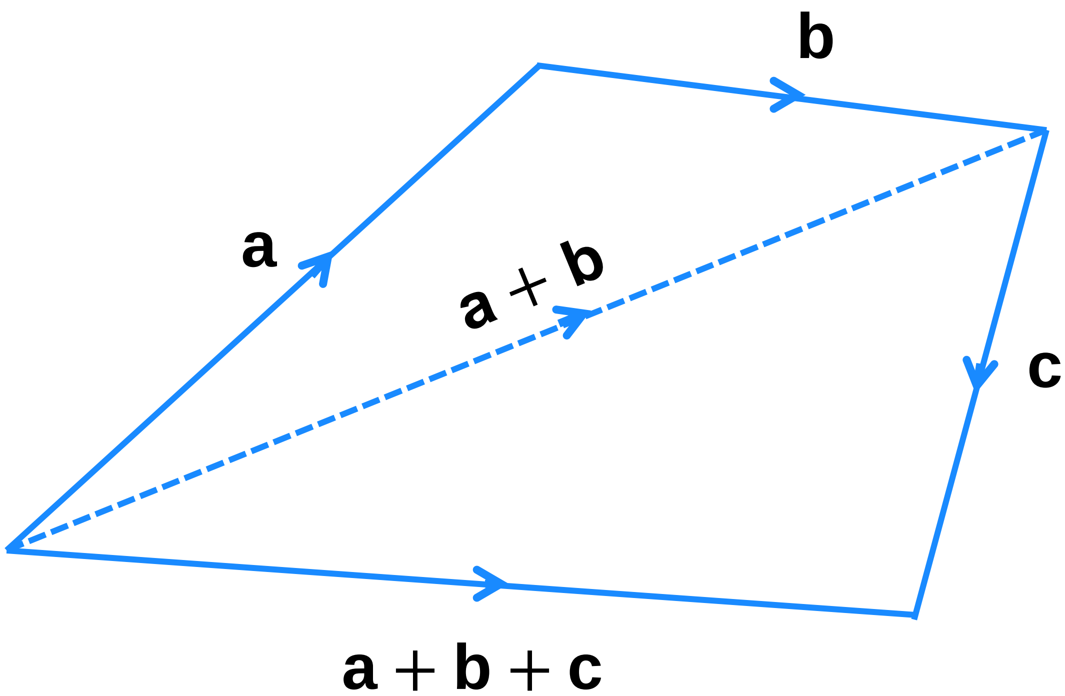
Example 2 – vector addition

The diagram shows the vectors **a, b** and **c**.

Draw another diagram to illustrate the vector addition **a** + **b** + **c**



Solution



First use the triangle law for **a** + **b**, then use it again for (**a** + **b**) + **c.**

The resultant goes from the start of **a** to the finish of **c.**

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9.1.3 Using the magnitude of a vector

The **magnitude** of a vector is the **distance** between its start point and end point.

The **magnitude** of a vector **a** is written |**a**|.

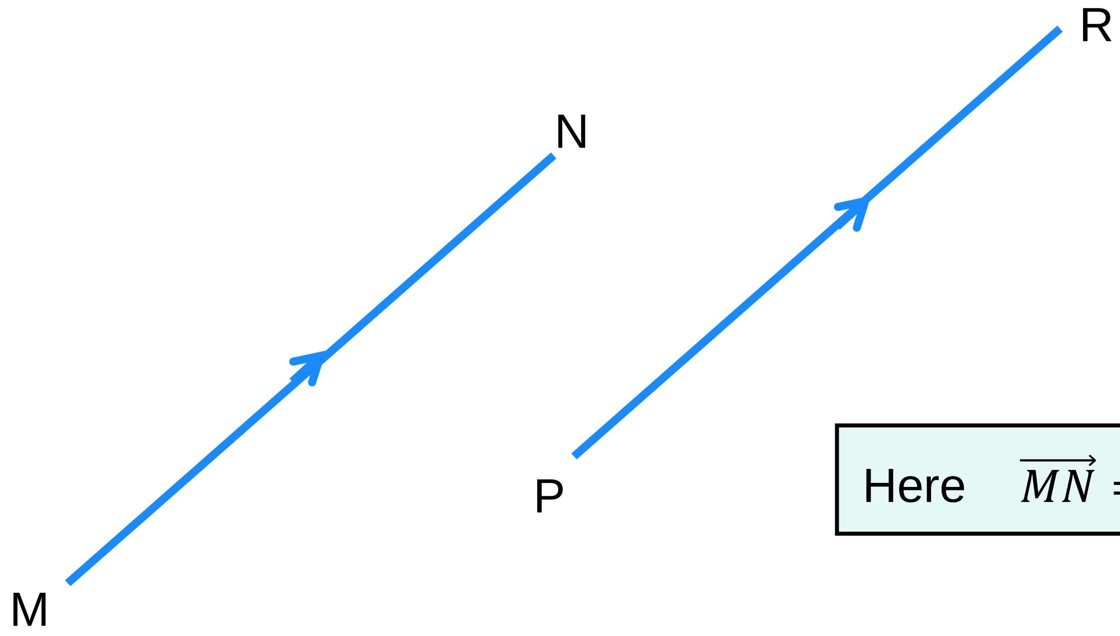
The **magnitude** of a vector $\overrightarrow{AB}$ is written $\left|\overrightarrow{AB}\right|$.

It's sometimes called **modulus** instead of magnitude. Magnitude is **a scalar**, and it's **always non-negative.**

We'll learn how to calculate magnitude in a little while but for now we just need the definition

Equal vectors

Vectors that are **equal** have both **the same magnitude** and **the same direction**

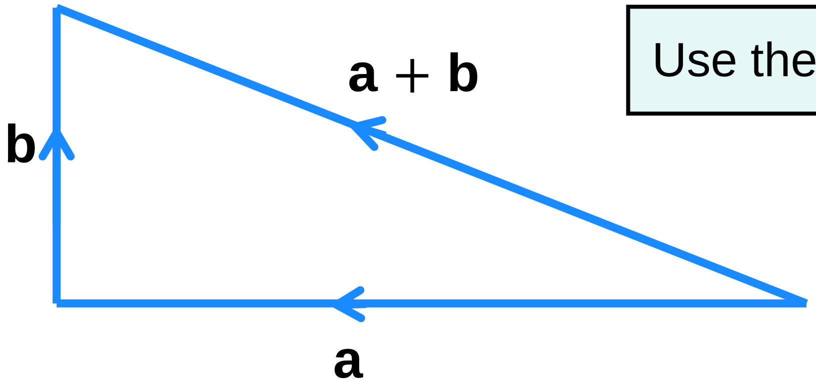


Here $\overrightarrow{MN} = \overrightarrow{PR}$

Calculations involving the magnitude (modulus)

The vector a is directed due west and $|\boldsymbol{a}| = 15$.
The vector b is directed due north and $|\boldsymbol{b}| = 8$.

Find $|\boldsymbol{a} + \boldsymbol{b}|$



Use the triangle law for adding the vectors **a** and **b**

Use Pythagoras' theorem

$$|a + b|^2 = 15^2 + 8^2 = 289$$

$$|a + b| = 17$$

9.1.4 Find a unit vector

A **unit vector** is any vector with a **length of 1 unit.**

Example 3

The vector **a** has magnitude 16 units.
Write down a unit vector that is parallel to **a**.

The unit vector is $\frac{\mathbf{a}}{16}$ or $\frac{1}{16}\mathbf{a}$

Divide **a** by its magnitude.

In general, the unit vector is $\frac{\mathbf{a}}{|\mathbf{a}|}$

9.1.5/6 Vector arithmetic & vector diagrams

In the diagram, $\overrightarrow{QP} = \mathbf{a}$, $\overrightarrow{QR} = \mathbf{b}$, $\overrightarrow{QS} = \mathbf{c}$ **and** $\overrightarrow{RT} = \mathbf{d}$

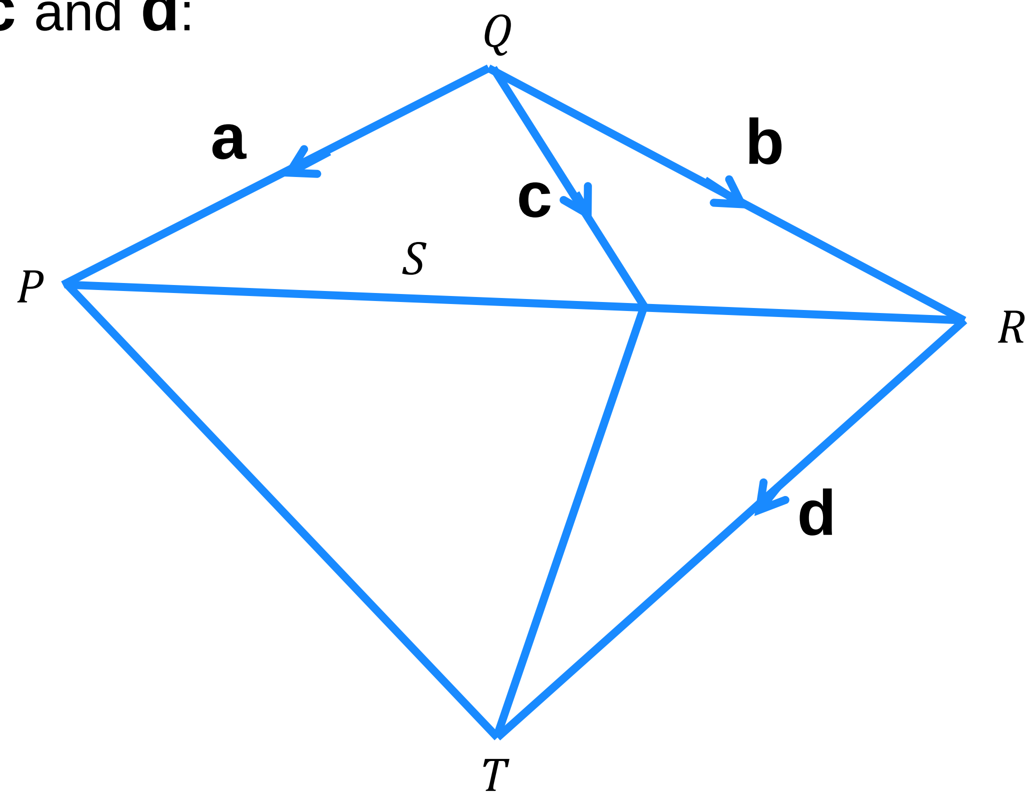
Find in terms of **a, b, c** and **d**:

a) $\overrightarrow{PS}$

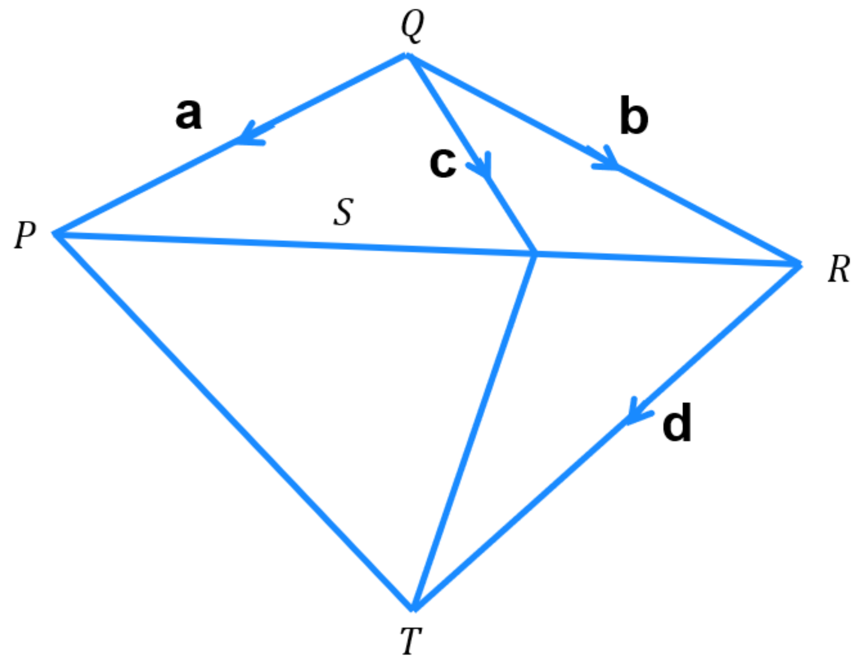
b) $\overrightarrow{RP}$

c) $\overrightarrow{PT}$

d) $\overrightarrow{TS}$



Solution



a) $\overrightarrow{PS} = \overrightarrow{PQ} + \overrightarrow{QS} = -\mathbf{a} + \mathbf{c} = \mathbf{c} - \mathbf{a}$

b) $\overrightarrow{RP} = \overrightarrow{RQ} + \overrightarrow{QP} = -\mathbf{b} + \mathbf{a} = \mathbf{a} - \mathbf{b}$

c) $\overrightarrow{PT} = \overrightarrow{PR} + \overrightarrow{RT} = (\mathbf{b} - \mathbf{a}) + \mathbf{d} = \mathbf{b} + \mathbf{d} - \mathbf{a}$

d) $\overrightarrow{TS} = \overrightarrow{TR} + \overrightarrow{RS} = -\mathbf{d} + \left(\overrightarrow{RQ} + \overrightarrow{QS}\right) = -\mathbf{d} + (-\mathbf{b} + \mathbf{c}) = \mathbf{c} - \mathbf{b} - \mathbf{d}$

Your turn!(2)

In the diagram, $\overrightarrow{PQ} = \mathbf{a}$, $\overrightarrow{QS} = \mathbf{b}$, $\overrightarrow{SR} = \mathbf{c}$ and $\overrightarrow{PT} = \mathbf{d}$.

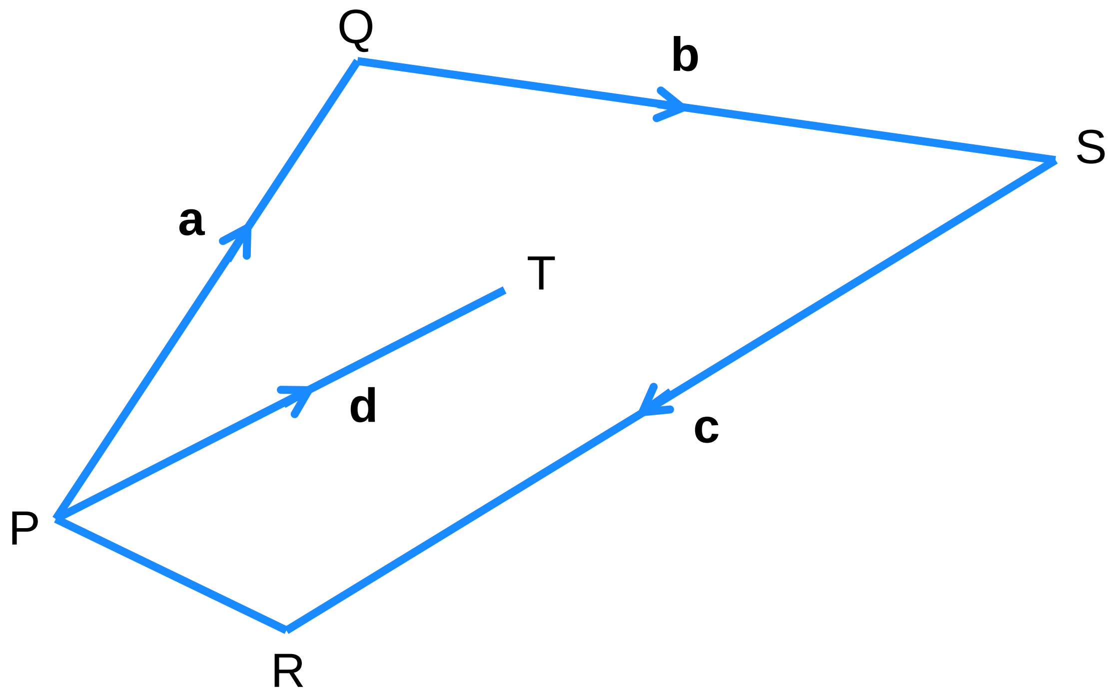
Find in terms of **a, b, c** and **d:**

a) $\overrightarrow{QT}$

b) $\overrightarrow{PR}$

c) $\overrightarrow{TS}$

d) $\overrightarrow{TR}$



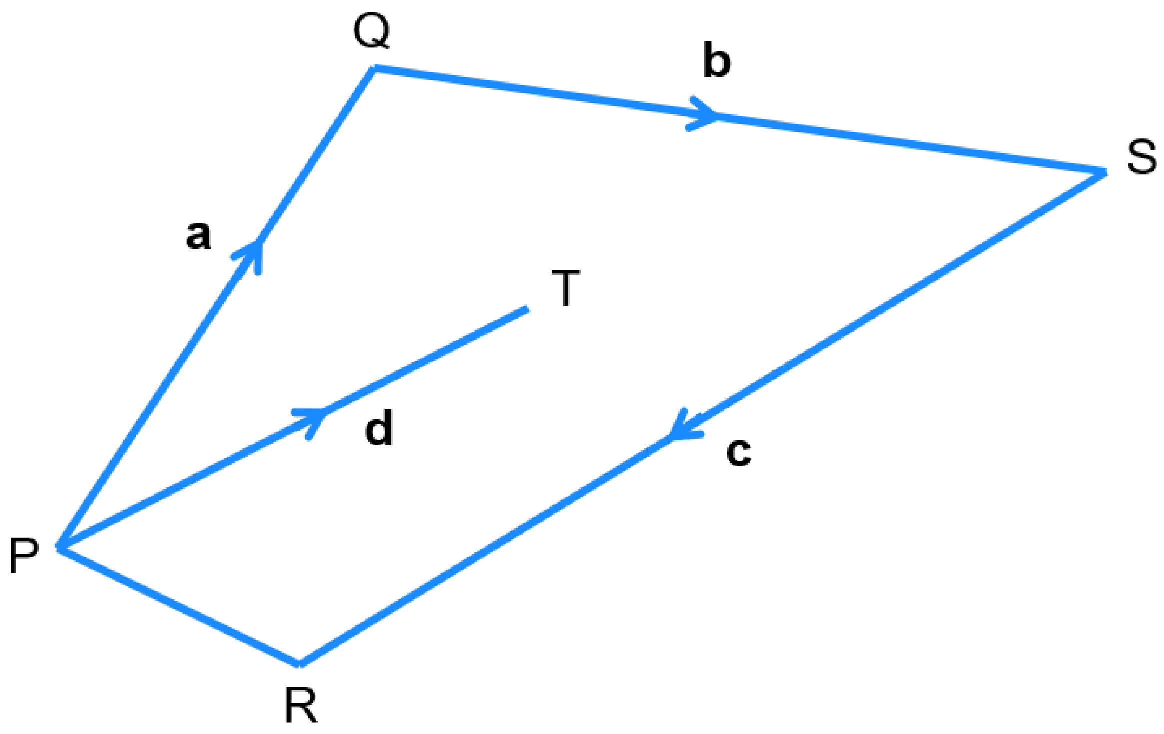
Solution

a) $\overrightarrow{QT}$ = **d** – **a**

b) $\overrightarrow{PR}$ = **a** + **b** + **c**

c) $\overrightarrow{TS}$ = **a** + **b** – **d**

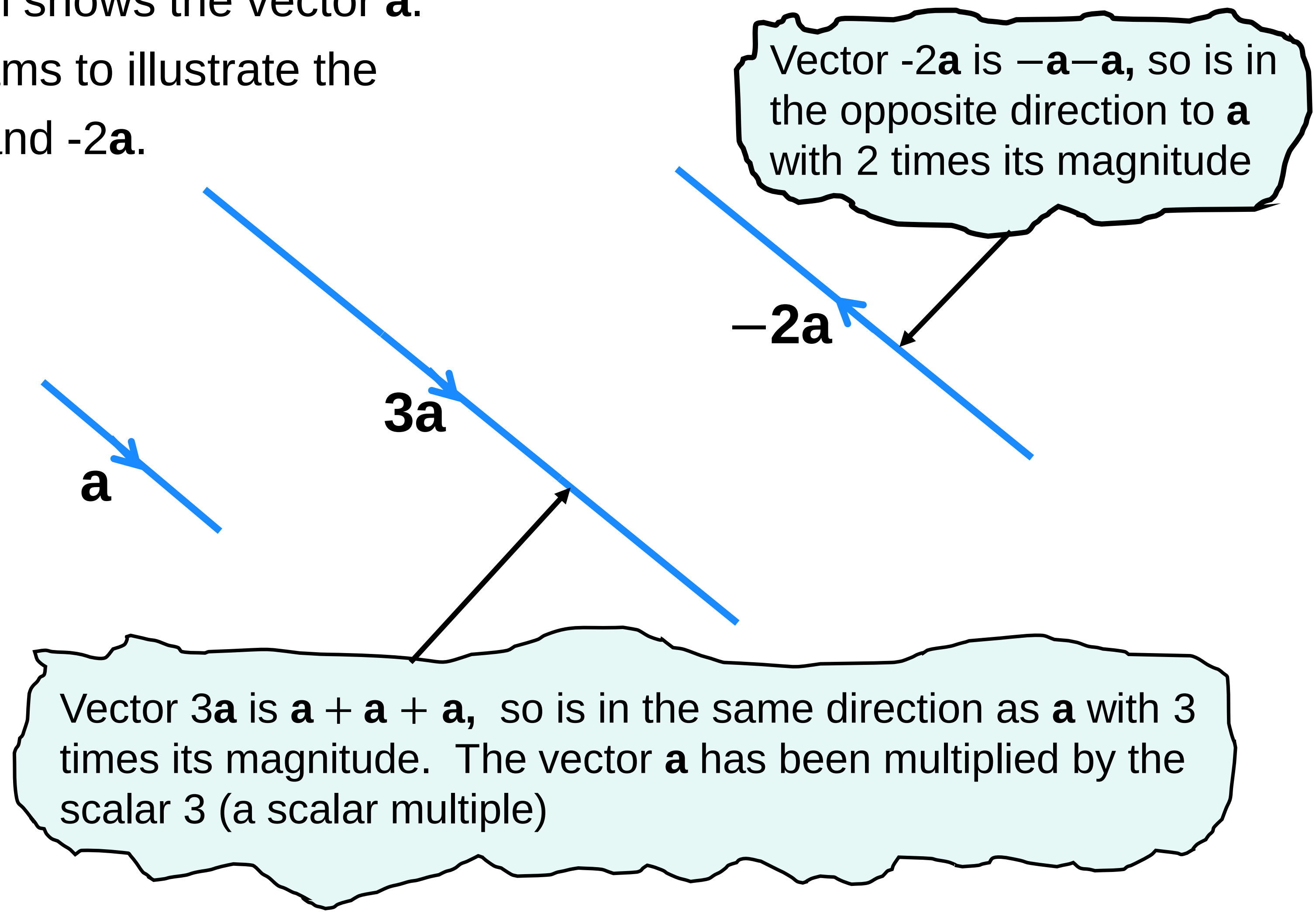
d) $\overrightarrow{TR}$ = **a** + **b** + **c** - **d**



Example – multiplying a vector by a scalar

The diagram shows the vector **a**.

Draw diagrams to illustrate the vectors 3**a** and -2**a**.



Parallel vectors

Any vector parallel to the vector a may be written as $\boldsymbol{\lambda}\mathbf{a}$, where $\boldsymbol{\lambda}$ is a non-zero scalar

Example

Show that the vectors $6\mathbf{a} + 8\mathbf{b}$ and $9\mathbf{a} + 12\mathbf{b}$ are parallel

$$9\mathbf{a} + 12\mathbf{b} = \frac{3}{2}(6\mathbf{a} + 8\mathbf{b})$$

$\therefore$ the vectors are parallel where $\lambda = \frac{3}{2}$

If $\boldsymbol{\lambda}\mathbf{a} + \boldsymbol{\mu}\mathbf{b} = \boldsymbol{\alpha}\mathbf{a} + \boldsymbol{\beta}\mathbf{b}$, and the non-zero vectors **a** and **b** are not parallel, then $\boldsymbol{\lambda} = \boldsymbol{\alpha}$ and $\boldsymbol{\mu} = \boldsymbol{\beta}$

The above result can be shown as follows:

$$\boldsymbol{\lambda}\mathbf{a} + \boldsymbol{\mu}\mathbf{b} = \boldsymbol{\alpha}\mathbf{a} + \boldsymbol{\beta}\mathbf{b}$$

$$(\boldsymbol{\lambda} - \boldsymbol{\alpha})\mathbf{a} = (\boldsymbol{\beta} - \boldsymbol{\mu})\,\mathbf{b}$$

The two vectors cannot be equal unless they are parallel or zero.
Since **a** and **b** are not parallel or zero:

$(\boldsymbol{\lambda} - \boldsymbol{\alpha}) = 0$ and $(\boldsymbol{\beta} - \boldsymbol{\mu}) = 0$, so $\boldsymbol{\lambda} = \boldsymbol{\alpha}$ and $\boldsymbol{\beta} = \boldsymbol{\mu}$

Example 6

Given that $5\mathbf{a} - 4\mathbf{b} = (2s + t)\mathbf{a} + (s - t)\mathbf{b}$, where $\mathbf{a}$ and $\mathbf{b}$ are non-zero, non-parallel vectors find the value of the scalars s and t.

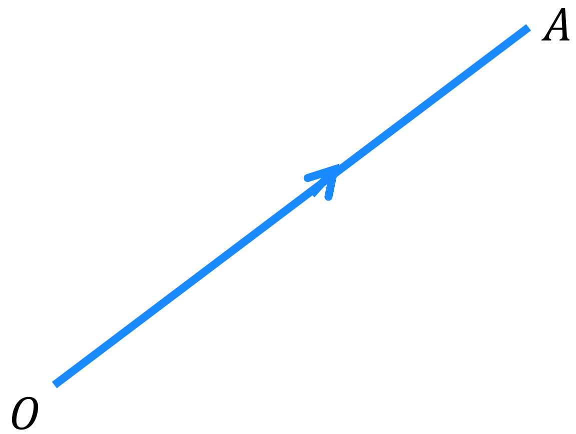
Equating coefficients and solving simultaneously gives:

$$2s + t = 5$$
$$s - t = -4$$

$$3s = 1$$
$$s = \frac{1}{3}$$
$$t = 4\frac{1}{3}$$

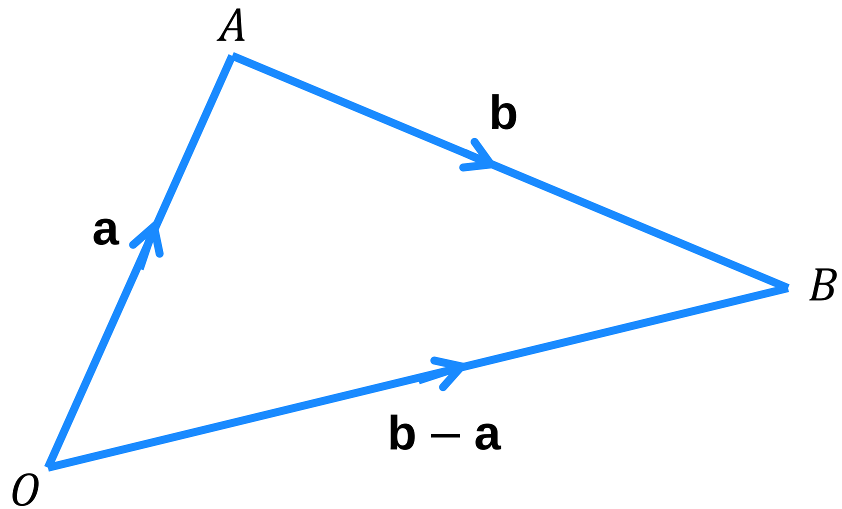
9.1.7 The position of a point

The position vector of a point A is the vector $\overrightarrow{OA}$, where O is the origin. $\overrightarrow{OA}$ is usually written as vector **a**



$$\overrightarrow{OA} = \mathbf{a}$$

$\overrightarrow{\boldsymbol{AB}} = \mathbf{b} - \mathbf{a}$**,** where **a** and **b** are the position vectors of A and B respectively.



Using the triangle law gives:

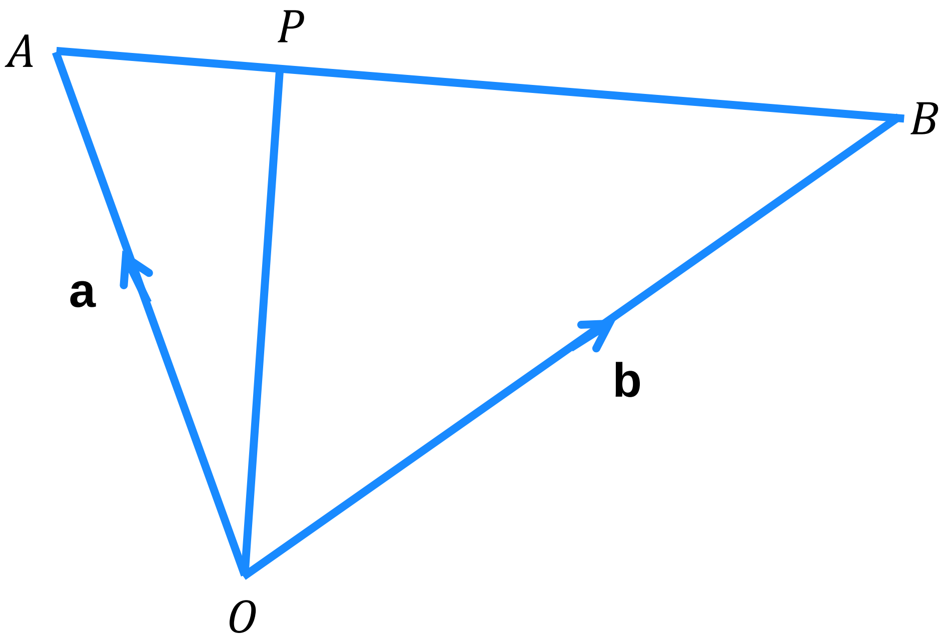
$\overrightarrow{\boldsymbol{AB}} = \overrightarrow{\boldsymbol{AO}} + \overrightarrow{\boldsymbol{OB}} = -\mathbf{a} + \mathbf{b}$

So $\overrightarrow{\boldsymbol{AB}} = \mathbf{b} - \mathbf{a}$

Example 7

In the diagram the points A and B have position vectors a and b respectfully (referred to the origin O).
The point P divides AB in the ratio $1 : 2$

Find the position vector of P.



$\overrightarrow{AB} = \mathbf{b} - \mathbf{a}$

$\overrightarrow{OP} = \overrightarrow{OA} + \overrightarrow{AP}$

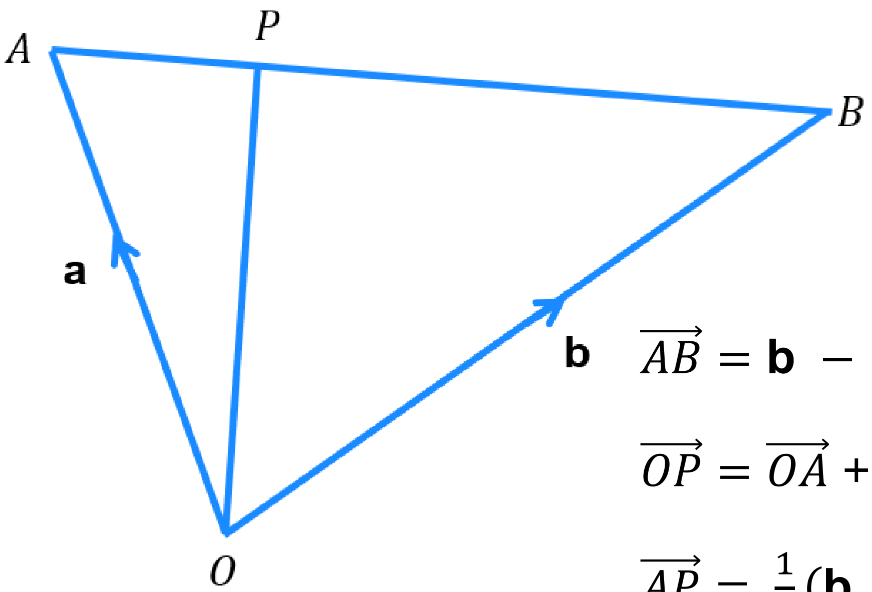
$\overrightarrow{OP}$ is the position vector of P

$\overrightarrow{AP} = \frac{1}{3}(\mathbf{b} - \mathbf{a})$

$\overrightarrow{OP} = \mathbf{a} + \frac{1}{3}(\mathbf{b} - \mathbf{a})$

$\overrightarrow{OP} = \frac{2}{3}\mathbf{a} + \frac{1}{3}\mathbf{b}$

Solution



$\overrightarrow{AB} = \mathbf{b} - \mathbf{a}$

$\overrightarrow{OP} = \overrightarrow{OA} + \overrightarrow{AP}$

$\overrightarrow{OP}$ is the position vector of P

$\overrightarrow{AP} = \frac{1}{3}(\mathbf{b} - \mathbf{a})$

Use the 1 : 2 ratio

$\overrightarrow{OP} = \mathbf{a} + \frac{1}{3}(\mathbf{b} - \mathbf{a})$

$\overrightarrow{OP} = \frac{2}{3}\mathbf{a} + \frac{1}{3}\mathbf{b}$

You could write $\mathbf{p} = \frac{2}{3}\mathbf{a} + \frac{1}{3}\mathbf{b}$

9.1.8 Cartesian components

i is a unit vector in the direction of the **positive x-axis**, and **j** is a unit vector in the direction of the **positive y-axis.**

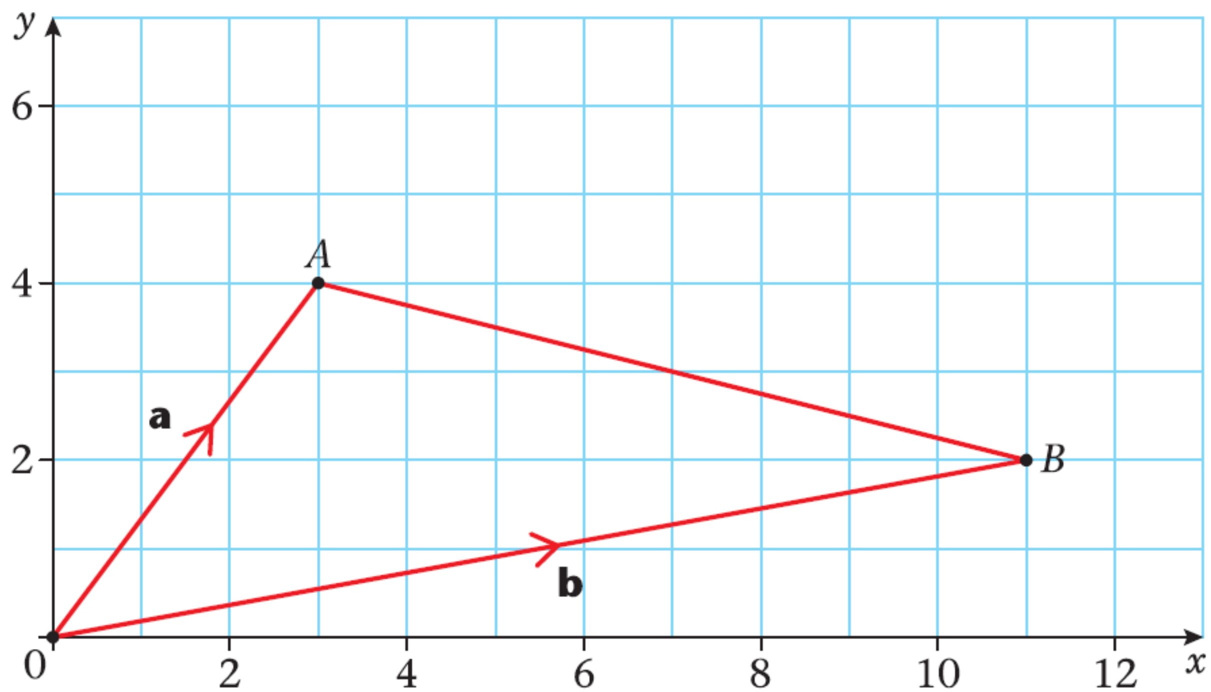
The vectors **i** and **j** are called **standard unit vectors**, and they each have a length of 1 unit.

Example 8

The points A and B in the diagram have coordinates $(3, 4)$ and $(11, 2)$ respectfully.

Find, in terms of **i** and **j**:

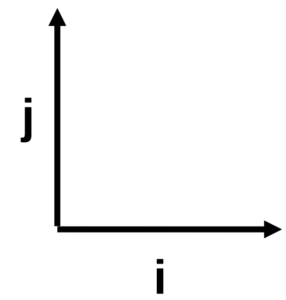
a) the position vector of A b) the position vector of B c) the vector $\overrightarrow{AB}$



Solution

a) $\mathbf{a} = \overrightarrow{OA} = 3\mathbf{i} + 4\mathbf{j}$

i goes 1 unit 'across'. **j** goes 1 unit 'up'



b) $\mathbf{b} = \overrightarrow{OB} = 11\mathbf{i} + 2\mathbf{j}$

c) $\overrightarrow{AB} = \mathbf{b} - \mathbf{a}$

$$= (11\mathbf{i} + 2\mathbf{j}\ ) - (3\mathbf{i} + 4\mathbf{j}\ )$$

$$= 8\mathbf{i}\ - 2\mathbf{j}$$

You can see from the diagram that the vector $\overrightarrow{AB}$ goes 8 units 'across' and 2 units 'down'.

Column vectors

Column vectors are another way of writing vectors in terms of their **horizontal** and **vertical components**.

You just write the **horizontal (i) component** on **top** of the **vertical (j) component** and put a **bracket** around them:

$$x\mathbf{i} + y\mathbf{j} = \binom{x}{y}$$

Your turn!(3)

Given that $\mathbf{a} = 2\mathbf{i} + 5\mathbf{j}$, $\mathbf{b} = 12\mathbf{i} - 10\mathbf{j}$ and $\mathbf{c} = -3\mathbf{i} + 9\mathbf{j}$ find $\mathbf{a} + \mathbf{b} + \mathbf{c}$,

using column matrix notation in your working

Solution

Given that $\mathbf{a} = 2\mathbf{i} + 5\mathbf{j}$, $\mathbf{b} = 12\mathbf{i} - 10\mathbf{j}$ and $\mathbf{c} = -3\mathbf{i} + 9\mathbf{j}$ find $\mathbf{a} + \mathbf{b} + \mathbf{c}$, using column matrix notation in your working

$$\mathbf{a} + \mathbf{b} + \mathbf{c} = \begin{pmatrix} 2 \\ 5 \end{pmatrix} + \begin{pmatrix} 12 \\ -10 \end{pmatrix} + \begin{pmatrix} -3 \\ 9 \end{pmatrix}$$

$$= \begin{pmatrix} 11 \\ 4 \end{pmatrix}$$

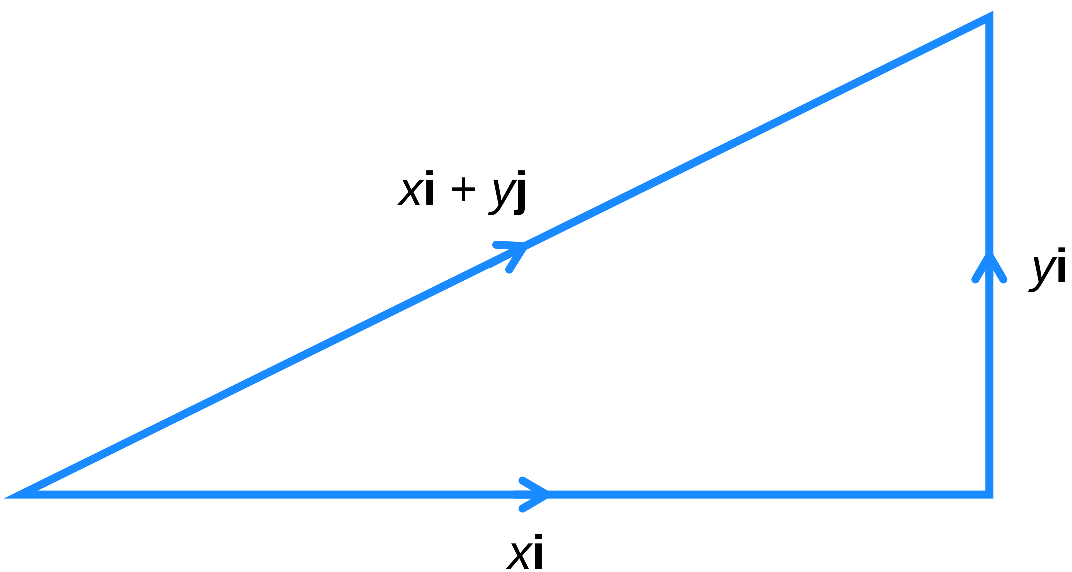
Add the numbers in the top line to get 11 (the x component), and the bottom line to get 4 (the y component).

This is $(11\mathbf{i} + 4\,\mathbf{j})$

Calculating the modulus in component form

The modulus (or magnitude)

of $x\boldsymbol{i}$ + y**j** is $\sqrt{x^2 + y^2}$



From Pythagoras' Theorem, the magnitude of x**i** + y**j**, represented by the hypotenuse, is $\sqrt{x^2 + y^2}$

Example 9

The vector a is equal to $5\mathbf{i} - 12\mathbf{j}$.

Find $|\mathbf{a}|$, and find a unit vector in the same direction as **a.**

$$|\mathbf{a}| = \sqrt{5^2 + (-12)^2} = \sqrt{169} = 13$$

Unit vector is $\frac{\mathbf{a}}{|\mathbf{a}|} = \frac{5\mathbf{i}-12\mathbf{j}}{13} = \frac{1}{13}(5\mathbf{i} - 12\mathbf{j})$

or $\frac{5}{13}\mathbf{i} - \frac{12}{13}\mathbf{j}$

or $\frac{1}{13}\begin{pmatrix} 5 \\ -12 \end{pmatrix}$

Your turn!(4)

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Given that $\mathbf{a} = 5\mathbf{i} + \mathbf{j}$ and $\mathbf{b} = -2\mathbf{i} - 4\mathrm{j}$, find the exact value of $|2\mathbf{a} + \mathbf{b}|$

Solution

Given that $\mathbf{a} = 5\mathbf{i} + \mathbf{j}$ and $\mathbf{b} = -2\mathbf{i} - 4\mathrm{j}$, find the exact value of $|2\mathbf{a} + \mathbf{b}|$

$$|2\mathbf{a} + \mathbf{b}| = 2(5\mathbf{i} + \mathbf{j}) + (-2\mathbf{i} - 4\mathbf{j})$$

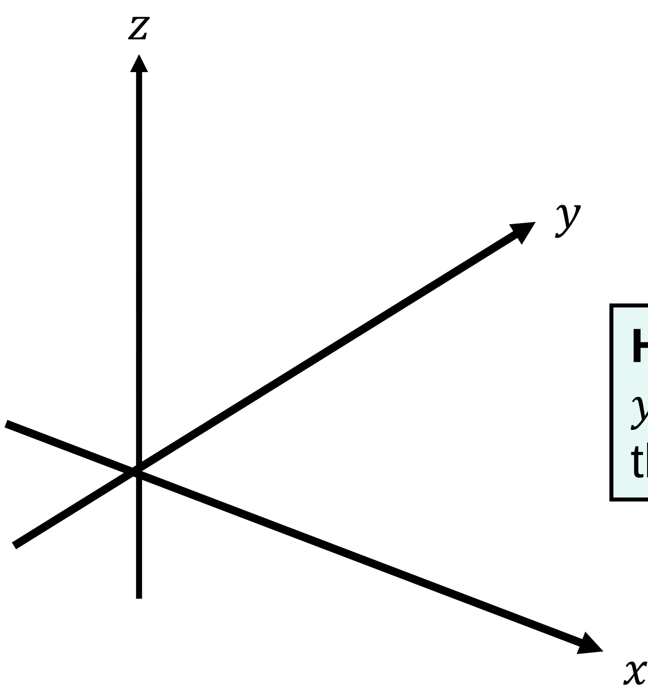
$$= 10\mathbf{i} + 2\mathrm{j} - 2\mathbf{i} - 4\mathbf{j}$$

$$= 8\mathbf{i} - 2\mathbf{j}$$

9.1.9 Extending to 3 dimensions

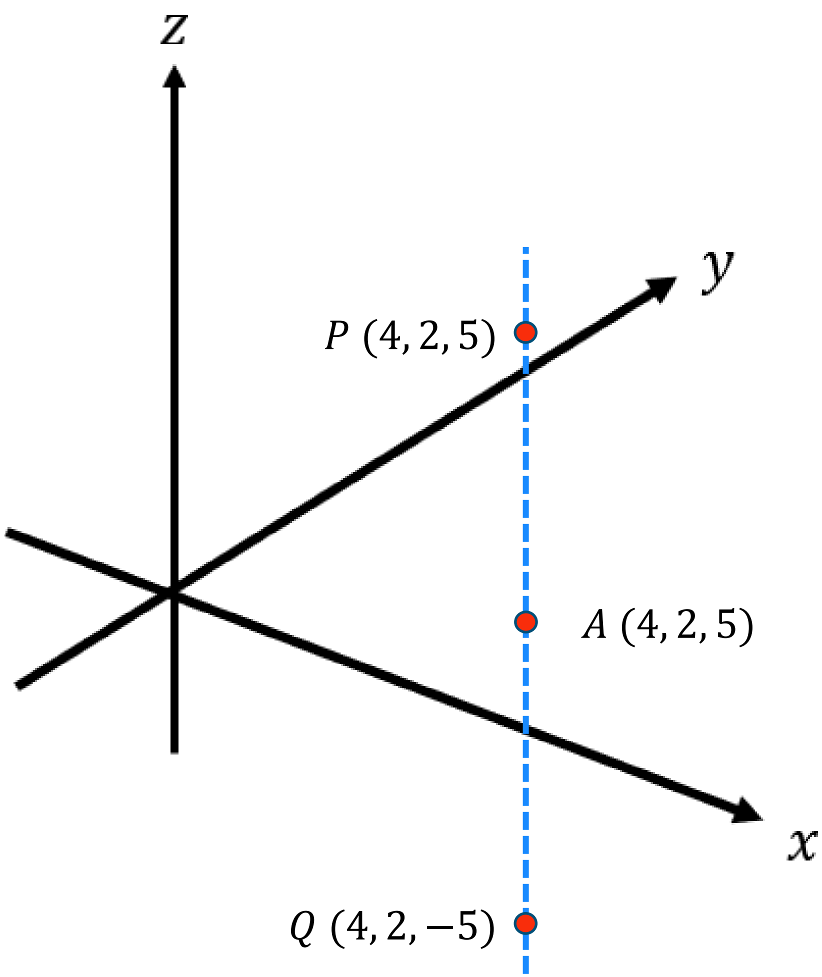
Cartesian coordinates axes in three dimensions are usually called x, y and z axes, each being at right angles to each other.

The coordinates of a point in three dimensions are written as (x, y, z)



Hint: To visualise this, think of the x and y axes being drawn on a flat surface and the z axis sticking up from the surface.

Points in 3D – Cartesian form



The *point* $A\ (4, 2, 0)$ is on the 'flat surface' (the x, y plane)

The point $P\ (4, 2, 5)$ is 5 units 'above the surface'

The point $Q\ (4, 2, -5)$ is 5 units 'below the surface'

So the line joining the points *P* and Q
Is parallel to the z-axis

Distance between points in 3 - dimensions

The magnitude of as 3-dimensional vector **a** in the component form

$\mathbf{a} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ is $|\mathbf{a}| = \sqrt{x^2 + y^2 + z^2}$

The distance between the points (x_1, y_1, z_1) and (x_2, y_2, z_2) is

$$\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}$$

This is the three-dimensional version of the formula

$$\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

Example 10

Find the distance between the points $A\ (1, 3, 4)$ and $B\ (8, 6, -5)$, giving your answer to one decimal place.

Solution:

$$AB = \sqrt{(1-8)^2 + (3-6)^2 + (4 - -5)^2}$$

$$= \sqrt{(-7)^2 + (-3)^2 + (9)^2}$$

$$= \sqrt{139} = 11.8\ (1\ \text{d.p.})$$

Component form of vectors in 3D

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- The vectors **i, j** and **k** are unit vectors parallel to the x-axis, the y-axis and the z-axis and in the direction of x increasing, y increasing and z increasing. Respectively

- The vector $x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, may be written as a column matrix $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$.

- The modulus (or magnitude) of $x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ is $\sqrt{x^2 + y^2 + z^2}$

Example 11

The points A and B have position vectors $4\mathbf{i} + 2\mathbf{j} + 7\mathbf{k}$ and $3\mathbf{i} + \mathbf{4j} - 1\mathbf{k}$ respectively, and O is the origin.

Find $\overrightarrow{|AB|}$ and show that $\triangle\, OAB$ is isosceles.

$$\overrightarrow{|OA|} = \mathbf{a} = \begin{pmatrix} 4 \\ 2 \\ 7 \end{pmatrix}, \quad \overrightarrow{|OB|} = \mathbf{b} = \begin{pmatrix} 3 \\ 4 \\ -1 \end{pmatrix},$$

Write down the position vectors

$$\overrightarrow{AB} = \mathbf{b} - \mathbf{a} = \begin{pmatrix} 3 \\ 4 \\ -1 \end{pmatrix} - \begin{pmatrix} 4 \\ 2 \\ 7 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \\ -8 \end{pmatrix}$$

Use $\overrightarrow{AB} = \mathbf{b} - \mathbf{a}$

$$|\overrightarrow{AB}| = \sqrt{(-1)^2 + 2^2 + (-8)^2} = \sqrt{69}$$
$$|\overrightarrow{OA}| = \sqrt{4^2 + 2^2 + 7^2} = \sqrt{69}$$
$$|\overrightarrow{OB}| = \sqrt{3^2 + 4 + (-1)^2} = \sqrt{26}$$

Use the vector magnitude formula

Find the lengths of OA and OB

So $\triangle\, OAB$ is isosceles, with $AB = OA$

9.1 Learning objectives

You should now be able to

9.1.1 Know the difference between scalar and vector quantities

9.1.2 Construct a simple vector diagram and apply the triangle law

9.1.3 Find the magnitude (modulus) of a vector

9.1.4 Find a unit vector

9.1.5 Perform simple vector arithmetic

9.1.6 Solve simple geometric problems in vector diagrams

9.1.7 Use vectors to describe the location of a point in a 2 and 3 dimensional Cartesian framework

9.1.8 Use the Cartesian component form of a vector in 2 and 3 dimensions

9.1.9 Extend 2 dimensional operations such as modulus, to 3 dimensions

Figure references

Some of the figures in the slides for this unit, as listed in the table below

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have been reproduced from the following text book series:

Attwood, G., Macpherson, A., Moran, B., Petran, J., Pledger, K., Staley, G. and Wilkins, D. (2008), Edexcel AS and A Level Modular Mathematics series C1-C4, Pearson, Harlow, UK.

You may wish to refer to these text books for further information, examples and practice questions.

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