# Lectures on Quantum Decoherence

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# Lecture 1 – Basic Concepts

## 1.1 Introducing decoherence

Classical physics has led us to the strategy of dealing with idealised isolated systems. This has been very successful and got incorporated into quantum mechanics without much scrutiny.

But isolated quantum systems have been an obstacle in understanding the quantum-to-classical transition. We therefore require open quantum systems, not as an add-on or a 'fix', but as an essential and more complete description of many systems of interest today, from quantum optics to quantum computing, and even (possibly) quantum biology.

### On locality

Quantum mechanics is a local theory (in the sense that all interactions are local and so there is no *physical* action at a distance). But, the states that can be generated by these local interactions are distinctly *non-local* due to entanglement. We talk about this in terms of non-local correlations.

*Entanglement* means the outside environment no longer just perturbs the system, but rather *defines* the observable physical properties of the system.

Entanglement with an environment does two things:

1. It causes irreversible loss of coherence from the quantum system
2. It limits/selects a small number of possible observables

**Quantum decoherence (loss of information) is a separate process from classical dissipation (loss of energy).**

For macroscopic bodies, decoherence is incredible fast, and it happens everywhere. Even the CMB will cause decoherence.

The first paper on decoherence was by Zeh [Zeh, H.D., "On the interpretation of measurement in quantum theory", *Found Phys* **1**, 69–76 (1970)]. That was over half a century ago and yet standard textbooks on quantum mechanics still largely ignore the subject, claiming (correctly) that the measurement of physical quantities (observables) are represented by Hermitian operators, but that these somehow *instantaneously* change the quantum state of the system into one of the eigenstates of that operator. We will see in these lectures that that is not the case, but rather that measurement involves several stages, and that it is a dynamical process.

Decoherence does not destroy superpositions, it simply extends them to include the environment. So, while decoherence is something that happens to the system of interest, what is really going on is that the environment is encoding, via quantum correlations, information about the system.

## 1.2 Quantum entanglement

Consider a quantum system, $S$, described by state vector $|\Psi\rangle$. It is composed of two subsystems (so, it is called a bipartite quantum system): $S_1$ and $S_2$, with state vectors $|\Psi_1\rangle$, $|\Psi_2\rangle$, respectively.

If $S$ cannot be split into a tensor product of the states of $S_1$ and $S_2$, i.e. if $|\Psi\rangle \neq |\psi\rangle_1 \otimes |\psi\rangle_2$, then we say that $S$ is entangled with respect to $S_1$ and $S_2$.

[Note: we will be using the shorthand notation $|\psi\rangle_1 \otimes |\psi\rangle_2 \equiv |\psi\rangle_1 |\psi\rangle_2$.]

Consider for example that $S_1$ and $S_2$ are two spin-$\frac{1}{2}$ particles described by basis states $|0\rangle$ and $|1\rangle$.

That is, $|0\rangle_1$ corresponds to particle 1 pointing in the spin up direction with respect to some axis, say the z-axis, and $|1\rangle_1$ corresponds to particle 1 pointing spin down along the same axis. Thus,

$$\sigma_z |0\rangle = |0\rangle \qquad \text{and} \qquad \sigma_z |1\rangle = -|1\rangle \ , \tag{1.1}$$

where $\sigma_z$ is the Paul spin operator. If $|\psi\rangle_1$ and $|\psi\rangle_2$ are each in a superposition of these spin states, then we can write

$$|\psi\rangle_1 = \alpha_1 |0\rangle_1 + \beta_1 |1\rangle_1 \ , \tag{1.2}$$

$$|\psi\rangle_2 = \alpha_2 |0\rangle_2 + \beta_2 |1\rangle_2 \ . \tag{1.3}$$

Note that we would normally write these two wave functions with the subscripts inside the kets (that is, $|\psi_1\rangle$ and $|\psi_2\rangle$), but I am keeping them outside for clarity later on.

If the particles are <u>not</u> initially entangled, then their combined system is

$$|\Psi\rangle = |\psi\rangle_1 |\psi\rangle_2 = \Big(\alpha_1 |0\rangle_1 + \beta_1 |1\rangle_1\Big)\Big(\alpha_2 |0\rangle_2 + \beta_2 |1\rangle_2\Big) \ . \tag{1.4}$$

Let us keep things simple and assume both particles are in 50/50 superpositions of being spin up and down. So,

$$|\Psi\rangle = \frac{1}{\sqrt{2}}\Big(|0\rangle_1 + |1\rangle_1\Big) \cdot \frac{1}{\sqrt{2}}\Big(|0\rangle_2 + |1\rangle_2\Big) \ . \tag{1.5}$$

Clearly, here the two particles' superpositions are not entangled and they are each described by separate (uncorrelated) quantum states (wave functions). The opposite extreme is if they are *maximally entangled*. This is called a **Bell state**:

$$|\Psi\rangle = \frac{1}{\sqrt{2}}\Big(|0\rangle_1 |0\rangle_2 + |1\rangle_1 |1\rangle_2\Big) \ , \tag{1.6}$$

which says that if particle 1 is spin up, then so is particle 2, and vice versa.

Note that they could also be maximally entangled such that they always have opposite spin and the sign between them could be a plus or a minus. These would still be Bell states:

$$|\Phi\rangle_\pm = \frac{1}{\sqrt{2}}\Big(|0\rangle_1 |1\rangle_2 \pm |0\rangle_1 |1\rangle_2\Big) \ . \tag{1.7}$$

The degree of entanglement between the two particles could be anything between unentangled (Eq.(1.5)) and fully entangled (eqs.(1.6) or (1.7)).

What is the difference between these two extremes when it comes to measurement?

<u>Answer</u>:
If we measure the spin of particle 2 in a Bell state and find it to have spin up ($|0\rangle_2$ then this immediately tells us the spin of particle 1. But for an uncorrelated (unentangled) pair, measuring the spin of particle 2 tells us nothing about particle 1 and just leaves it in its original superposition.

Now here is the subtlety: What if the quantum states of particles 1 and 2 are each in superpositions of different basis states $\{|\psi_i\rangle\}$ and $\{|\phi_i\rangle\}$ , respectively, that do not necessarily form orthonormal sets like spin states. We will assume that they are in a general entangled Bell state defined as

$$|\Psi\rangle = \frac{1}{\sqrt{2}}\Big(|\psi_1\rangle\,|\phi_1\rangle + |\psi_2\rangle\,|\phi_2\rangle\Big)\ . \tag{1.8}$$

Consider particle 2: if $|\phi_1\rangle$ and $|\phi_2\rangle$ *are* orthogonal – for example eigenstates of an operator corresponding to some observable, $A$, and so macroscopically distinguishable (such as their spin in a magnetic field or their positions measured on a dial on a classical detector). Then a measurement of $A$ would yield an eigenvalue corresponding to one or other of them: $|\phi_1\rangle$ or $|\phi_2\rangle$. This will immediately tell us the state of particle 1 because one of the two terms in Eq.(1.8) will disappear.

However, if $|\phi_1\rangle_2$ and $|\phi_2\rangle_2$ are not orthogonal – that is they have non-zero overlap – then we will only get partial information on particle 1. In fact, if they overlap completely then this means that $|\phi_1\rangle$ and $|\phi_2\rangle$ are the same and can be factorised out:

$$|\Psi\rangle = \frac{1}{\sqrt{2}}\Big(|\psi_1\rangle + |\psi_2\rangle\Big)\,|\phi_1\rangle\ , \tag{1.9}$$

and so we learn nothing about particle 1.

Thus the more the elements of the superposition of particle 2 overlap, the less distinguishable they are and the less information we get about particle 1 by measuring particle 2.

## 1.3 The density matrix

For a pure state $|\Psi\rangle$, the density matrix operator (or just 'density matrix') is a very useful quantity if we wish to go beyond the unitary dynamics of the Schrödinger equation to describe open quantum systems, mixed states, ensembles and the measurement process. It is defined as

$$\hat{\rho} = |\Psi\rangle\langle\Psi|\ . \tag{1.10}$$

If $|\Psi\rangle$ is expressed as a superposition states in some orthonormal basis

$$|\Psi\rangle = \sum_i c_i\,|\psi_i\rangle\ , \tag{1.11}$$

then

$$\hat{\rho} = \sum_{ij} c_i c_j^* \left|\psi_i\right\rangle\left\langle\psi_j\right| \; . \tag{1.12}$$

Let us consider the simple case of $i, j = 1, 2$ (eg spin states of a spin-1/2 particle or a particle in a superposition of two energy states) then we can easily expand out the sums to get four terms:

$$\hat{\rho} = |c_1|^2 \left|\psi_1\right\rangle\left\langle\psi_1\right| + |c_2|^2 \left|\psi_2\right\rangle\left\langle\psi_2\right| + \underbrace{c_1 c_2^* \left|\psi_1\right\rangle\left\langle\psi_2\right| + c_1^* c_2 \left|\psi_2\right\rangle\left\langle\psi_1\right|}_{\text{interference terms}} \; . \tag{1.13}$$

We can see here that we can write $\hat{\rho}$ as a $2 \times 2$ matrix in the $\{\left|\psi_i\right\rangle\}$ basis (which has matrix elements $\left\langle\psi_i\right| \hat{\rho} \left|\psi_j\right\rangle$) to obtain (since $\left\langle\psi_i\right|\left.\psi_j\right\rangle = \delta_{ij}$):

$$\begin{pmatrix} |c_1|^2 & c_1 c_2^* \\ c_1^* c_2 & |c_2|^2 \end{pmatrix}$$

In this sense you can see that the interference terms are the off-diagonal ones in the matrix. Often, we talk about decoherence (or measurement) destroying the interference terms in Eq.(1.13) or the off-diagonal elements in the matrix, leaving just the diagonal elements. We will see later that this is what is referred to as a mixed state density matrix.

Note however that whether or not there is any interference (non-zero off-diagonal elements in $\hat{\rho}$) depends on our choice of basis. **Just because a density matrix is diagonal is some basis does *not* mean the system is behaving classically.**

## The Trace of $\hat{\rho}$

Unlike the above statement in bold, the *trace* of density matrix is independent of the basis chosen when we write it as an actual matrix (rather than just an operator). To see this, let's do a simple example:

Consider a state that is in a superposition of spin up and down with respect to the z-axis:

$$\left|\Psi\right\rangle = \alpha \left|0\right\rangle_z + \beta \left|1\right\rangle_z \; . \tag{1.14}$$

Clearly, our density matrix operator is similar to that in Eq.(1.13):

$$\begin{aligned} \hat{\rho} &= \left|\Psi\right\rangle\left\langle\Psi\right| \\ &= |\alpha|^2 \left|0\right\rangle_z \left\langle 0\right|_z + |\beta|^2 \left|1\right\rangle_z \left\langle 1\right|_z + \alpha\beta^* \left|0\right\rangle_z \left\langle 1\right|_z + \alpha^*\beta \left|1\right\rangle_z \left\langle 0\right|_z . \end{aligned} \tag{1.15}$$

But we can always write our two eigenvector $|0\rangle_z$ and $|1\rangle_z$ in terms of superpositions of eigenstates of $\sigma_x$:

$$|0\rangle_z = \frac{1}{\sqrt{2}}\Big(|0\rangle_x + |1\rangle_x\Big)\,, \tag{1.16}$$

$$|1\rangle_z = \frac{1}{\sqrt{2}}\Big(|0\rangle_x - |1\rangle_x\Big)\,. \tag{1.17}$$

Substituting these into Eq.(1.14),

$$\begin{aligned} |\Psi\rangle &= \frac{\alpha}{\sqrt{2}}\Big(|0\rangle_x + |1\rangle_x\Big) + \frac{\beta}{\sqrt{2}}\Big(|0\rangle_x - |1\rangle_x\Big) && (1.18)\\ &= \frac{1}{\sqrt{2}}\Big[(\alpha+\beta)\,|0\rangle_x + (\alpha-\beta)\,|1\rangle_x\Big] && (1.19) \end{aligned}$$

and the density matrix is then

$$\begin{aligned} \hat{\rho} = |\Psi\rangle\langle\Psi| = \frac{1}{2}\Big[ & |\alpha+\beta|^2\,|0\rangle_x\langle 0|_x + |\alpha-\beta|^2\,|1\rangle_x\langle 1|_x \\ & + (\alpha+\beta)(\alpha-\beta)^*\,|0\rangle_x\langle 1|_x \\ & + (\alpha+\beta)^*(\alpha-\beta)\,|1\rangle_x\langle 0|_x \Big]\,. \end{aligned} \tag{1.20}$$

Now let us evaluate the trace of the density matrix in the two bases:

In the basis of $\sigma_z$ eigenstates, it is trivial:

$$\mathrm{Tr}(\hat{\rho}) = \langle 0|_z\,\hat{\rho}\,|0\rangle_z + \langle 1|_z\,\hat{\rho}\,|1\rangle_z = |\alpha|^2 + |\beta|^2\,. \tag{1.21}$$

And in the basis of $\sigma_x$ eigenstates:

$$\begin{aligned} \mathrm{Tr}(\hat{\rho}) &= \langle 0|_x\,\hat{\rho}\,|0\rangle_x + \langle 1|_x\,\hat{\rho}\,|1\rangle_x \\ &= \frac{1}{2}|\alpha+\beta|^2 + \frac{1}{2}|\alpha-\beta|^2 \\ &= \frac{1}{2}|\alpha|^2 + \frac{1}{2}|\beta|^2 + \cancel{\frac{1}{2}\alpha\beta^*} + \cancel{\frac{1}{2}\alpha^*\beta} \\ &\quad + \frac{1}{2}|\alpha|^2 + \frac{1}{2}|\beta|^2 - \cancel{\frac{1}{2}\alpha\beta^*} - \cancel{\frac{1}{2}\alpha^*\beta} \\ &= |\alpha|^2 + |\beta|^2 \end{aligned} \tag{1.22}$$

A simpler way to see that $\mathrm{Tr}(\hat{\rho})$ is basis independent is look at what its diagonal elements represent: they are simply the probabilities of getting the different outcomes (eigenvalues) of a particular observable whose eigenstates make up that particular basis, and their sum must add up to 1. It doesn't matter what basis we choose the trace is just a sum of probabilities and we have

$$\boxed{\mathrm{Tr}(\hat{\rho}) = 1} \tag{1.23}$$

### The trace as an expectation value

Consider some observable, $O$, with corresponding operator $\hat{O}$, and let us derive the quantity $\mathrm{Tr}(\hat{\rho}\hat{O})$. We choose an orthonormal basis for the matrix corresponding to the product $\hat{\rho}\hat{O}$ as the basis of eigenstates $\{|\psi_i\rangle\}$ of $\hat{O}$:

$$\hat{O}\,|\psi_i\rangle = \alpha_i\,|\psi_i\rangle\ . \tag{1.24}$$

Thus,

$$\begin{aligned} \mathrm{Tr}(\hat{\rho}\hat{O}) &= \sum_i \langle\psi_i|\,\hat{\rho}\hat{O}\,|\psi_i\rangle = \sum_i \langle\psi_i|\,\hat{\rho}\;\alpha_i\,|\psi_i\rangle \\ &= \sum_i \langle\psi_i|\,|\Psi\rangle\langle\Psi|\,\alpha_i\,|\psi_i\rangle = \sum_i \alpha_i|\langle\psi_i|\,|\Psi\rangle\,|^2\ . \end{aligned} \tag{1.25}$$

However, imagine expanding the wave function $|\Psi\rangle$ in the basis of $|\psi_i\rangle$: ($|\Psi\rangle = \sum_i c_i\,|\psi_i\rangle$). Then the coefficients, $c_i$, are just the overlap amplitude $\langle\psi_i|\,|\Psi\rangle$, and $|c_i|^2$ is just the probability of finding a system described by $|\Psi\rangle$ with eigenvalue $\alpha_i$ (that is, in the eigenstate $|\psi_i\rangle$) if we were to measure observable $O$.

So we can instead write our trace as

$$\mathrm{Tr}(\hat{\rho}\hat{O}) = \sum_i \alpha_i|c_i|^2\ . \tag{1.26}$$

But you should recall (yes, you really should) that this is simply the definition of the expectation value of an operator (the average of all the eigenvalues, each weighted by the probability of getting it on measurement).

$$\left[\langle\hat{O}\rangle = \langle\Psi|\,\hat{O}\,|\Psi\rangle = \textstyle\sum_{ij} c_i^*\,\langle\psi_i|\,\hat{O}\;c_j\,|\psi_j\rangle = \sum_{ij} c_i^* c_j\,\langle\psi_i|\,\alpha_j\,|\psi_j\rangle = \sum_i \alpha_i|c_i|^2\right].$$

We have shown the very useful relation involving the density matrix:

$$\boxed{\langle\hat{O}\rangle\ = \mathrm{Tr}(\hat{\rho}\hat{O})} \tag{1.27}$$

# 2

# Lecture 2 – Mixed States

## 2.1 Defining mixed states

Consider a system that is prepared in a pure state (i.e. one that is described by a single wave function), but we don't know what this pure state is. It might be $|\Psi_1\rangle$ or it might be $|\Psi_2\rangle$ with (let us assume) equal probability (where here the probability reflects our ignorance rather than a property of the system itself). We express this ignorance as a mixed state density matrix

$$\hat{\rho} = \frac{1}{2} \underbrace{|\Psi_1\rangle\langle\Psi_1|}_{\hat{\rho}_1} + \frac{1}{2} \underbrace{|\Psi_2\rangle\langle\Psi_2|}_{\hat{\rho}_2} , \tag{2.1}$$

where the factors of $\frac{1}{2}$ are classical probabilities. This is a true mixed state density matrix (to be compared with an 'improper' mixed state density matrix we will encounter in the next lecture). Note we can also use it to describe an ensemble of systems rather than just one, where half of them are in pure state $|\Psi_1\rangle$ and other half are in pure state $|\Psi_2\rangle$.

In general, we can write a density matrix as

$$\boxed{\hat{\rho} = \sum_i p_i \, |\Psi_i\rangle\langle\Psi_i|} , \tag{2.2}$$

which can now describe a mixed state. Again, this can either describe a single particle/system that is in some pure state $|\Psi_i\rangle$, but we just don't know which one (and the $p_i$s reflect our ignorance), or of an ensemble of many particles/systems, some of which are in one pure state while others are in different pure states, where the $p_i$s now reflect the fraction of the ensemble population in any given pure state.

Note that if $\hat{\rho}$ is a pure state density matrix then there is just one value of $p$ that is unity and we just have $\hat{\rho} = |\Psi\rangle\langle\Psi|$ (see Eq.(1.10)).

What about the expectation value of some operator $\langle\hat{O}\rangle$ now that we are dealing with the more general case of mixed states?
That is, can we still write it as the trace of the product of a mixed density matrix and $\hat{O}$? Does Eq.(1.28) still hold for mixed states?

It turns out that the answer is yes, and this is quite straightforward to prove:

Consider the mixed state density matrix in Eq.(2.1). If it is really in state $|\Psi_1\rangle$ then the expectation value of an operator, $\hat{O}$ is

$$\langle\hat{O}\rangle_1 = \langle\Psi_1|\,\hat{O}\,|\Psi_1\rangle = \mathrm{Tr}(\hat{\rho}_1\hat{O})\ . \tag{2.3}$$

But likewise if it is actually in state $|\Psi_2\rangle$ then the expectation value is

$$\langle\hat{O}\rangle_2 = \langle\Psi_2|\,\hat{O}\,|\Psi_2\rangle = \mathrm{Tr}(\hat{\rho}_2\hat{O})\ . \tag{2.4}$$

Since we don't know which state it is in, the classical probabilities of Eq.(2.1) are reflected in the average expectation value

$$\begin{aligned}\langle\hat{O}\rangle &= \tfrac{1}{2}\langle\hat{O}\rangle_1 + \tfrac{1}{2}\langle\hat{O}\rangle_2\\ &= \tfrac{1}{2}\big(\mathrm{Tr}(\hat{\rho}_1\hat{O}) + \mathrm{Tr}(\hat{\rho}_2\hat{O})\big)\\ &= \mathrm{Tr}\big(\tfrac{1}{2}\hat{\rho}_1\hat{O}\big) + \mathrm{Tr}\big(\tfrac{1}{2}\hat{\rho}_2\hat{O}\big)\\ &= \mathrm{Tr}\big((\tfrac{1}{2}\hat{\rho}_1 + \tfrac{1}{2}\hat{\rho}_2)\hat{O}\big)\\ &= \mathrm{Tr}(\hat{\rho}\hat{O})\ . \end{aligned}\tag{2.5}$$

So, yes, even though $\hat{\rho}$ here is a mixed state density matrix, the expectation value formula still works.

$\Big[$ Similarly, even for a mixed $\hat{\rho}$, $\mathrm{Tr}(\hat{\rho}) = 1$, since $\mathrm{Tr}(\hat{\rho}_1) = \mathrm{Tr}(\hat{\rho}_2) = 1$. $\Big]$

## 2.2 Mixed states as ensembles

We've said that the definition of a general density matrix in Eq.(2.2) means it may have been prepared in some pure state but we just don't know which one. However, if we have an *ensemble* of identical systems (particles) with some of them prepared in pure state $|\Psi_1\rangle$ and some in pure state $|\Psi_2\rangle$, etc, then the '$p_i$'s are now no longer probabilities, but tell us what fraction of the ensemble is in state $|\Psi_i\rangle$. Often, this ensemble interpretation is what is assumed when we talk about density matrices.

## 2.3 The difference between pure and mixed states

Recall that if a system described by $|\Psi\rangle$ is made up of two entangled subsystems then we *cannot* write them as a tensor product: $|\Psi_1\rangle \otimes |\Psi_2\rangle$. Similarly, we cannot write their density matrices as $\hat{\rho}_1 \otimes \hat{\rho}_2$.

Thus, if we start off at $t = 0$ with $\hat{\rho}(0) = \hat{\rho}_1(0) \otimes \hat{\rho}_2(0)$, then we say that subsystems 1 and 2 are initially uncorrelated (not entangled). Also, both $\hat{\rho}_1(0)$ and $\hat{\rho}_2(0)$ describe pure states and the overall $\hat{\rho}(0)$ will also be a pure state. But, as systems 1 and 2 become entangled with each other they will individually become mixed states in the sense that we can no longer define either of them by a state vector (wave function). However, there is no reason why the overall combined system described by $\hat{\rho}(t)$ cannot remain in a pure state.

If the combined system's state vector is expanded in some basis (repeating Eq.(1.11)):

$$|\Psi\rangle = \sum_i c_i |\psi_i\rangle \ , \tag{2.6}$$

then the corresponding, pure, density matrix (Eq.1.12) is

$$\hat{\rho}_{\text{pure}} = |\Psi\rangle\langle\Psi| = \sum_{ij} c_i c_j^* |\psi_i\rangle\langle\psi_j| \ . \tag{2.7}$$

See how the right hand side above differs from the sum in Eq.(2.2). Here we are summing over two indices and $c_i c_j^*$ is not necessarily a real probability. In fact, we can rewrite (2.7) as

$$\hat{\rho}_{\text{pure}} = \underbrace{\sum_i |c_i|^2 |\psi_i\rangle\langle\psi_i|}_{\text{diagonal}} + \underbrace{\sum_{j\neq i} c_i c_j^* |\psi_i\rangle\langle\psi_j|}_{\text{offdiagonal}} \ . \tag{2.8}$$

You can now see that the first term (corresponding to just the diagonal terms in the density matrix) looks like a mixed state, whereas the second term is the interference and gives the off-diagonal elements. It is this second term that disappears during decoherence, turning a pure state (quantum) with off-diagonal elements present to a mixed state (classical) with diagonal elements only.

For a pure state

$$\hat{\rho}^2 = \big( |\Psi\rangle\langle\Psi| \big)^2 = |\Psi\rangle\langle\Psi|\Psi\rangle\langle\Psi| = |\Psi\rangle\langle\Psi| = \hat{\rho}\ , \tag{2.9}$$

(provided of course that $|\Psi\rangle$ is normalised).

In fact, $\boxed{\hat{\rho}^2 = \rho}$ is a <u>definition</u> of a pure state.

But this relation does not hold for a mixed state,

$$\rho^2 = \big( \sum_i p_i\, |\Psi_i\rangle\langle\Psi_i| \big)^2 = \sum_{ij} p_i p_j\, |\Psi_i\rangle\langle\Psi_i|\Psi_j\rangle\langle\Psi_j|\ . \tag{2.10}$$

We have two options:

(a) if $\langle\Psi_i|\Psi_j\rangle \neq 0$ (and why should they be orthogonal?) then Eq.(2.10) cannot be simplified and we cannot have $\hat{\rho}^2 = \rho$;

(b) Even if $\langle\Psi_i|\Psi_j\rangle = \delta_{ij}$ then

$$\rho^2 = \sum_i p_i^2\, |\Psi_i\rangle\langle\Psi_i|\ , \tag{2.11}$$

which is compared with our definition from earlier:

$$\rho = \sum_i p_i\, |\Psi_i\rangle\langle\Psi_i|\ . \tag{2.12}$$

So, again, $\hat{\rho}^2 \neq \rho$

Therefore, $\boxed{\hat{\rho}^2 \neq \rho \qquad \text{for a mixed state}}$.

## 2.4 Quantifying 'mixedness'

There are two simple measures of purity (or mixedness) of a density matrix. Both of these quantities measure the amount of entanglement between the system and the environment as the system progressively decoheres and evolves from an initially pure state to a final mixed state:

1. **Purity**: $\zeta = \mathrm{Tr}(\hat{\rho}^2)$

   For a pure state, clearly

$$\zeta = \mathrm{Tr}(\hat{\rho}^2) = \mathrm{Tr}(\hat{\rho}) = 1\ . \tag{2.13}$$

The opposite extreme is a maximally mixed state of the form of Eq.(2.2) in which the different states $\{|\Psi_i\rangle\}$ form an orthonormal basis in the system's Hilbert space, then from Eq.(2.11) we see that

$$\zeta = \sum_i^N p_i^2 \,, \tag{2.14}$$

and it can easily be shown that the minimum value the purity can be is $\frac{1}{N}$.

Proof:

$$\text{Since} \qquad \sum_i^N p_i = 1$$

then for maximum mixing (minimum purity in Eq.(2.14)) we would want to distribute the probabilities equally

$$p_1 = p_2 = p_3 = \cdots = p_N = \frac{1}{N}$$

Therefore

$$\zeta = N \times \big(\frac{1}{N}\big)^2 = \frac{1}{N} \,.$$

2. **Von Neumann entropy**: $S(\hat{\rho})$

   This is defined as

$$\boxed{S(\hat{\rho}) = -\mathrm{Tr}\big(\hat{\rho}\log_2\hat{\rho}\big)} \,. \tag{2.15}$$

   It can also be written in the form:

$$\boxed{S(\hat{\rho}) = -\sum_i \lambda_i \log_2 \lambda_i} \,, \tag{2.16}$$

   where $\lambda_i$ are the eigenvalues of $\hat{\rho}$. Note that this is a non-trivial statement. It follows from the fact that the trace of any matrix, $A$, is equal to the sum of its eigenvalues. We will not prove that here as it would take a little longer than these lectures have time for. Suffice it to say that it relies on properties of matrices involving what are called upper triangular matrices and the Schur decomposition. You can look it up if you want.

Anyway, if a pure state density matrix, $\hat{\rho} = |\Psi\rangle\langle\Psi|$, acts on the state vector $|\Psi\rangle$, then trivially

$$\hat{\rho}\,|\Psi\rangle = |\Psi\rangle\langle\Psi|\,\Psi\rangle = |\Psi\rangle \ , \qquad (2.17)$$

which is an eigenvalue equation with eigenvalue of $\hat{\rho}$ being $\lambda = 1$.

Therefore

$$S(\hat{\rho}_{\text{pure}}) = -1\ \log_2 1 = 0\,. \qquad (2.18)$$

That is, the von Neumann entropy of a pure state is zero.

But for the maximally mixed case, with $\{|\Psi_i\rangle\}$ forming an orthonormal set, we can see that they are the eigenvectors of $\hat{\rho}$:

$$\hat{\rho}\,|\Psi_i\rangle = \sum_j p_j\,|\Psi_j\rangle\langle\Psi_j|\,\Psi_i\rangle = p_i\,|\Psi_i\rangle \ , \qquad (2.19)$$

and in this basis, $\hat{\rho}$ will be a diagonal matrix whose eigenvalues, $\lambda_i$, are just the classical probabilities, $p_i$:

$$\begin{pmatrix} p1 & & & & & \\ & p_2 & & \bigcirc & & \\ & & p_3 & & & \\ & \bigcirc & & \bullet & & \\ & & & & \bullet & \\ & & & & & \bullet \end{pmatrix}$$

Therefore, using the definition of the entropy in Eq.(2.16)

$$\begin{aligned} S(\hat{\rho}_{\text{mixed}}) &= -\sum_i \frac{1}{N}\log_2\left(\frac{1}{N}\right) \\ &= -N \times \frac{1}{N}\big(\log_2 1 - \log_2 N\big) = \log_2 N\,. \qquad (2.20) \end{aligned}$$

Thus the von Neumann entropy ranges from zero (for a pure state) up to a maximum of $\log_2 N$ for a mixed state. For just two mixed states, $S = 1$, for three, $S = 1.58$, for four, $S = 2$, and so on. The more states make up $\hat{\rho}$, the more scope for entanglement and the higher the entropy.

So far, we've said that the probabilities in the definition of $\hat{\rho}_{\text{mixed}}$ are just a measure of our ignorance regarding which *pure* state, $|\Psi_i\rangle$, the system is really in. But, **just writing our mixed-state density matrix in the form of**

**Eq.(2.2) does not mean that it is actually in one of those pure states. We also need to *know* that it was indeed prepared in one of those states**.

Example

Consider a spin state that has previously been measured by a detector (prepared) along the $z$-axis. It is thus in a pure state of either $|0\rangle_z$ or $|1\rangle_z$. If we don't know which of the states it is in, we must describe it by a mixed state,

$$\hat\rho = \tfrac{1}{2}\,|0\rangle_z\,\langle 0|_z + \tfrac{1}{2}\,|1\rangle_z\,\langle 1|_z \ . \tag{2.21}$$

But, as we said in the last lecture in Eqs.(1.16) and (1.17), each of $|0_z\rangle$ and $|1_z\rangle$ can itself be written as a linear superposition of eigenstates of $\sigma_x$. Therefore

$$\begin{aligned}
\hat\rho &= \frac{1}{2}\cdot\frac{1}{\sqrt{2}}\Big(|0\rangle_x + |1\rangle_x\Big)\cdot\frac{1}{\sqrt{2}}\Big(\langle 0|_x + \langle 1|_x\Big) + \frac{1}{\sqrt{2}}\Big(|0\rangle_x - |1\rangle_x\Big)\cdot\frac{1}{\sqrt{2}}\Big(\langle 0|_x - \langle 1|_x\Big)\\
&= \frac{1}{4}\Big[|0\rangle_x\,\langle 0|_x + \cancel{|0\rangle_x\,\langle 1|_x} + \cancel{|1\rangle_x\,\langle 0|_x} + |1\rangle_x\,\langle 1|_x\\
&\qquad\qquad + |0\rangle_x\,\langle 0|_x - \cancel{|0\rangle_x\,\langle 1|_x} - \cancel{|1\rangle_x\,\langle 0|_x} + |1\rangle_x\,\langle 1|_x\Big]\\
&= \frac{1}{2}\,|0\rangle_x\,\langle 0|_x + \frac{1}{2}\,|1\rangle_x\,\langle 1|_x \ .
\end{aligned} \tag{2.22}$$

This looks just like Eq.(2.21), but now it no longer reflects our ignorance of which pure state the system is in. A mixed-state density matrix can be written in any number of ways and its partition into a set of states is arbitrary. So, unless we know the physical axis along which the spin state has been prepared, then the density matrix alone can only give us probabilities of *different sets* of pure states. Thus, interpreting a mixed-state density matrix simply as a measure of our ignorance of what pure state the system is not reliable.

## 3

# Lecture 3 – Reduced density matrices

## 3.1 Definition

Consider a system, $S$, made of two entangled parts, $A$, $B$, and that everything we can know about $S$ comes from the measurement of some property of $A$ alone.

This means we need a quantity that describes what can know about $A$. However, since $A$ and $B$ are entangled we cannot separate them. Instead, we calculate what is called the *reduced density matrix* of $A$ by averaging over the degrees of freedom of $B$. The way we do this is to start with the full density matrix of $S$, $\hat{\rho}_S$. [Note, sometimes it is called $\hat{\rho}_{AB}$.] Remember that since $A$ and $B$ are entangled,

$$\hat{\rho}_S \neq \hat{\rho}_A \otimes \hat{\rho}_B \ . \tag{3.1}$$

The reduced density matrix of $A$ is defined as

$$\tilde{\rho}_A = \mathrm{Tr}_B(\hat{\rho}_S) \ . \tag{3.2}$$

Note that if $A$ and $B$ were not entangled (uncorrelated) then $\hat{\rho}_S = \hat{\rho}_A \otimes \hat{\rho}_B$ and we could write

$$\mathrm{Tr}_B(\hat{\rho}_S) = \mathrm{Tr}_B(\hat{\rho}_A \otimes \hat{\rho}_B) = \mathrm{Tr}_B(\hat{\rho}_A)\mathrm{Tr}_B(\hat{\rho}_B) \ . \tag{3.3}$$

But we cannot do this now and have to derive the reduced density matrix instead.

## 3.2 Deriving the reduced $\hat{\rho}$

Consider our overall (pure) state $S$ made up of entangled states $A$ and $B$, each of which, were we able to disentangle them, being in a superposition of two equal amplitude states, $|a_1\rangle\,, |a_2\rangle$, and $|b_1\rangle\,, |b_2\rangle$, respectively. The overall state of $S$ is

$$|\Psi_S\rangle = \frac{1}{\sqrt{2}}\big(\,|a_1\rangle\,|b_1\rangle + |a_2\rangle\,|b_2\rangle\big)\ . \tag{3.4}$$

If these states were orthogonal then this would be a Bell state. Let us assume they are normalised ($\langle a_i|\,a_i\rangle = \langle b_i|\,b_i\rangle = 1$, but not orthogonal ($\langle a_1|\,a_2\rangle \neq 0$, $\langle b_1|\,b_2\rangle \neq 0$).

Therefore, the density matrix is

$$\begin{aligned}
\hat{\rho}_S &= |\Psi_S\rangle\langle\Psi_S| \\
&= \frac{1}{2}\big(\,|a_1\rangle\langle a_1|\otimes|b_1\rangle\langle b_1| + |a_1\rangle\langle a_2|\otimes|b_1\rangle\langle b_2| + \cdots\big) \\
&= \frac{1}{2}\sum_{i,j}^{2}|a_i\rangle\langle a_j|\otimes|b_i\rangle\langle b_j|\ .
\end{aligned} \tag{3.5}$$

Note that we have had to put back in the tensor product symbol, $\otimes$, since without it the terms would look like they had overlap amplitudes $\langle a_j|\,b_i\rangle$. Also, take a moment to compare the structure of the density matrix above with that of the state vector in Eq.(3.4).

Therefore, the reduced density matrix of $A$ is

$$\begin{aligned}
\tilde{\rho}_A &= \mathrm{Tr}_B\left(\frac{1}{2}\sum_{i,j}^{2}|a_i\rangle\langle a_j|\otimes|b_i\rangle\langle b_j|\right) \\
&= \frac{1}{2}\sum_{i,j}^{2}\Big[\,|a_i\rangle\langle a_j|\,\mathrm{Tr}_B\big(\,|b_i\rangle\langle b_j|\,\big)\Big]\ .
\end{aligned} \tag{3.6}$$

Now let us define an orthonormal basis in the Hilbert spaces $H_B$ as $\{|\phi_\ell\rangle\}$. We can therefore write

$$|b_i\rangle = \sum_\ell c_\ell^{(i)}\,|\phi_\ell\rangle\ , \tag{3.7}$$

and

$$\begin{aligned}
\tilde{\rho}_A &= \frac{1}{2}\sum_{i,j}^{2}\Bigg[\,|a_i\rangle\langle a_j|\overbrace{\sum_k\langle\phi_k|\,\big(\,|b_i\rangle\langle b_j|\,\big)\,|\phi_k\rangle}^{\text{def. of a trace}}\Bigg] \\
&= \frac{1}{2}\sum_{i,j}^{2}\Bigg[\,|a_i\rangle\langle a_j|\sum_k\langle\phi_k|\left(\sum_{\ell\ell'}c_\ell^{(i)}c_{\ell'}^{(j)*}\,|\phi_\ell\rangle\langle\phi_{\ell'}|\right)|\phi_k\rangle\Bigg] \\
&= \frac{1}{2}\sum_{i,j}^{2}\Bigg[\,|a_i\rangle\langle a_j|\sum_k c_k^{(i)}c_k^{(j)*}\Bigg]\ ,
\end{aligned} \tag{3.8}$$

But this is very nice because the product of the coefficients is just an overlap:

$$\langle b_j | b_i \rangle = \sum_{\ell\ell'} c_\ell^{(i)} c_{\ell'}^{(j)^*} \langle \phi_{\ell'} | \phi_\ell \rangle = \sum_\ell c_\ell^{(i)} c_\ell^{(j)^*} , \tag{3.9}$$

and so we can replace the coefficients with the overlap to give us our final expression for the reduced density matrix of $A$:

$$\boxed{\tilde{\rho}_A = \frac{1}{2} \sum_{i,j}^{2} |a_i\rangle \langle a_j| \langle b_j | b_i \rangle} . \tag{3.10}$$

Writing this out in full (the four terms) recalling that we said the $|b_i\rangle$ states were normalised but not necessarily orthogonal:

$$\begin{aligned} \tilde{\rho}_A \;=\;& \tfrac{1}{2} |a_1\rangle\langle a_1| + \tfrac{1}{2} |a_1\rangle\langle a_2| \langle b_2 | b_1\rangle \\ & + \tfrac{1}{2} |a_2\rangle\langle a_1| \langle b_1 | b_2\rangle + \tfrac{1}{2} |a_2\rangle\langle a_2| . \end{aligned} \tag{3.11}$$

Since this is the reduced density matrix for system $A$ we can see that the two middle terms are the interference terms (off-diagonals), whereas the first and last are the diagonal terms. If $|b_1\rangle$ and $|b_2\rangle$ are orthogonal then they kill off the interference terms. Note that for now we need only think of $B$ as another quantum system with few degrees of freedom. Later we will regards it as the environment of $A$ (or a measuring device) and saying that $|b_1\rangle$ and $|b_2\rangle$ are orthogonal amount to saying that they correspond to macroscopically distinguishable states. So killing the interference terms amounts to what we refer to as *decoherence*. However, for now, all we need say is orthogonality of the states making up $B$ mean that $\tilde{\rho}_A$ reduces to

$$\tilde{\rho}_A = \tfrac{1}{2} |a_1\rangle\langle a_1| + \tfrac{1}{2} |a_2\rangle\langle a_2| . \tag{3.12}$$

This looks just like the mixed state density matrix we first defined in Eq.(2.1) where the two terms terms represented different pure states that the system could be in but that we didn't know which one. The density matrix above may look similar, but it is a very different beast. It is *not* the case here that $A$ is in a pure state of either $|a_1\rangle$ or $|a_2\rangle$ and we just don't know which. Here, both states still exist and we refer to this reduced density matrix as an *improper mixed state*.

## 3.3 The measurement problem

In fact, there are really three measurement problems:

1. The problem of preferred basis
2. The non-observability of interference

3. The problem of outcomes

We will see that decoherence has solved problems (1) and (2), while the solution of problem (3) relies on our choice of interpretation and we will say little about this.

Let us lay out the problem simply and carefully.
Consider a quantum system of interest, $S$, with basis states $\{|s_i\rangle\}$. We also have some measuring apparatus (a detector), $A$, with basis states $\{|a_i\rangle\}$.

Start $A$ in some prepared (or "ready") state, $|a_r\rangle$.

If the system is in one of the basis states, $|s_i\rangle$, and if the detector does not disturb this state when it measures it then we have an *ideal* measurement:

$$|s_i\rangle\,|a_r\rangle \to |s_i\rangle\,|a_i\rangle\ . \tag{3.13}$$

But if $S$ starts in a superposition

$$|\psi\rangle = \sum_i c_i\,|s_i\rangle\ , \tag{3.14}$$

then
$$|\psi\rangle\,|a_r\rangle \to \underbrace{\sum_i c_i\,|s_i\rangle\,|a_i\rangle}_{|\Psi\rangle}\,, \tag{3.15}$$

where $|\Psi\rangle$ is the final *entangled* state.

Note that getting this entangled state, $|\Psi\rangle = c_1\,|s_1\rangle\,|a_1\rangle + c_2\,|s_2\rangle\,|a_2\rangle + \cdots$, is just a 'pre-measurement'. That is, we cannot say that a measurement has happened yet:
Entanglement of system and detector is not a measurement.

Now we can examine the three measurement problems:

## (1) The problem of preferred basis

Clearly, our system, $S$, could have been written as a superposition of any number of basis states

$$|\psi\rangle = \sum_i c_i\,|s_i\rangle = \sum_i c_i'\,|s_i'\rangle = \sum_i c_i''\,|s_i''\rangle = \cdots\ . \tag{3.16}$$

Now, for the detector to give us a classical measurement – that is, an eigenvalue that is macroscopically distinguishable from other eigenvalues of the observable being measured by the detector (different spins, positions, energies, etc) – then we require its basis states to be orthogonal: $\langle a_i | a_j \rangle = 0$ for $i \neq j$.

Recall from Lecture 1 (or Eq.(3.4)) that we have an entangled Bell state here (consider just two terms)

$$|\Psi\rangle = c_1 |s_1\rangle |a_1\rangle + c_2 |s_2\rangle |a_2\rangle \ , \tag{3.17}$$

then, when $|a_1\rangle$ and $|a_2\rangle$ are orthogonal (eigenstates of an observable measured by the detector and so macroscopically distinguishable), then the measurement would force the detector into one of these eigenstates, which immediately tells us the state of the system.

Of course, if states of the detector $A$ are not macroscopically distinguishable, then neither will the states of $S$ be. And that would be a rubbish detector!

[ NOTE: I will discuss this in more detail later, but just to flag it here, there is still a big problem in the foundations of QM about what we mean by macroscopic. Thus, what does macroscopically distinguishable mean here anyway? This problem is referred to as the 'Heisenberg cut'. To my mind, and to many philosophers of physics who have thought about this longer than I have, the idea of 'macroscopically distinguishable' is no less arbitrary than Bohr's irreversible act of amplification, or the arbitrary split between the quantum world and the classical world of detectors, observables and observers. This is what Bell refers to as the "shifty split" (J. Bell, Against Measurement, Physics World, August, 1990) between system and detector; or, if we insist on a combination of entangled system plus detector into one larger quantum system, then the split between this and the environment. But where does one end and the other start? It is arbitrary and subjective. Note, this is not the same issue as coarse graining in classical statistical mechanics. I will come back to this later.]

Given that $|\psi\rangle$ can be written in any one of an infinite number of basis state expansions (Eq.3.16), how then do we fix a unique basis? The answer of course is the detector, which must be prepared (set up in advance). For example, in a Stern-Gerlach apparatus, $A$ might be set up to measure spin along the $z$-axis.
This fixes the basis set of $A$: $\{|a_z\rangle\} = |0_z\rangle_A \, , \ |1_z\rangle_A$.

Therefore, while it is true that we could write $|\psi\rangle$ as

$$|\psi\rangle = \frac{1}{\sqrt{2}}\Big(|0_z\rangle_s + |1_z\rangle_s\Big) \tag{3.18}$$

or

$$|\psi\rangle = \frac{1}{\sqrt{2}}\Big(|0_x\rangle_s + |1_x\rangle_s\Big) \,, \tag{3.19}$$

the final entangled state with the detector would then be

$$|\Psi\rangle = \frac{1}{\sqrt{2}}\Big(|0_z\rangle_s\,|0_z\rangle_A + |1_z\rangle_s\,|1_z\rangle_A\Big) \tag{3.20}$$

or

$$|\Psi\rangle = \frac{1}{\sqrt{2}}\Big(|0_x\rangle_s\,|0_x\rangle_A + |1_x\rangle_s\,|1_x\rangle_A\Big) \,. \tag{3.21}$$

But this suggests $A$ could simultaneously measure observables corresponding to two non-commuting operators, $\hat{\sigma}_z$ em and $\hat{\sigma}_x$, which is impossible as the apparatus can only be set up in one way at a time.

Therefore, the choice of the set-up of $A$ uniquely fixes the basis of $S$ that we measure. And we can generalises this beyond spin directions of course to any non-commuting quantities: set up a detector to measure position picks out eigenstates of position and we measure position eigenvalues, and likewise if we set it up to measure momentum.

## 3.4 The non-observability of interference

This is solved by decoherence, which we will come to shortly.

## 3.5 The problem of outcomes

Here is where things get interesting if you enjoy the philosophy of physics, because Eq.(3.20) is now an entangled state. You might think that surely we've just jumped over decoherence and isn't that meant to solve the problem, changing the superposition of outcome 1 AND outcome 2 to the statistical mixture of outcome 1 OR outcome 2, provided the detector is classical (a la Bohr) or, more precisely, it's eigenstates are macroscopically distinguishable in order to cause decoherence? It turns out this is not what decoherence does. For even after decoherence, we still need to appeal to one or other of the various interpretations of quantum mechanics to solve the problem of how we end up getting just one of the possible outcomes. This requires a more careful assessment of the famous measurement problem.

## 4

# Lecture 4 – Measurement

To recap, the entangled combined state of system and detector is still in a superposition (*even after decoherence has taken place*).

$$|\psi\rangle = \sum_i c_i \, |s_i\rangle \, |a_i\rangle \ . \tag{4.1}$$

The issue is therefore that even if the detector states $|a_i\rangle$ are orthogonal (macroscopically distinguishable), this does not tell us *how* just one of the states of the system $|s_i\rangle$ is selected on measurement (the one we observers, or 'perceive'). This is why the Copenhagen school require the state to 'collapse', and the Everettian school require branching of the Universe, and why the 'spontaneous collapse model' is favoured by some.

In the language of Schrödinger's cat, we can think of the cat as being part of the extended detector before the box is opened:

$$|\Psi\rangle_{\text{Box}} = |\text{undecayed}\rangle_{\text{atom}} \, |\text{alive}\rangle_{\text{cat}} + |\text{decayed}\rangle_{\text{atom}} \, |\text{dead}\rangle_{\text{cat}} \ . \tag{4.2}$$

So it is not that the cat is in a superposition of Dead *and* Alive or even that it is either Dead *or* Alive, but rather that the box contains a superposition of the composite system of Cat + Atom. That is, the superposition of the atom is now *delocalised* and infects the larger system that includes everything inside the box, including the cat.

This is why it is useful to analyse the measurement problem in terms of reduced density matrices. When decoherence takes place and the reduced density matrix of the atom loses its diagonal terms, this does not mean that there is no longer any interference. As Joos and Zeh state: "the interference still exists, but it is not *there*" [Joos, E., Zeh, H.D. *The emergence of classical properties through interaction with the environment*, Z. Physik B - Condensed Matter **59**, 223–243 (1985)].

Here is a gentle reminder that, even though they look similar in structure, the diagonal/decohered reduced density matrix

$$\tilde{\rho} = \sum_i |c_i|^2 \, |s_i\rangle\langle s_i| \tag{4.3}$$

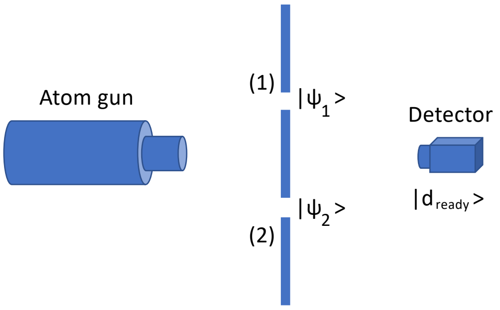
is not the same as a mixed (ensemble) density matrix

$$\rho = \sum_j p_j \, |\psi_j\rangle\langle\psi_j| \ , \tag{4.4}$$

where the state vectors, $|\psi_j\rangle$, and the basis states, $|s_i\rangle$, are related via Eq.(4.1). That is, the system is not in some pure state and we just don't know which one and therefore assign probabilities, but rather that all outcomes/options are still there.

## 4.1 The Two-Slit Experiment

Let us examine the problem in terms of the most famous example of all.



The detector starts off in the 'ready' state, $|d_{\text{ready}}\rangle$ and if the atom definitely goes through slit (1) only (for example if slit (2) is closed) then the combined system of atom plus detector evolves as $|\psi_1\rangle|d_{\text{ready}}\rangle \rightarrow |\psi_1\rangle|d_1\rangle$.

However, if both slits are open, but with no detector present, we will have interference:

$$|\Psi\rangle = \frac{1}{\sqrt{2}}\Big(|\psi_1\rangle + |\psi_2\rangle\Big) \ , \tag{4.5}$$

where $|\psi_{1,2}\rangle$ are the spatial wave functions emerging from the two slits, starting off for example as Gaussian wave packets and spreading outwards to the right of the slits until they overlap and interfere.

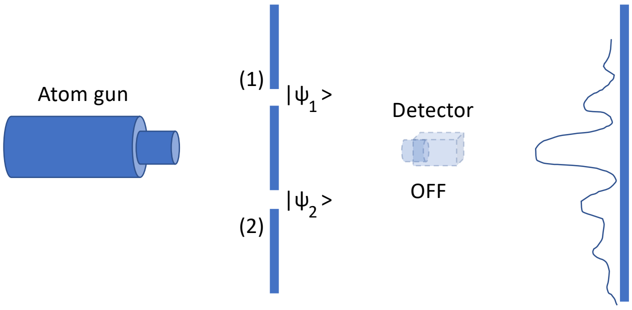
Now, put detector in position,

$$|\Psi\rangle\, |d_{\text{ready}}\rangle = \frac{1}{\sqrt{2}}\Big(\, |\psi_1\rangle + |\psi_2\rangle \Big)|d_{\text{ready}}\rangle \to \frac{1}{\sqrt{2}}\Big(\, |\psi_1\rangle\, |d_1\rangle + |\psi_2\rangle\, |d_2\rangle \Big)\,. \tag{4.6}$$

This is an entangled pure state. But to develop the formalism for a measurement (since we want to know something about which way the atom went) we write the reduced density matrix for the atom. If we assume the detector states are normalised for convenience ($\langle d_1|\, d_1\rangle = \langle d_2|\, d_2\rangle = 1$) then we know that the reduced density matrix for the quantum system of interest (the atom) is

$$\begin{aligned}\tilde{\rho}_{\text{atom}} = \tfrac{1}{2}\, |\psi_1\rangle\langle\psi_1| + \tfrac{1}{2}\, |\psi_2\rangle\langle\psi_2| \\ + \tfrac{1}{2}\, |\psi_1\rangle\langle\psi_2|\, \langle d_2|\, d_1\rangle + \tfrac{1}{2}\, |\psi_2\rangle\langle\psi_1|\, \langle d_1|\, d_2\rangle\,. \end{aligned} \tag{4.7}$$

If the detector is switched off, or it cannot distinguish which slit/path, then $|d_1\rangle$ and $|d_2\rangle$ are not orthogonal and we will still observe an interference pattern build up on the back screen after many atoms are fired through the slits



This would be described by a probability density distribution defined by the diagonal elements of the atom's reduced density matrix in configuration space basis:

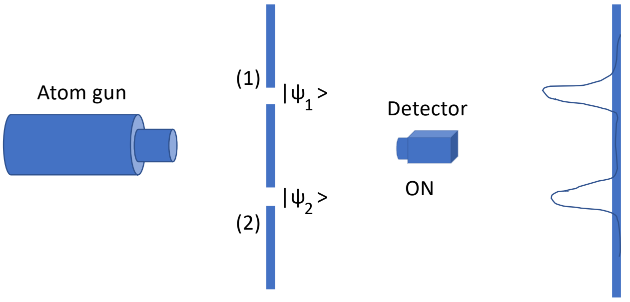
$$\langle x|\, \tilde{\rho}_{\text{atom}}\, |x\rangle = \tilde{\rho}_{\text{atom}}(x, x)\,, \tag{4.8}$$

where the density matrix on the left should really have a hat to indicate that it is an operator inside a matrix element, whereas the $\rho$ on the right is the matrix element (a number).

As a probability density, this is

$$\varrho(x) = \tfrac{1}{2}|\psi_1(x)|^2 + \tfrac{1}{2}|\psi_2(x)|^2 + \underbrace{\Re\Big\{\psi_1(x)\psi_2^*(x)\,\langle d_2|\, d_1\rangle\Big\}}_{\text{interference term}}\,, \tag{4.9}$$

and you can clearly see that if detector is now switched on and it is a decent detector – that is, having clearly distinguishable dial positions or different coloured lights for detecting a particle coming from slit 1 or 2 – then $\langle d_2 | d_1 \rangle = 0$ and there is no interference, so we just see just two peaks corresponding to $|\psi_1(x)|^2$ and $|\psi_2(x)|^2$.



In terms of probability density, when the detector was off or completely unable to say anything about which slit/path then $\langle d_2 | d_1 \rangle = 1$. and we can see that Eq. (4.9) can be written as

$$\varrho(x) = \tfrac{1}{2} |\psi_1(x) + \psi_2(x)|^2 \,, \tag{4.10}$$

which has interference.

Crucially, however, there is an in-between situation in which the overlap of detector states is between 0 and 1. In reality, a good detector when switched on will quickly become entangled with the atom states and gain information so $\langle d_2 | d_1 \rangle$ drops from 1 to 0. *This is decoherence.*

## 4.2 The Environment

When it comes to decoherence, it is the environment surrounding a quantum system can do the measuring, storing information about it as system and environment become unavoidably entangled. By 'environment', we do not just mean macroscopic objects like detectors or cats, but the scattering of air molecules, photons, even those of the cosmic microwave background. This information may not be easily retrievable, but in principle it is there in the environment whether or not we are able to access it. The formalism in Eq.(4.9) remains the same but we will just replace $|d_1\rangle$ and $|d_2\rangle$ by $|E_1\rangle$ and $|E_2\rangle$ to denote the two environment states (such as differently scattered photons off the atom depending on which slit it is observed by the environment to go through. The environment can thus be regarded as a ubiquitous 'measuring device' encoding which-path (which-state) information about a system. This is *encoded* in the environment and does not need a human observer. This is why, if we wish to see an interference pattern in the two-slit experiment we have been make sure it is adequately shielded from its environment. This cannot be done perfectly (that is $\langle E_1 | E_2 \rangle$ never overlap perfectly), but enough for us to see an interference pattern. This is why it becomes increasingly difficult to see interference as we move to more massive particles going through the slits (C60 molecules, viruses, etc). It is *not* because the slit width has to be small enough to observe de Broglie wavelike behaviour (although that is still needed) but because more massive objects interact more strongly with the environment.

It is also worth stressing again that the interference never really goes away. But if we have any measuring system (even the environment) then in order to say something about the system we must trace over the environment's degrees of freedom to get the reduced density matrix of our system of interest. Once decoherence has taken place we do not see this interference. But the total system+environment (plus us, since we are part of the extended environment) is still in an (entangled) pure state.

Because of the huge number of degrees of freedom of the environment, this entanglement is virtually impossible to undo and we cannot return to the pure state of the quantum system alone. This is why we think of decoherence as a time-asymmetric process with non-unitary dynamics for the local system alone. **Measurement brings about irreversibility FAPP**. However, in reality, the environment is not just 'switched on at some time' but is continuously monitoring systems within it. But what is 'system' and what is 'environment' is arbitrary and down to what we trace over and what we wish to learn something about.

## 4.3 Environment-induced superselection

The way an environment couples to our system of interest makes some bases more preferable than others as they will be ones in which the environment is more able to distinguish between states.

For example, in the two-slit experiment, states $|\psi_1\rangle$ and $|\psi_2\rangle$ are spatially separated. But we could just as easily have written our overall state of the atom in superposition of different basis states.

Define

$$|\psi_\pm\rangle = \frac{1}{\sqrt{2}}\Big( |\psi_1\rangle \pm |\psi_2\rangle \Big) \,, \tag{4.11}$$

Hence, $|\psi_1\rangle = \frac{1}{\sqrt{2}}(|\psi_+\rangle + |\psi_-\rangle)$ and $|\psi_2\rangle = \frac{1}{\sqrt{2}}(|\psi_+\rangle - |\psi_-\rangle)$.

It so happens that for our two-slit set-up, $|\Psi\rangle = |\psi_+\rangle$. We could have set up our initial state as

$$|\Psi\rangle = \frac{1}{\sqrt{2}}\Big( |\psi_+\rangle + |\psi_-\rangle \Big) , \tag{4.12}$$

but this is just $|\psi_1\rangle$ and we know that for an environment starting off in an initial state, $|E_0\rangle$,

$$|\psi_1\rangle |E_0\rangle \rightarrow |\psi_1\rangle |E_1\rangle = \frac{1}{\sqrt{2}}\Big( |\psi_+\rangle + |\psi_-\rangle \Big) |E_1\rangle . \tag{4.13}$$

This single term means environment and atom are not entangled, so there is no interference and hence no decoherence in this basis.

That is, the interaction between the atom and environment is such that the environment can distinguish between $|\psi_1\rangle$ and $|\psi_2\rangle$, so if the atom's quantum state is in a superposition of these two states (as is the case with both slits open) then the environment becomes entangled with it and causes decoherence. But if the atom is in a superposition of states in the $\{|\psi_\pm\rangle\}$ basis then there is no entanglement and the environment cannot distinguish between them or gain information about them. We therefore say that the $\{|\psi_{1,2}\rangle\}$ basis is a **preferred basis**.

In this example, $|\psi_1\rangle$ and $|\psi_2\rangle$ represent spatially well-localised states and so the 'environment-superselected' *preferred observable* is the *position* of the atom.

This is why we don't see spread-out particles in the real world: If the particle is already localised (via some measurement for example) then the environment does not get entangled with it. But if it is in a superposition of position states then the environment gets entangled with them, causing decoherence into localised positions.

[Note that we still have the problem of outcomes to deal with: i.e. which localised position ends up being selected/observed?]

## 4.4 Two limits of quantum measurement

Let us now consider the interaction between system and environment. Take what is called the **quantum measurement limit** (in which the system–environment interaction is so strong it dominates their internal energies. That is, if we write the total hamiltonian for System, $S$, and Environment, $E$ as

$$\hat{H} = \hat{H}_S + \hat{H}_E + \hat{H}_I \approx \hat{H}_I , \tag{4.14}$$

where $\hat{H}_I$ is the interaction Hamiltonian.

We often write $\hat{H}_I = \hat{S} \otimes \hat{E}$, where $\hat{S}$ and $\hat{E}$ are operators acting in the Hilbert spaces of the system and environment. We really only need to worry about $\hat{S}$, which will correspond to some system observable, like its position, that is superselected by the environment (i.e. constantly being monitored by the environment).

Let $\hat{S}$ be the position operator, $\hat{x}$, where

$$\hat{x} = \sum_i x_i |X_i\rangle\langle X_i| \ , \tag{4.15}$$

and $x_i$ are position eigenvalues and $|X_i\rangle$ are position eigenstates (called 'pointer states'). Note the above equation defining the operator is just the equivalent of the eigenvalue equation $\hat{x}|X_i\rangle = x_i|X_i\rangle$.

If $\hat{H}_I = \hat{x} \otimes \hat{E}$ then it follows that $\left[\hat{H}_I, \hat{x}\right] = 0$.

The above commutation relation is Zurek's *commutativity criterion* and is straightforward to see since system and environment operators act in different Hilbert spaces and therefore $\left[\hat{x}, \hat{E}\right] = 0$. So

$$\left[(\hat{x} \otimes \hat{E}), \hat{x}\right] = (\hat{x} \otimes \hat{E})\hat{x} - \hat{x}(\hat{x} \otimes \hat{E}) = \hat{x}\hat{x} \otimes \hat{E} - \hat{x}\hat{x} \otimes \hat{E} = 0 \, . \tag{4.16}$$

Therefore, while in general the position operator does not commute with the Hamiltonian (we cannot measure the position and energy of a quantum system simultaneously) it holds in this particular limit (the quantum measurement limit) of $\hat{H} = \hat{H}_I = \hat{x} \otimes \hat{E}$.

So, $\hat{H}_I$ and $\hat{x}$ have common eigenstates, $|X_i\rangle$.

If we start system in some position eigenstate, $|X_i\rangle$, and the environment in initial state, $|E_0\rangle$, then at $t = 0$ the combined state is $|X_i\rangle|E_0\rangle$. An evolution operator, $U$, will take this forward to time $t$:

$$U|X_i\rangle|E_0\rangle = e^{-i\hat{H}_I t}|X_i\rangle|E_0\rangle = |X_i\rangle e^{-ix_i\hat{E}t}|E_0\rangle = |X_i\rangle|E_{x_i}\rangle \ , \tag{4.17}$$

where $|E_{x_i}\rangle$ is the state of the environment now containing information about the position of the quantum system (particle).

What we see in this last equation is that the system and environment are still not entangled. So $|X_i\rangle$ represents an *environmentally superselected preferred state*.

Now, let our system be in a superposition of pointer states:

$$|\psi\rangle = \sum_i c_i |X_i\rangle \ . \tag{4.18}$$

Now

$$\begin{aligned} e^{-i\hat{H}_I t}|\psi\rangle|E_0\rangle &= e^{-i\hat{x}\otimes\hat{E}t}\Big(\sum_i c_i|X_i\rangle\Big)|E_0\rangle \\ &= \Big(c_1|X_1\rangle e^{-ix_1\hat{E}t} + c_2|X_2\rangle e^{-ix_2\hat{E}t} + \cdots\Big)|E_0\rangle \\ &\rightarrow c_1|X_1\rangle|E_1\rangle + c_2|X_2\rangle|E_2\rangle + \cdots\ , \end{aligned} \tag{4.19}$$

where we now have an entangled state of system and environment and $|E_1\rangle$ etc is the state of the environment that contains information about system being in position $x_1$.

We can imagine that the initial state of the system $|\psi\rangle$ is an energy eigenstate. If we saying that such a state typically decoheres very quickly due to its environment into definite positions, how come we see atoms with definite energies? The answer is because in such cases $\hat{H}_I$ does not dominate in Eq. (4.14), but rather it is the system's Hamiltonian, $H_S$ that is important. We can think of this in terms of timescales: or conversely that the frequencies associated with the environment (think of it as collection of oscillators) are much lower than those associated with the system of interest.

So we have two extreme cases:

1. The quantum measurement limit
   Evolution dominated by $\hat{H}_I$. Preferred states are eigenstates of $\hat{H}_I$ (i.e. position).
2. The quantum limit of decoherence
   Evolution dominated by $\hat{H}_S$. Environment slow relative to system. Preferred states are eigenstates of $\hat{H}_S$ (i.e. energy).
3. intermediate regime ($\hat{H}_I \approx \hat{H}_S$)
   e.g Brownian motion – Caldeira-Leggett model
   Compromise is that preferred states are localised in phase space (i.e. in both position and momentum/energy – within the bounds of the Uncetainty Principle).

**A further comment on the measurement problem:**

It is worth stressing once again that measurement does not require a conscious observer. Think of a uranium atom trapped in a rock on a planet orbiting a star on the other side of the galaxy. Not only is there uncertainty in the time at which its nucleus spits out an alpha particle, but that particle could be emitted in any direction. If we left things for long enough that we know the alpha particle has almost certainly been emitted, are we to believe that it exists in a superposition of all directions at once (an outgoing spherical wave, $\exp(ikr)$), until we visit the planet to observe it? And if we don't, does it remain forever in this 'unmeasured' superposition? Of course not. The rock is a measuring device like any other macroscopic object and it will record a particular path for the alpha particle that we could – if we wished to and were able to – go and see for ourselves. You see, the spherically symmetric alpha particle wave function, when it was emitted, quickly became entangled with the state of the rock. If we trace over the degrees of freedom of the rock to get the reduced density matrix of the alpha particle we find that the tracks that the alpha would form in the rock are macroscopically distinguishable from each other and so kill the off-diagonal elements of the alpha's density matrix, leaving just the (very) large number of possible directions the alpha could go.

But here is where we again hit the problem of outcomes. All physically realisable results (directions) still exist. What is it that picks out one of them? Surely it is not waiting for us. It's one thing talking about opening Schrödinger's cat's box, but quite another to expect us to travel thousands of lightyears to examine a rock and kill off the other branches/options. Whatever needs doing has to be done by the rock.

Are we forced then to adopt the Everettian view in which there are now many universes, with the alpha moving in a different direction in each? This implies that, without ever visiting that planet and observing the rock, suddenly there are multiple versions of us, identical in every way other than in each universe a rock on a distant planet has a tiny alpha particle trail in a different direction. And the same goes for every other alpha particle emitted from every other uranium atom in every other rock on every other planet, etc. Maybe I just misunderstanding the MWI.

At this point, one might wonder whether there is a need for something like Rovelli's relational quantum mechanics. But we are straying into interpretational issues here and need to pull back. Nevertheless we still have to face up to the ambiguity of the dividing line between the quantum and classical domains. What does macroscopically distinguishable mean anyway? This seems no less arbitrary than Bohr's "irreversible act of amplification". This is what Bell refers to as the"shifty split"* between system and detector.

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* J.S. Bell, *Against Measurement, Physics World*, August, 1990.

# 5

# Lecture 5 – Decoherence

## 5.1 A simple model for decoherence

Physical systems exhibiting decoherence are varied. Luckily – and perhaps surprisingly – a small set of simple *canonical models* can describe a wide range of phenomena and physical systems. Thus, the system of interest can be modelled as either a spin-1/2 particle (qubit) or as having continuous phase space variables and moving in some potential (H-O or double well are popular examples). The environment likewise can be modelled either as collection of qubits or as a heat bath of harmonic oscillators.

We will consider one such example: we will take our quantum system of interest, $S$, to be a qubit (a spin-1/2 particle) with basis states $|0\rangle$ and $|1\rangle$ denoting spin up and down with respect to the $z$-axis. This qubit interacts with a large number of other qubits, which we denote by $|\uparrow\rangle_i$ and $|\downarrow\rangle_i$ , $i = 1 \cdots N$ representing the environment (with the up and down arrows distinguishing them from the 0 and 1 of $S$. The total system+environment combination is described by a $2^{N+1}$-dimensional tensor product Hilbert space

$$\mathscr{H} = \mathscr{H}_S \otimes \mathscr{H}_{\epsilon_1} \otimes \mathscr{H}_{\epsilon_2} \otimes \cdots \otimes \mathscr{H}_{\epsilon_N} \tag{5.1}$$

where $\mathscr{H}_S$ denotes the Hilbert space of the system and $\mathscr{H}_{\epsilon_i}$ denotes the Hilbert space of the $i$th environmental qubit.

We will also assume, as we did earlier, that we are in the so-called quantum measurement limit described in the previous lecture. That is, the interaction Hamiltonian, $H_I$ completely dominates the evolution and both the system and environment are completely static with the only dynamical process being the formation of correlations between system and environment. Our total Hamiltonian is thus chosen to be of the form

$$\begin{aligned} H = H_I &= \tfrac{1}{2}\big(\,|0\rangle\langle 0| - |1\rangle\langle 1|\,\big) \otimes \Big(\sum_{i=1}^{N} g_i\big(\,|{\uparrow}\rangle_i\langle{\uparrow}|_i - |{\downarrow}\rangle_i\langle{\downarrow}|_i\,\big) \bigotimes_{j\neq i} \hat{I}_j\Big) \\ &= \tfrac{1}{2}\hat{\sigma}_z \otimes \Big(\sum_{i=1}^{N} g_i \hat{\sigma}_z^{(i)} \bigotimes_{j\neq i} \hat{I}_j\Big)\,, \qquad (5.2) \end{aligned}$$

where we've used the definition $\hat{\sigma}_z = |0\rangle\langle 0| - |1\rangle\langle 1|$, which you can check satisfies the eigenvalue equations, $\hat{\sigma}_z\,|0\rangle = |0\rangle$ and $\hat{\sigma}_z\,|1\rangle = -\,|1\rangle$. Also, the large tensor product symbol, $\bigotimes$, is the equivalent of the more familiar scalar product symbol, $\prod$ (that is, $\bigotimes \hat{I}_j = \hat{I}_1 \otimes \hat{I}_2 \otimes \cdots$), where $\hat{I}_j = |{\uparrow}\rangle_j\langle{\uparrow}|_j + |{\downarrow}\rangle_j\langle{\downarrow}|_j$ is the identity operator for the $j$th qubit of the environment. Note the $'+'$ sign in the definition of $\hat{I}_j$, which means that, unlike $\hat{\sigma}_z^{(i)}$, this operator does nothing when it acts on the environment spin state, $|{\downarrow}\rangle_j$. Henceforth we will drop the explicit mention of these identity operators. So Hamiltonian for the combined system+environment is

$$H = H_I = \tfrac{1}{2}\hat{\sigma}_z \otimes \sum_{i=1}^{N} g_i\,\hat{\sigma}_z^{(i)} \equiv \tfrac{1}{2}\hat{\sigma}_z \otimes \hat{E}\,. \qquad (5.3)$$

Now, when we act with the evolution operator involving the above Hamiltonian on an initial unentangled state of system and environment we see

$$\begin{aligned} e^{-i\hat{H}_I t}\,|0\rangle\,|E_0\rangle &= e^{-\frac{i}{2}\hat{\sigma}_z \otimes \sum g_i\,\hat{\sigma}_z^{(i)} t}\,|0\rangle\,|E_{\text{initial}}\rangle \\ &= |0\rangle e^{-\frac{i}{2}\sum g_i\,\hat{\sigma}_z^{(i)} t}\,|E_{\text{initial}}\rangle = |0\rangle\,|E(t)\rangle\,, \qquad (5.4) \end{aligned}$$

where the state of the environment can start off as complicated as we wish, with each qubit in a superposition:

$$|E_{\text{initial}}\rangle = \big(\alpha_1\,|{\uparrow}\rangle_1 + \beta_1\,|{\downarrow}\rangle_1\,\big) \otimes \cdots \otimes \big(\alpha_N\,|{\uparrow}\rangle_N + \beta_N\,|{\downarrow}\rangle_N\,\big)\,, \qquad (5.5)$$

and the state at time $t$, $|E(t)\rangle$, may therefore be very messy and entangled, but that is not of concern to us just yet. We can immediately see from Eq.(5.4) that we do not get an entangled state of system and environment and that therefore the system's eigenbasis $\{|0\rangle, |1\rangle\}$ of $\hat{\sigma}_z$ is a dynamically selected preferred basis of the system. However, if the system were to start off in some superposition of $|0\rangle$ and $|1\rangle$, then the environment will become entangled with it and cause decoherence.

In summary then, if the system starts off in an eigenstate of the operator in the interaction Hamiltonian that links it to the environment and which sits in the evolution operator, then the system will not become entangled with its environment.

An equally important point is that with this definition of the Hamiltonian, it commutes with the spin operator, $\hat{\sigma}_z$ and thus they share eigenstates. So, the system is also an energy eigenstate and hence energy is conserved. That is, there is no exchange of energy between the system and environment (no dissipation). This is true whatever state the system starts in (as the above argument relates to the definition of the Hamiltonian) and so energy is conserved even if the system starts off in a superposition and we get entanglement with environment and decoherence. Therefore we can see that with such a definition of $\hat{H}_I$ the action of the environment (whether or not it becomes entangled with the system) only influences the coherence, and that decoherence is a purely quantum phenomenon with no classical counterpart.

Let us now examine the state of the environment. Recall that if the system starts off in a superposition then

$$e^{-i\hat{H}_I t}\big(a\,|0\rangle + b\,|1\rangle\big)\,|E_{\text{initial}}\rangle \rightarrow a\,|0\rangle\,|E_0(t)\rangle + b\,|1\rangle\,|E_1(t)\rangle\ . \tag{5.6}$$

We have seen already that the rate of decoherence depends on the overlap of the environment states that are entangled with each of the system states and the degree to which they are orthogonal (distinguishable)

$$r(t) = \langle E_1(t)|\,E_0(t)\rangle\,. \tag{5.7}$$

But this is of course a complicated beast. Recall from Eqs.(5.4) and (5.5) that

$$\begin{aligned} |E_0(t)\rangle &= e^{-\frac{i}{2}\sum_i^N g_i \hat{\sigma}_z^{(i)} t}\Big[\big(\alpha_1\,|{\uparrow}\rangle_1 + \beta_1\,|{\downarrow}\rangle_1\big)\otimes\cdots\otimes\big(\alpha_N\,|{\uparrow}\rangle_N + \beta_N\,|{\downarrow}\rangle_N\big)\Big] \\ &= e^{-\frac{i}{2}\sum_i^N g_i \hat{\sigma}_z^{(i)} t}\Big[\alpha_1\alpha_2\cdots\alpha_N\,|{\uparrow}\rangle_1\,|{\uparrow}\rangle_2\cdots|{\uparrow}\rangle_N \\ &\qquad +\alpha_1\alpha_2\cdots\alpha_{N-1}\beta_N\,|{\uparrow}\rangle_1\,|{\uparrow}\rangle_2\cdots|{\uparrow}\rangle_{N-1}\,|{\downarrow}\rangle_N \\ &\qquad +\cdots \\ &\qquad +\beta_1\beta_2\cdots\beta_N\,|{\downarrow}\rangle_1\,|{\downarrow}\rangle_2\cdots|{\downarrow}\rangle_N\Big]\,, \end{aligned} \tag{5.8}$$

where we now have $2^N$ terms inside the brackets (for example, for $N = 2$ we have 4 terms: $\alpha_1\alpha 2\cdots$, $\alpha_1\beta 2\cdots$, $\beta_1\alpha 2\cdots$ and $\beta_1\beta 2\cdots$, and for $N = 3$ there are 6 terms, etc.)

But we can write this more compactly as

$$|E_0(t)\rangle = \sum_{j=1}^{2^N} e^{-i\epsilon_j t/2} c_j\,|n\rangle_j\ . \tag{5.9}$$

We have $2^N$ terms and have used simplifying notation as follows: the product states are defined as

$$|n\rangle_j = \prod_i^N |\updownarrow\rangle_i \tag{5.10}$$

where $|\updownarrow\rangle_i$ means the $i$th state is either up or down. Thus

$$|n\rangle_1 = |\uparrow_1\uparrow_2\uparrow_3 \cdots \uparrow_N\rangle \ , \qquad |n\rangle_2 = |\uparrow_1\uparrow_2\uparrow_3 \cdots \uparrow_{N-1}\downarrow_N\rangle \text{ etc.} \tag{5.11}$$

The $c_j$ coefficients are each a product of $N$ $\alpha$'s and $\beta$'s (for example, $c_1 = \alpha_1\alpha_2 \cdots \alpha_N$); and finally, the energy $\epsilon_j$ is

$$\epsilon_j = \sum_k^N (-)^{n_j} g_k, \ \ n_j = \begin{cases} 0 & \text{(for an even number of } |\downarrow\rangle \\ & \quad \text{states in the product } |n\rangle_j) \\ 1 & \text{(for an odd number of } |\downarrow\rangle \\ & \quad \text{states in the product } |n\rangle_j) \end{cases} \tag{5.12}$$

Now we can look at the overlap of two environment states to see the structure of the decoherence rate defined in Eq.(5.7). First, if we check the action of the evolution operator on $|1\rangle|E_{\text{initial}}\rangle$ (where again $|E_{\text{initial}}\rangle$ is just the environment-ready state at $t = 0$) we see that the equivalent of Eq.(5.4) is

$$e^{-i\hat{H}_I t}\,|1\rangle\,|E_{\text{initial}}\rangle = |1\rangle e^{+\frac{i}{2}\sum g_i \hat{\sigma}_z^{(i)} t}\,|E_{\text{initial}}\rangle = |1\rangle\,|E_1(t)\rangle \ , \tag{5.13}$$

but due to the minus sign in the exponential from the action of $\hat{\sigma}_z$ on the system state $|1\rangle$ we see that

$$|E_1(t)\rangle = \sum_j^{2^N} e^{+i\epsilon_j t/2} c_j \,|n\rangle_j \ . \tag{5.14}$$

Therefore, taking the overlap means we have two minus signs

$$r(t) = \langle E_1(t)|\, E_0(t)\rangle = \sum_{i,j}^{2^N} e^{-i\epsilon_i t/2} e^{-i\epsilon_j t/2} c_i^* c_j \delta_{ij} = \sum_{i=1}^{2^N} e^{-i\epsilon_i t} |c_i|^2 \ . \tag{5.15}$$

It was shown by Zurek in his classic paper (*Phys. Rev. D* **26**, 1862 (1982)) that evolution of $r(t)$ reduces to a random walk problem in the 2-D complex plane and that the time averaged modulus square of the complex vector $r(t)$ scales as

$$\langle |r(t)|^2 \rangle \propto 2^{-N} \xrightarrow{N\to\infty} 0 \ . \tag{5.16}$$

That is, the rate of decoherence scales exponentially with the size of the environment. We will not prove this here but you can clearly see how the size of the environment affects the decoherence rate since recall that the $c_i$ coefficients in Eq.(5.15) are each a product of $N$ amplitudes, $\alpha$ and $\beta$. That is $|c_i|^2$ is a product of $N$ probabilities, each $< 1$. So the larger the environment (size of $N$), the smaller the value of $|c_i|^2$ in (5.14).

For very large $N$, the decoherence rate is roughly a Gaussian decay:

$$r(t) \approx e^{-\Gamma^2 t^2} . \tag{5.17}$$

The decay constant, $\Gamma^2$, depends on the distribution of the coupling strengths, $g_i$, between the system and each of the qubits in the environment. You see, for our model, each of the $2^N$ terms in the sum in (5.15) a different phase since $\epsilon_j$ is a sum (Eq.(5.12)) of coupling strengths whose sign depends on whether the qubit in the environment is spinning up or down.

## 5.2 Decoherence versus dissipation

We define the *relaxation time*, $\tau_r$, as the time taken for a system to dissipate thermal energy into its environment until the two are in thermal equilibrium. It depends on the interaction strength between system and environment. However, as we saw in the previous example, there can be decoherence even if there is no energy dissipation at all, which means the environment can gain information about the system without any exchange of energy – although usually in real physical systems, one is accompanied by the other.

But decoherence and dissipation have very different timescales:

$$\tau_d \ll \tau_r . \tag{5.18}$$

To get a feel for what these time scales are, Zurek (*Physics Today* **44**, 36–44 (1991)) started from the well-known Caldeira and Leggett master equation (which we will meet in the next lecture) to show that for an object of mass $m$ that is in a superposition of two spatial locations separated by a distance $\Delta x$ the ratio of the two time is

$$\frac{\tau_r}{\tau_d} \approx \left(\frac{\Delta x}{\lambda_{dB}}\right)^2 , \tag{5.19}$$

where $\lambda_{dB}$ is the thermal de Broglie wavelength of the object, which for massive non-interacting particles can be derived from the partition function (see standard textbooks) and has the formula

$$\lambda_{dB} = \frac{\hbar}{\sqrt{2mk_B T}} . \tag{5.20}$$

For $m = 1\text{g}$ and $T = 300\text{K}$ (room temperature), $\lambda_{dB} \approx 10^{-23}$ m, which is clearly very small, as one might expect the de Broglie wavelength of a classical object to be. But if $\Delta x$ is 1cm, then $\tau_r / \tau_d \approx 10^{42}$, which is why people talk about decoherence happening so fast for classical objects like cats, and why we observe macroscopic objects to follow perfectly Newtonian paths.

## 5.3 Decoherence versus classical noise

Often, people will use density matrix formalism to describe ensembles and treat ensemble averaging over different noisy systems (leading to diagonalisation of density matrix) as a decoherence process because the density matrix looks just like the reduced density matrix of an individual system subject to quantum decoherence (delocalisation of phase coherence) to become an improper mixed state. However, this ensemble averaging is known as 'fake decoherence' as there is no system-environment entanglement.

The physical processes are different:

- Classical noise: environment perturbs system, can in principle be undone by local operations. Very slow.
- Decoherence: system perturbs environment (gives it information). FAPP irreversible process. Very fast.

Sometimes stochastic fluctuations (kicks) are used to *simulate* decoherence. But here we really only have unitary dynamics in which the Hamiltonian has some random fluctuation term in the potential. Whereas quantum decoherence needs to be described by an open quantum system in which the system becomes entangled with environment.