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**EFFICIENT SIMULATION OF DSGE MODELS WITH  
INEQUALITY CONSTRAINTS**

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# Efficient simulation of DSGE models with inequality constraints

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**Abstract:** *This paper presents a fast, simple and intuitive algorithm for simulation of linear dynamic stochastic general equilibrium models with inequality constraints. The algorithm handles both the computation of impulse responses, and stochastic simulation, and can deal with arbitrarily many bounded variables. Furthermore, the algorithm is able to capture the precautionary motive associated with the risk of hitting such a bound. To illustrate the usefulness and efficiency of this algorithm we provide a variety of applications including to models incorporating a zero lower bound (ZLB) on nominal interest rates. Our procedure is much faster than comparable methods and can readily handle large models. We therefore expect this algorithm to be useful in a wide variety of applications.*

**Keywords:** *Inequality constraints, Zero lower bound, Precautionary motives, Two-country models*

**JEL Classification:** *C63, E32, E43, E52*

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## 1. Introduction

New Keynesian (NK) Dynamic Stochastic General Equilibrium (DSGE) models are today's standard framework for analysing central bank policies.<sup>3</sup> The nominal and real rigidities in these models mean central banks may improve welfare through monetary policy. Traditionally, DSGE models are log-linearised, which results in both computational and analytic tractability. However, the simplicity of this approach also neglects important non-linearities, not-least inequality constraints, of which the zero lower bound (ZLB) on interest rates is the most prominent example. In this paper, we present an efficient algorithm for simulating DSGE models subject to arbitrarily many inequality constraints, at arbitrary accuracy, at least away from the bound.

Macroeconomic analysis had ignored the zero lower bound almost completely before the experience of Japan in the 1990s, since the bulk of macroeconomists believed the constraint would bind only for a short time span (if at all). Under this presumption, the effects would be negligible, and so ignoring the bound seemed to be a reasonable simplification. However, the interest rates in Japan during the 1990s, as well as those in the US over the last few years, disabused researchers and policymakers of this popular fallacy.

In the aftermath of the crisis, the transmission of monetary policy under the ZLB became an important matter for central banks and academia. To cope with the bound, researchers either solve such non-linear models using global approximation methods (which come at a dramatic increase in computational costs, and scale exceptionally poorly) or use deterministic setups. This paper provides a fast, simple and intuitive algorithm to deal with inequality constraints in perturbation approximations to DSGE models, which correctly captures the precautionary motives associated with such bounds. The code is designed to work with Dynare (Adjemian et al. 2011), so incorporating it into existing models is trivial. The method is not only useful for deriving impulse response functions (IRFs), but can also be used for stochastic simulations, opening up the possibility of particle filter based estimation of models with inequality constraints. The method endogenously determines when the constraint will bind, and can handle constraints that may bind in multiple disjoint runs, or that may not begin to bind until long after the initial impulse. The general idea is to introduce “shadow price shocks”, which hit the bounded variables every time the constraint is violated, and “push” these variables back to zero. To ensure the solution is consistent with rational expectations, these shocks are expected by agents in advance, so they may be thought of as a kind of endogenous news shock.

This algorithm is not solely useful for modelling a ZLB on interest rates, but can be used for any model including constrained variables. Holden (2010), for example, uses it to constrain invention rates to be positive in a model of endogenous growth. In addition, Funke and Paetz (2012) use this technique to evaluate threshold loan-to-value policies in Hong Kong, where policymakers decrease the loan-to-value ratio, when property price inflation exceeds a certain value. In Chen et al. (2012), the same method is used to model the People's Bank of China's interest rate corridor on retail lending and deposit rates.

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<sup>3</sup> See Clarida et al. (1999) for an early literature review on NK models.

The rest of the paper is organized as follows. In section 2, the algorithm is described and related to the existing literature. Section 3 assesses the accuracy of the procedure in a variety of models for which high accuracy solutions are available, and section 4 goes on to provide some sample applications to larger models. The final section concludes.

## 2. The numerical method

### 2.1. The existing literature

Due to the recent experience of the US and Europe, the literature on (stochastic) simulation of models with a ZLB has grown rapidly in the past few years. The most important contributions include Eggertsson and Woodford (2003), Erceg and Lindé (2010), Braun and Körber (2011), Christiano et al. (2011) and Fernández-Villaverde et al. (2012). In what follows, we highlight the similarities and differences between the approaches employed in these papers and the method presented in this paper.<sup>4</sup>

The first generation of papers used variants of the method proposed by Eggertsson and Woodford (2003), in an appendix. This relied on a piecewise linear approximation to the model, with the model being driven by a two-state Markov chain with an absorbing state. Once the ZLB is hit, there is a positive probability in each period that the discount factor jumps to its long run value, at which point the ZLB will never be hit again. Obviously, this is a highly restrictive assumption. A version of this algorithm without the restriction has been proposed by Jung et al. (2005), and implemented in full generality in Dynare by Guerrieri and Iacoviello (2012). Nonetheless, the algorithm still relies on a linear approximation, which the results of Braun et al. (2012) suggest may lead to unreliable conclusions in the presence of the ZLB.

The next generation of papers used nonlinear perfect foresight solvers. These include Coenen et al. (2004), Braun and Körber (2011) and the “extended path” method of Adjemian and Juillard (2011). These solve the model’s nonlinear equations, under the assumption that eventually (e.g. after 100 periods) the model is guaranteed to have returned to steady state. Such methods fully capture the nonlinearities of the model, but because they solve under perfect foresight, they omit any “precautionary motives” including those that arise from the risk of hitting the ZLB. Furthermore, since the model has to have returned to steady state up to machine precision by the final period considered, they require a very large number of nonlinear equations to be solved. This means they tend to both be prohibitively slow, and unstable, with the algorithm frequently failing to find a solution to the equations.

A third strand of the literature considers global approximations to models containing inequality constraints, with Fernández-Villaverde et al. (2012) doing this for a small scale NK model, using the Smolyak collocation method of Krueger et al. (2011). Global methods successfully capture both the model’s nonlinearities and precautionary motives; however, they are subject to a curse of

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<sup>4</sup> See the introduction in Braun et al. (2012) for a survey on models including a zero lower bound. An early analysis of the zero lower bound in a deterministic model can also be found in Fuhrer and Madigan (1997).

dimensionality that renders them infeasible in the medium scale NK models we usually consider. While global methods that avoid the curse of dimensionality have been developed by Maliar et al. (2011), these rely on an endogenous grid constructed from the model's ergodic set, which is likely to lead to low accuracy at the ZLB if this bound is only hit occasionally.

Our method represents a compromise between the accuracy of global methods, and the speed and scalability of linear ones, much like standard high order perturbation approximations. The paper that is probably most closely related to our work is Erceg and Lindé (2010), which we were not aware of until after the completion of the first version of our algorithm in Holden (2010). The authors rely on techniques, explained in an unpublished mimeo of James Hebden, Jesper Lindé and Lars Svensson, which, like our algorithm, are based on the idea of adding shocks to the bounded variable. Since we have not seen this mimeo, we are unable to relate our work to this algorithm, but we are confident that our method is novel in several respects. Firstly, it is designed to take advantage of existing algorithms both for simulating DSGE models (e.g. those of Dynare), and algorithms for quadratic programming, leading to its high speed. Secondly, it is generalized to permit any number of constrained variables. Thirdly, it is extended for use in stochastic simulations, permitting us to derive average IRFs, and opening up the possibility of estimating bounded models. Finally, it is generalized to perturbation approximations of arbitrary order, which is what enables the algorithm to capture precautionary incentives, thus improving on the accuracy of nonlinear perfect foresight algorithms.

## 2.2. Our basic IRF algorithm with a single bound

Suppose we have a rational expectations model in the variables  $x_{1,t}, \dots, x_{n,t}$ , and we are interested in the response to the shock,  $\epsilon_t$ . Initially, we will suppose further that all of the model's equations are linear, except one that takes the form:

$$x_{1,t} = \max\{0, \mu_1 + \phi'_{-1}x_{t-1} + \phi'_0x_t + \phi'_1\mathbb{E}_tx_{t+1} - (\phi'_{-1} + \phi'_0 + \phi'_1)\mu\}, \quad (2.1)$$

where  $x_t$  is the vector  $[x_{1,t}, x_{2,t}, \dots, x_{n,t}]'$ ,  $\mu = [\mu_1, \mu_2, \dots, \mu_n]'$  is a vector stacking the variables' steady state values and  $\mu_1 > 0$ . We can transform any linear model with a bound into this form through the addition of appropriate auxiliary variables.<sup>5</sup>

Now, a shock that drives  $x_{1,t}$  to 0 for some number of periods is like a combination of the original shock, and a news shock stating  $x_{1,t}$  will be higher than expected for some duration. We call these news shocks "shadow price shocks" as in a model with bounded assets, they represent the Lagrange multiplier on the constraint. The key to the simplicity of our algorithm is the fact that in

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<sup>5</sup> So, if the bounded equation stated that  $x_t = \max\{\tilde{x}_t, y_t\}$  (where  $\sim$  denotes unconstrained variables), with  $x_t > y_t$  in steady state, we would add an auxiliary variable defined as  $\tilde{x}_t - y_t$ , noting that  $x_t = y_t + \max\{0, \tilde{x}_t - y_t\}$ . The models of Funke and Paetz (2012) and Chen et al. (2012) include variables that are bounded at a positive value, for example. When  $x_t = y_t$  in steady state, we instead add the variable  $y_t - \tilde{x}_t$ , and note that  $x_t = \tilde{x}_t + \max\{0, y_t - \tilde{x}_t\}$ . If the model is in levels, rather than in logs, it may be preferable to define auxiliary variables as ratios rather than differences. In this case, rather than adding shadow shocks, we must multiply by their exponentials.

linear models, the IRF to a linear combination of shocks is equal to the same linear combination of each shock's IRF.<sup>6</sup>

Let us start by defining  $v_j$  to be the column vector containing the relative impulse response of variable  $x_j$  to the shock, ignoring the ZLB. Let  $T^*$  be the number of periods after which the constraint is no longer expected to bind. Note that this will in general be much smaller than the time it takes to return to steady state. We assume that the IRF vectors are of length  $T$ , where  $T \geq T^*$ .

The first step of our algorithm is to replace equation (2.1) of the model with:

$$x_{1,t} = \mu_1 + \phi'_{-1}x_{t-1} + \phi'_0x_t + \phi'_1\mathbb{E}_t x_{t+1} - (\phi'_{-1} + \phi'_0 + \phi'_1)\mu + \sum_{s=0}^{T^*-1} \epsilon_{s,t-s}^{\text{SP}}, \quad (2.2)$$

where  $\epsilon_{s,t}^{\text{SP}}$  is a newly introduced shadow price shock at horizon  $s$ , for  $s = 0, \dots, T^* - 1$ . Since  $\epsilon_{s,t}^{\text{SP}}$  is known at  $t$ , but does not "hit" until  $t + s$ , it will function as a news shock, as required. In an impulse response exercise, all news arrives in period 0, so we want to find values for these shadow shocks such that  $\epsilon_{s,t}^{\text{SP}} = 0$  except perhaps when  $t = 0$ .

In order to exploit the linearity of IRFs in linear models, we also store the impulse responses to these new shadow price shocks. In particular, we let  $m_{j,s}$  be a column vector containing the relative impulse response of variable  $x_{j,t}$  to the shock  $\epsilon_{s,t}^{\text{SP}}$ . We then horizontally stack these vectors into the matrix  $M_j = [m_{j,0}, m_{j,1}, \dots, m_{j,T^*-1}]$ . With this matrix, we can now calculate the impulse responses to a simultaneous shock to  $\epsilon_t$  of magnitude 1 and to  $\epsilon_{s,t}^{\text{SP}}$  of magnitude  $\alpha_s$  (for each  $s = 0, \dots, T^* - 1$ ), for an arbitrary  $\alpha = [\alpha_0, \alpha_1, \dots, \alpha_{T^*-1}]'$ , without any further simulation of the model. In particular, the IRF to this combination of shocks for variable  $x_j$  will be equal to  $\mu_j + v_j + M_j\alpha$ .

Our task then is just to find a value for  $\alpha$  that is consistent with the model and with rational expectations. For the constraint to be satisfied, it certainly has to be the case that  $\mu_1 + v_1 + M_1\alpha \geq 0_T$  (where  $0_T$  is a length  $T$  vector of 0s). Additionally, we require that  $\alpha \geq 0_{T^*}$ , as shadow shocks must increase the (lower) bounded variable.<sup>7</sup> Finally, a complementary slackness type condition must hold: the shadow shock at horizon  $s$  can only be non-zero if the bound binds at that horizon.

We can express this as:

$$\alpha'[\mu^* + v^* + M^*\alpha] = 0_{T^*}, \quad (2.3)$$

where  $\mu^* := \mu_1 1_{T^*}$ ,  $v^*$  is the first  $T^*$  elements of  $v_1$  and  $M^*$  is the upper  $T^* \times T^*$  sub-matrix of  $M_1$ . To solve for an  $\alpha$  that satisfies these constraints, we run the following quadratic programming problem:

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<sup>6</sup> The algorithm presented here was first explained in the appendix of Holden (2010). The various extensions we describe were not covered in that paper, however.

<sup>7</sup> We assume throughout that the diagonal of the  $M_1$  matrix is strictly positive, so a shadow shock at horizon  $s$  increases the bounded variable at horizon  $s$ . When this does not hold, (as it may not, for example, at long horizons in sufficiently rich NK models), the sign with which the shadow shock enter equation (2.2) must be flipped.

$$\alpha^* := \arg \min_{\substack{\alpha \geq 0_{T^*} \\ \mu^* + v^* + M^* \alpha \geq 0_{T^*}}} \left[ \alpha'(\mu^* + v^*) + \frac{1}{2} \alpha'(M^* + M^{*'})\alpha \right], \quad (2.4)$$

where the solution is considered admissible if the minimand is 0 at the optimum (i.e. equation (2.3) is satisfied).<sup>8</sup> Since there are well-established, fast, robust algorithms for quadratic programming, this is then a straightforward problem.<sup>9</sup>

The standard properties of quadratic programming problems imply that a sufficient condition for the existence of a unique solution to (2.4) is that  $M^* + M^{*'}$  is positive definite. In our experience, this is satisfied in only the simplest models. When  $M^* + M^{*'}$  is not positive definite, we cannot rule out the existence of multiple solutions. In these cases, which solution is returned will depend on the precise properties of the quadratic programming algorithm used. However, for most models the construction of our problem will lead the algorithm to select the solution in which the components of  $\alpha$  are as small as possible, which will also tend to minimise the amount of time the constraint binds. If desired, explicit guarantees on the solution selected may be enforced via homotopy methods,<sup>10</sup> though this will increase the time cost of our algorithm.

It is also possible that there will be no admissible solution to (2.4), at least for sufficiently large shocks. This will happen if there are “complementarities” between constraints: e.g. hitting one constraint increases the chance of hitting another constraint *and vice versa*. Obviously, this is much more likely when there are multiple bounds, rather than merely multiple horizons. A necessary condition for the existence of an admissible solution to (2.4) for arbitrarily large shocks is that there exists some  $\alpha \geq 0_{T^*}$  such that  $M^* \alpha \geq 1_{T^*}$ . In simple models the anticipation effects of hitting the bound in future are weak, so this condition will be satisfied, but in medium scale models, we will generally not be able to provide such a guarantee.

### 2.3. Dealing with multiple bounds

The algorithm previously described may be readily generalised to cases with multiple bounds. Suppose that in the set-up above, rather than just  $x_{1,t}$  being bounded, each of the variables  $x_{1,t}, x_{2,t}, \dots, x_{n^*,t}$  is bounded, with a corresponding equation taking the form of (2.1). Much as before, we add shadow shocks to each of these equations (giving a total of  $n^*T^*$  extra shocks), and we horizontally concatenate the impulse responses of variable  $x_{j,t}$  to each of the shadow shocks in the equation for  $x_{l,t}$  into the matrix  $M_{j,l}$ . We then define  $M_{j,l}^*$  to be the upper  $T^* \times T^*$  sub-matrix of  $M_{j,l}$ , and  $M^*$  to be the  $n^*T^* \times n^*T^*$  block-matrix with  $(j, l)^{\text{th}}$  block  $M_{j,l}^*$  for  $j, l \in \{1, \dots, n^*\}$ . Likewise, we define  $\mu_l^* := \mu_l 1_{T^*}$ ,  $v_l^*$  to be the first  $T^*$  elements of  $v_l$ , and  $\mu^*$  and  $v^*$  to be the length  $n^*T^*$  block vectors with  $l^{\text{th}}$  block  $\mu_l^*$  and  $v_l^*$ , respectively. With these (re-)definitions, an admissible

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<sup>8</sup> The constraint  $\mu^* + v^* + M^* \alpha \geq 0_{T^*}$  may also be replaced with  $\mu_1 + v_1 + M_1 \alpha \geq 0_T$  to check there are no bound violations after  $T^*$ .

<sup>9</sup> In MATLAB, these are provided by the “quadprog” command.

<sup>10</sup> To guarantee selecting the equilibrium in which  $\|\alpha\|_2$  is minimal, we replace  $M^* + M^{*'}$  by  $M^* + M^{*' + \lambda I}$ , where  $\lambda \rightarrow 0$  is the homotopy parameter. To guarantee selecting the equilibrium in which  $\|\alpha\|_1$  is minimal, we replace  $\mu^* + v^*$  by  $\mu^* + v^* + \lambda 1_{T^*}$ , where  $\lambda \rightarrow 0$  is the homotopy parameter.

solution to (2.4) again gives us the required combination of shadow shocks to enforce all bounds, and the uniqueness and existence conditions are identical as well.

## 2.4. Stochastic simulation

The algorithm of the previous sections may be readily extended to the stochastic simulation of bounded, linear models.<sup>11</sup> As before we begin by adding shadow price shocks to the equations defining bounded variables. Now suppose we have simulated up to period  $t - 1$ . In a linear model, agents' expectations at  $t$  of the state of the economy at  $t + 1, t + 2, \dots$  are the same as they would be were the variance of all shocks equal to 0 from  $t + 1$  onwards. This will not be exactly true in a model with bounds, since the bounds will tend to increase the means of lower bounded variables. However, since (log-)linearisation has already removed any effects of uncertainty on the mean, if we solve assuming there are no shocks after period  $t$ , our approximation error is likely to be of the same order as that of a (log-)linearised model without constraints,<sup>12</sup> at least providing the precautionary incentives stemming from the risk of hitting the bound are fairly weak.

Thus, much as in the IRF case, we first simulate the model to find the path by which it would return to the steady-state, in the absence of bounds, and with no shocks arriving after period  $t$ . If the constraints are not violated along this path, then our simulated value for period  $t$  is fine, and we may move on to period  $t + 1$ . Otherwise, shadow shocks must be added. The algorithm for doing this is identical to that described above for IRFs, except that the simulated return paths of the economy's variables take the role of  $\mu^* + v^*$  above. (We again use the  $M_{j,l}^*$  matrices formed from the impulse responses to shadow shocks.) The found solution to the quadratic programming problem gives a valid, new value for variables at  $t$ , enabling us to go on to the next period. Note that it is now no longer the case that the "news" contained in shadow shocks is guaranteed to come true, since other shocks may arrive in the meantime pushing us away from the bounds. Consequently,  $\alpha_0$  no longer represents the found value of today's shadow price shock. Rather, it is equal to the cumulated history of shadow price shocks that hit in period 0 (i.e.  $\epsilon_{0,0}^{SP} + \epsilon_{1,-1}^{SP} + \dots + \epsilon_{T^*-1,-T^*+1}^{SP}$ ).

## 2.5. Average IRFs

Using our simulation algorithm we can go on to produce average IRFs, i.e. IRFs that give a measure of the average response of the model to a one standard deviation shock, rather than a measure of the response in steady-state. (The two measures agree in the absence of bounds.) To do this, we run a stochastic simulation of the model, and then rerun the same simulation with the same shocks in all periods except one, in which we add 1 to the shock of interest. The difference between these two simulations gives one sample IRF, and the average of many such sample IRFs gives us our average IRF.<sup>13</sup>

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<sup>11</sup> The algorithm described here was first publicly described by Tom Holden at <http://bit.ly/I0nAHf>.

<sup>12</sup> An identical approximation is made in the non-linear case by Adjemian and Juillard (2011).

<sup>13</sup> This is the algorithm used by Dynare for constructing IRFs in non-linear models. See <http://www.dynare.org/DynareWiki/IrFs>.

## 2.6. Estimating Bounded Models

Our method for simulating models incorporating a zero lower bound naturally leads to an algorithm for particle filter based estimation of them. Indeed, since the observation equations are still linear in the state, and the transition equations are near linear, this is likely to be far more efficient than particle filter estimation of second order approximations to standard DSGE models (as in Fernández-Villaverde and Rubio-Ramírez (2010)). Indeed, since our solution method readily delivers last period's expectation of today's shadow price shock, we can write down a close approximation to the model for which the transition equations are linear in today's shock. For this approximated model, the optimal particle filter "proposal distribution" may be derived analytically, giving us a near optimal proposal distribution for the actual model, and enabling us to get high accuracy out of a small number of particles. We intend to assess the practical performance of this method in future work.

## 2.7. Generalisation to approximations of arbitrary order

The recent work of Braun et al. (2012) brought to light some serious problems with log-linearised solutions to models with a ZLB. They illustrate that the log-linearised equilibrium conditions can be misleading with respect to the existence and uniqueness of equilibrium, and may lead to "wrong" dynamics under the ZLB. For example, they show that in a simple NK model with Rotemberg (1996) quadratic price adjustment cost, the "paradox of toil"<sup>14</sup> disappears in the fully nonlinear model, at least when solved under perfect foresight. The authors pinpoint the resource cost of price adjustment, which is zero in a linearised model, as being the key to this discrepancy. The paper shows that these costs work as automatic stabilizers that reduce the variation in marginal costs and inflation, and decrease the government spending multiplier at the ZLB.

In fact, even fully nonlinear perfect foresight solutions may be misleading at the ZLB. For example, Adjemian and Juillard (2011) evaluate the accuracy of the (fully non-linear perfect foresight) extended path approach in a small-scale DSGE model and show that the accuracy drops significantly when the ZLB is hit regularly. Furthermore, using global methods, Fernández-Villaverde et al. (2012) find that when the interest rate stays at the ZLB for a prolonged time period, the government spending multiplier does indeed become large, contrary to the claims of Braun et al. (2012), again suggesting that the assumption of perfect foresight may itself be a source of substantial inaccuracy in the presence of a ZLB. Our algorithm provides an answer to these worries, as it may be readily generalised to "pruned" perturbation approximations (Sunghyun Henry Kim et al. 2008) of arbitrary order, and these approximations may be so constructed as to capture the precautionary motive arising from the risk of hitting the ZLB in future.

The key to ensuring our algorithm works at higher orders, is that in a  $d^{\text{th}}$  order pruned perturbation approximation, shocks of the form  $\epsilon_t^d$  only enter linearly,<sup>15</sup> hence, if we use shocks of the form  $(\epsilon_{s,t}^{\text{SP}})^d$  as shadow shocks, then our algorithm will work much as before, with the expected path implied by the pruned perturbation approximation taking the place of the expected path under

<sup>14</sup> A fall in employment after a labour tax cut at the ZLB. See Eggertsson and Woodford (2003).

<sup>15</sup> See equation (36) of (Sunghyun Henry Kim et al. 2008).

perfect foresight. Using shocks of the highest available order as shadow shocks is also consistent with perturbation approximation theory, since with Gaussian shocks the probability of hitting the ZLB is  $o(\sigma^d)$  for any  $d \in \mathbb{N}$ , where  $\sigma$  is the perturbation parameter controlling the variance of shocks.

To capture the precautionary motive arising from the risk of hitting the ZLB in future we require  $d$  to be even, since in that case  $(\epsilon_{s,t}^{\text{SP}})^d$  is positive in expectation.<sup>16</sup> This enables us to capture the effects of the ZLB on each series' mean, removing the deficiency in our stochastic simulation algorithm previously noted. To do this requires us to first solve a fixed-point problem to ensure that the variance of  $\epsilon_{s,t}^{\text{SP}}$  used in constructing the perturbation approximation actually agrees with the variance that is implied by the simulated mean values of  $\alpha$ . In practice, the fixed-point problem is solved by standard numerical optimisation algorithms, at low accuracies since each residual evaluation requires the computation of simulated moments. We also found it helpful to treat the non-stochastic steady-state inflation target as an additional parameter to be optimised, to hold the mean level of inflation constant.

It is also possible to approximate around the risky steady state, following Juillard (2011), within our algorithm. This enables the model's responses to reflect better the differences in all equations' slope close to the ZLB.

At sufficiently high (even) orders of perturbation approximation, our algorithm (using either the non-stochastic steady state or the risky steady state) must beat any perfect foresight method in terms of accuracy, since it captures the effects of uncertainty on the variables' means. Indeed, we show in the next section that even at order 2 our algorithm beats the extended path in standard models. Our algorithm is likely to be particularly useful for the analysis of "paradox of toil"-type effects, since it enables the analysis of the effects of the ZLB even in second order approximations to large models.

### 3. Accuracy

#### 3.1. A borrowing constraints model

We begin with a very simple model taken from Guerrieri and Iacoviello (2012).

An individual's income follows the process  $\log Y_t = \rho \log Y_{t-1} + \sigma \sqrt{1 - \rho^2} \varepsilon_t$ , where  $\varepsilon_t \sim \text{NIID}(0,1)$ ,  $\sigma = 0.03$  and  $\rho = 0.9$ . This income may be used for consumption  $C_t$  or saving, and it also acts as collateral. There is a collateral constrain on total borrowing,  $B_t$ , that states that  $B_t \leq MY_t$ , with  $M = 2$ . Consumers maximise the utility function:

$$U = \mathbb{E}_t \sum_{s=0}^{\infty} \beta^s \log C_{t+s},$$

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<sup>16</sup> Our algorithm will not always generate positive values for  $(\epsilon_{s,t}^{\text{SP}})^d$  as news may not be realised. However, since  $\epsilon_{s,t}^{\text{SP}}$  only enters into the transition equations when raised to the power of  $d$ , this will not result in complex numbers entering into the simulated paths.

with  $\beta = 0.94$ , subject to the collateral constraint and subject to the budget constraint  $C_t = Y_t + B_t - RB_{t-1}$ , with  $R = 1.05$ . In normal times, the collateral constraint binds, so in place of the standard Euler equation we use the equations:

$$A_t = \max \left\{ 0, \frac{1}{(1+M)Y_t - RB_{t-1}} - \mathbb{E}_t \frac{\beta R}{C_{t+1}} \right\}$$

$$\frac{1}{C_t} = \mathbb{E}_t \frac{\beta R}{C_{t+1}} + A_t$$

where  $A_t$  is an auxiliary variable.

	$\Pr\left(\frac{B_t}{Y_t} > 1.98\right)$	$\mathbb{E} \log C_t$	$\sqrt{\text{Var} \log C_t}$	$\text{Corr}(\log C_t, \log Y_t)$
Linear <sup>17</sup>	100%	-0.1054	0.0480	0.75
Log-linear <sup>18</sup>	100%	-0.1043	0.0475	0.77
Piecewise linear <sup>17</sup>	88%	-0.1051	0.0426	0.82
Extended path	89%	<b>-0.1045</b>	0.0431	0.84
Our algorithm order 1 <sup>18</sup>	88%	-0.1040	0.0427	0.85
Our algorithm order 2 <sup>18</sup>	<b>85%</b>	-0.1046	<b>0.0420</b>	<b>0.86</b>
Value function <sup>17</sup>	79%	-0.1045	0.0403	0.86

**Table 1: An accuracy comparison in the simple borrowing constraints model of Guerrieri and Iacoviello (2012).**

In Table 1 below we present a comparison of the results of our algorithms with the piecewise linear algorithm of Guerrieri and Iacoviello (2012) and the extended path method of Adjemian and Juillard (2011). All values except those taken from Guerrieri and Iacoviello (2012) were generated from a run of 10000 periods. The shaded cells with bold text show the results coming closest to the value function iteration results, and the shaded cells without bold text show the next closest values. Our second order algorithm comes closest to the value function iteration results along three out of the four metrics considered, and is a runner up in that fourth case. Indeed, for this model, even our first order algorithm beats the extended path algorithm in three cases out of four. However, it is only really the second order version that can come close to matching the percentage of time in which the constraint binds, by virtue of taking into account the incentive to save to avoid it.

### 3.2. An NK model

We now turn to the NK model of Fernández-Villaverde et al. (2012). This is a standard nonlinear NK model with sticky prices, labour choice, flexible wages, Taylor rule monetary policy and a stochastic share of government spending in output. The equilibrium conditions for this model are given in the first appendix, section 7.1, and we calibrate to the same standard values as Fernández-Villaverde et al. (2012).

To assess our accuracy, we treat the solution of Fernández-Villaverde et al. (2012) as the “truth”, and measure the deviations between their simulated paths<sup>19</sup> and the paths generated by our

<sup>17</sup> Taken from Guerrieri and Iacoviello (2012).

<sup>18</sup> With  $Y_t$  and  $C_t$  simulated in logs.

<sup>19</sup> We would like to thank Grey Gordon for providing us with this data.

algorithms (with all variables in logs) and the extended path method of Adjemian and Juillard (2011). All paths were length 30000 periods. We report the norms of these errors for consumption, nominal interest rates and inflation in Tables 2, 3 and 4 respectively. (Cells are formatted as before.)

	1 norm	2 norm	$\infty$ norm
Log-linear	364.0	1.011	0.03816
Extended path	311.0	0.7575	0.01792
Our algorithm order 1	336.7	0.8120	0.03593
Our algorithm order 2	<b>276.1</b>	<b>0.6924</b>	<b>0.01780</b>

**Table 2: Norms of the approximation errors in log consumption in the NK model of Fernández-Villaverde et al. (2012).**

	1 norm	2 norm	$\infty$ norm
Capped log-linear <sup>20</sup>	314.4	0.8344	0.01096
Extended path	<b>311.6</b>	0.8240	0.01064
Our algorithm order 1	314.4	0.8344	0.01096
Our algorithm order 2	313.8	<b>0.8144</b>	<b>0.01050</b>

**Table 3: Norms of the approximation errors in log gross nominal interest rates in the NK model of Fernández-Villaverde et al. (2012).**

	1 norm	2 norm	$\infty$ norm
Log-linear	217.7	0.6021	0.01576
Extended path	<b>206.5</b>	<b>0.5555</b>	0.01387
Our algorithm order 1	212.8	0.5695	0.01381
Our algorithm order 2	208.2	0.5621	<b>0.009203</b>

**Table 4: Norms of the approximation errors in log inflation in the NK model of Fernández-Villaverde et al. (2012).**

At second order, our algorithm beats the extended path algorithm for consumption whichever norm is used. This is unsurprising since consumption is sensitive to risk. For the nominal interest rate, it beats the extended path method with respect to the 2 norm or the  $\infty$  norm, but not with respect to the 1 norm. This implies the extended path algorithm is capable of closely following interest rates a lot of the time, but occasionally it is a long way off, perhaps because of its difficulties in tracking consumption. Finally, for the inflation rate, the extended path method is more accurate with respect to all norms except the  $\infty$  norm. It is also only with respect to the  $\infty$  norm of the inflation error that our first order algorithm is capable of beating the extended path one.

The evidence then is a little mixed here, with six instances in which our second order algorithm wins, and three instances in which the extended path one does. However, whereas the extended path algorithm took 7 hours and 25 minutes, ours completed in only 2 hours and 34 minutes on an identical system, with the vast majority of that time in the “fixed cost” stage of finding the correct variance for the shadow shocks. In summary, then, we find that our second order algorithm is both marginally more accurate, and significantly faster, and so it seems right to conclude that our second order algorithm is the natural choice for NK models.

<sup>20</sup> We replace the interest rate generated by the log-linear simulation with the maximum of 0 and the generated value.

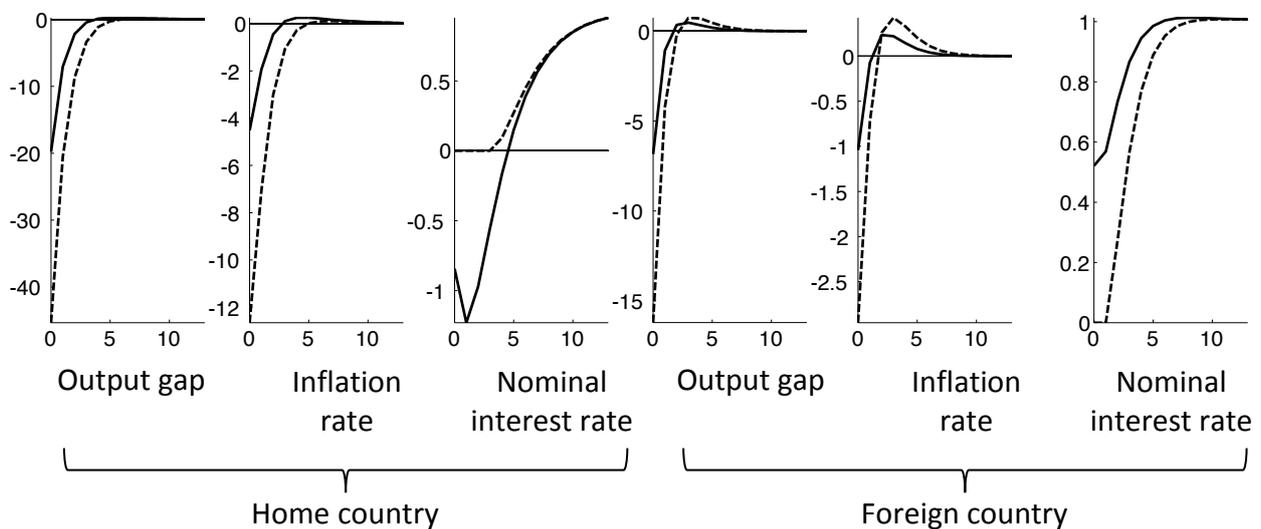
## 4. Sample applications

In order to illustrate the usefulness of the algorithm provided in section 2, we apply it to two popular linear DSGE models. The first is the stylised two-country setting of Clarida et al. (2002), which we choose in order to show that our method can easily handle multiple constraints. The second is the estimated Smets and Wouters (2003) model of the Euro area, which acts as an illustration of the speed of our algorithm, even for large models. In future versions of this paper we will also analyse second order approximations to these models.

### 4.1. The ZLB in the two-country model of Clarida et al. (2002)

The framework used in the following is a completely symmetric version of the Clarida et al. (2002) model with sticky prices in both countries, and perfect risk-sharing. We add domestic and foreign preference shocks as exogenous drivers, and introduce a fraction of backward-looking price-setters as in Galí and Gertler (1999). The model is described in full in the appendix, section 7.2, along with its calibration. This calibration is standard with the exception of the elasticity of intertemporal substitution which takes a high value (i.e. a low value for risk aversion) in order to generate strong co-movement across countries. As a result, we do not claim that the simulations provide a realistic story of the transmission of shocks during the recent (nor any other) financial crises; we provide these simulations purely for illustrative reasons.

In **Figure 1**, we show the IRFs of the benchmark model, ignoring the ZLB (solid line), and the model imposing the ZLB via our algorithm (dashed line), to a negative domestic preference shock of magnitude 0.65. We choose such a strong shock to ensure that both countries' ZLBs are hit.

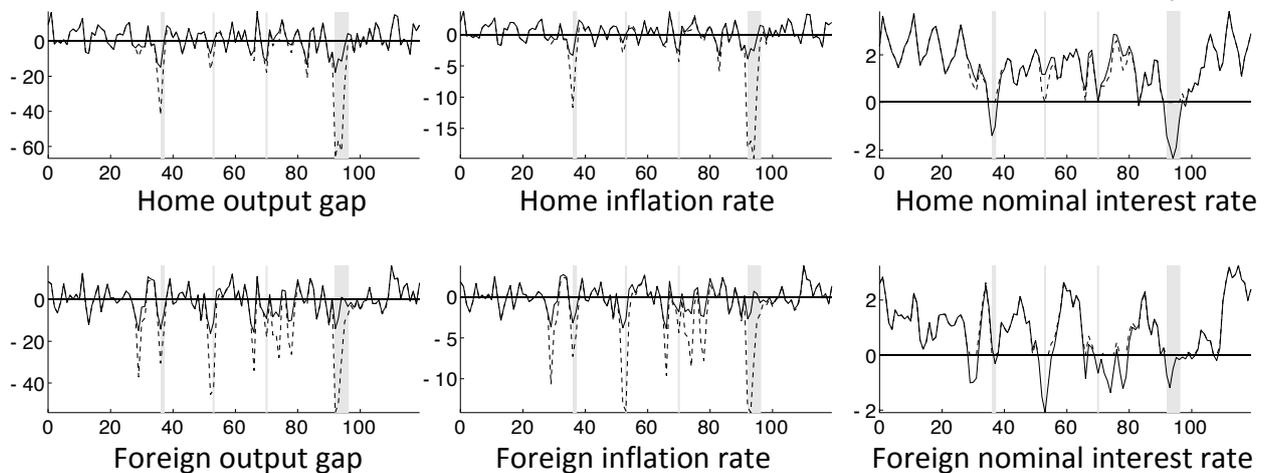


**Figure 1: Perfect foresight IRFs to a domestic demand shock in the two-country model of Clarida et al. (2002).** (Dashed line imposes the ZLB, solid line does not.)<sup>21</sup>

<sup>21</sup> Output gaps and inflation are measured in percentage deviations from equilibrium, and interest rates are measured in percent.

The fall in domestic demand induces output in both countries to fall. This leads to falling inflation in both countries and hence the central banks decrease the interest rates to boost demand. However, in the presence of the ZLB, the central banks are unable to decrease the interest rate strongly enough to generate a negative real interest rate, and consequently the recession is even more severe. This is further amplified by even larger downturns of both inflation rates. Since the foreign nominal interest rate hits the ZLB, a strong foreign recession is generated, despite the model's otherwise feeble transmission mechanism. Producing **Figure 1** took roughly a half second on a standard desktop PC, illustrating just how low is the computation cost of imposing the ZLB.

In Figure 2 we go on to simulate this model (with the standard deviation of both shocks set equal to 0.2), for 220 quarters, dropping the first 100, a process that took less than two seconds. The shaded areas of the figure show periods when both bounds are hit simultaneously, illustrating that recessions become substantially more severe in these situations.



**Figure 2: Simulations from the two-country model of Clarida et al. (2002).**

(Dashed line imposes the ZLB, solid line does not.)<sup>21</sup>

Finally, in Table 5 we show the simulated moments of both the benchmark model (ignoring the ZLBs) and the constrained one, evaluated from a sample of 50,000 periods. Using our algorithm this took only 7 minutes and 24 seconds, whereas the extended path algorithm took 37 minutes and 19 seconds to produce identical results, running on the same machine.<sup>22</sup>

	Home country			Foreign country		
	Output	Inflation	Nom. int.	Output	Inflation	Nom. int.
Benchmark mean	-0.07	-0.01	0.97	0.01	0.00	1.01
Constrained mean	-3.75	-1.17	1.12	-3.55	-1.10	1.14
Benchmark standard dev.	6.90	1.58	1.33	6.88	1.57	1.31
Constrained standard dev.	14.80	4.13	1.06	14.54	4.03	1.06

**Table 5: Moments from the two-country model of Clarida et al. (2002).**

<sup>22</sup> The relative speed of our algorithm becomes even more apparent if we decrease the standard deviations of both shocks to 0.1, as then our algorithm needs only 4m36s, while the extended path method still takes 31m52s.

Table 5 makes clear the magnitude of the increase in the volatility of both output gaps and inflation when we impose the ZLB on interest rates, as well as the large recessionary impact.<sup>23</sup> We stress again that we do not claim these values to be realistic. Rather, we presented the previous graphs and Table 5 to make clear the importance of the ZLB, and underline the tiny cost of imposing it using the method proposed in this paper.

Having shown the workings of the algorithm in a very stylized example, we now turn to a more realistic application, using the seminal Smets and Wouters (2003) model of the Euro Area.

#### **4.2. The ZLB in the Smets and Wouters (2003) model**

Modern macroeconomic models are getting progressively larger, so it is important that our algorithm can handle big models, of which the Smets and Wouters (2003) model is the canonical example. It also enables us to get a more realistic impression of the importance of the ZLB for the analysis of DSGE models.

We briefly recap the features of the model, but refer the reader to the paper for further details. It features sticky prices and wages, capital adjustment costs, variable capital utilisation and habit formation in consumption. The stochastic side of the model consists of ten exogenous shocks: six persistent shocks (technology, investment, preferences, labor supply, government spending and an inflation objective shock), all modelled as standard AR(1) -processes, and four short-run i.i.d. shocks (wage mark-ups, price mark-ups, Tobin's Q, and a monetary policy shock).

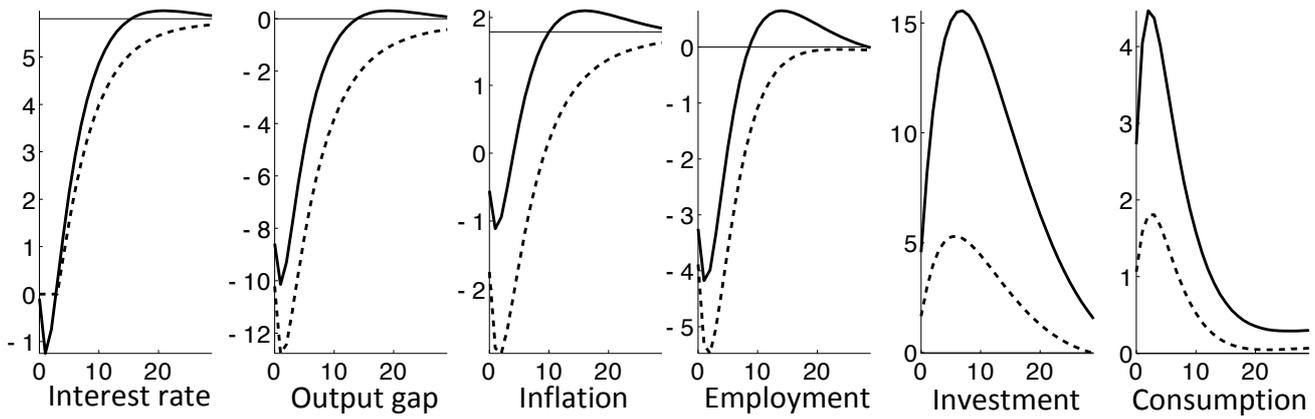
Since the original paper uses detrended data for estimation, the inflation target is set to zero. As this implies a very low steady state interest rate, we would find ourselves at the ZLB implausibly often. Hence, we add a positive annual inflation target of 1.8% to the Taylor rule, implying a quarterly steady state interest rate of 1.451%. Since the official target of the ECB is below, but near 2%, we believe this to be an adequate representation of Europe's monetary policy. All other model parameters are calibrated according to the reported posterior mode in Smets and Wouters (2003).<sup>24</sup>

Smets and Wouters (2003) find that preference and productivity shocks are the chief drivers of fluctuations in nominal interest rates. It makes sense then to look at productivity shocks to illustrate how things change when the ZLB is imposed. In Figure 3, we plot the IRF to a large positive productivity shock (4.9 times the shock's estimated standard deviation). Since natural output increases faster than actual output, this produces a large negative output gap, and the ZLB is hit. Hitting the ZLB dampens the response of investment, and so rather than output returning to trend within four years (as it would in the absence of the ZLB), instead it takes around eight years for the output gap to close.

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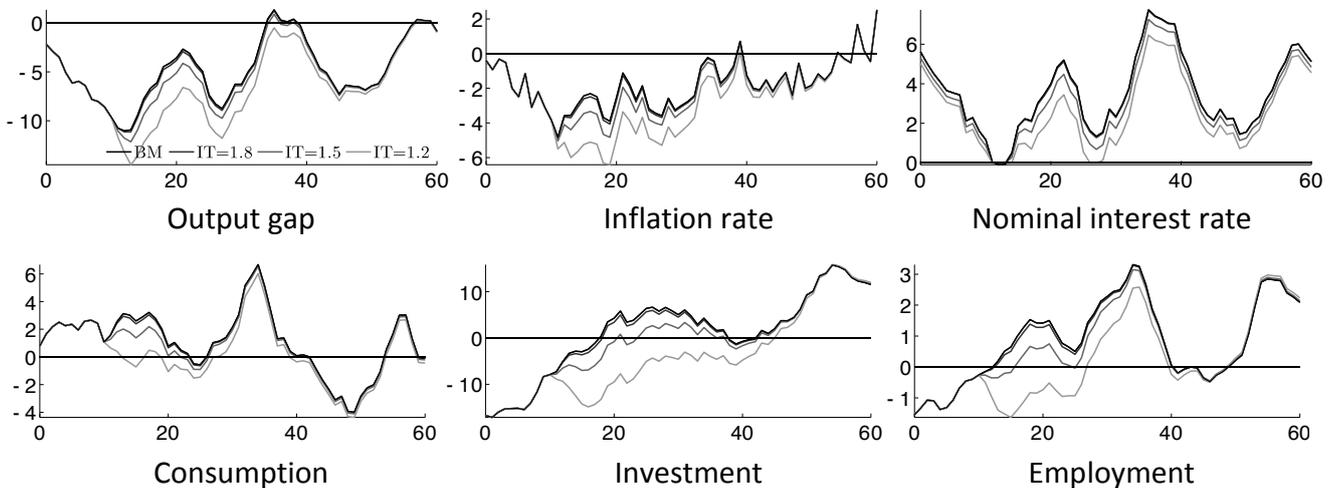
<sup>23</sup> Obviously, the assumption of completely symmetric countries implies that the moments of domestic and foreign variables should converge to equal values as we increase the number of periods.

<sup>24</sup> The model code is taken from the extensive model database described in Cwik et al. (2012).



**Figure 3: Impulse response to a large positive productivity shock in the Smets and Wouters (2003) model.** (Dashed line imposes the ZLB, solid line does not.)<sup>25</sup>

Obviously a realisation of a shock that is 4.9 times the shock's standard deviation is not particularly plausible, so one may wonder about the importance of the ZLB within the Smets and Wouters (2003) model. To assess this we ran a simulation of length 250,000 periods and noted when the ZLB was hit. With our preferred specification for the inflation target (1.8%), the implied probability of hitting the ZLB was only 0.01%. With an inflation target of 1.5% this increased to 0.02%, and with a target of 1.2% it hit 0.04%. This most likely reflects the absence of a financial accelerator mechanism from the Smets and Wouters (2003) model, and the fact that the data on which they estimated excluded the recent financial crisis.



**Figure 4: Simulations from the Smets and Wouters (2003) model.** (Dark line does not impose the ZLB, others correspond to imposing the ZLB with inflation targets of 1.8%, 1.5% and 1.2%)<sup>24</sup>

However, although the probability of hitting the ZLB is very low in the Smets and Wouters (2003) model, the implications when it is hit can be severe. This is illustrated in Figure 4, which presents

<sup>25</sup> The output gap, employment, investment and consumption are measured in percentage deviations from equilibrium, the interest rate and inflation are measured in percent; inflation and the interest rate are annualised.

simulated paths for a period of 15 years during which the ZLB was hit. This figure also illustrates the increasing effect of the ZLB as the inflation target is decreased.

The solid black line represents the benchmark scenario, ignoring the bound; the grey lines represent scenarios with a ZLB for different inflation targets (the lighter the line, the lower the inflation target). For a 1.8% percent inflation target, the ZLB is hit in the 13<sup>th</sup> quarter, and holds for only two periods. In this case, the simulated path is nearly identical to the benchmark one ignoring the ZLB. However, the lower the inflation target, the more severe is the fall in prices and aggregate activity. For an inflation target of 1.2%, the ZLB binds for one year. During this period, monetary policy is unable to prevent a drop in investment, resulting in investment staying below equilibrium for eleven and a half years. Moreover, employment stays below its long-term steady state for about nine years. The conclusion of this exercise is that with a high enough inflation target, hitting the ZLB is incredibly unlikely, and so the expected welfare loss from the bound is minimal. However, once the bound is hit, the impact can be very strong and highly persistent, as we are presently seeing in reality.

Finally, the Smets and Wouters (2003) model gives us another opportunity to present our algorithm's speed advantage. With the first two inflation targets, our algorithm only took 10 minutes and 33 seconds, for 250,000 periods, rising to 22 minutes and 17 seconds with an inflation target of 1.2%. By contrast, the extended path algorithm of Adjemian and Juillard (2011) took 3 hours and 45 minutes just to simulate 50000 periods, after which it crashed. Clearly, only our algorithm is viable on models of the scale of Smets and Wouters (2003).

## 5. Conclusion

This paper provides a fast, simple and intuitive method for the simulations of DSGE models with inequality constraints, based on the introduction of "shadow price shocks" which hit the bounded variables whenever the constraints are violated. We showed that at second order, the algorithm is more accurate than all methods except fully global ones, and we showed that the second order algorithm also leads in terms of speed, at least when compared to other algorithms of similar accuracy. At first order we illustrated that the speed was even greater, enabling it to be reliably used on the largest models around today.

We believe our algorithm will prove useful to a very wide variety of models including bounded variables, and we hope to investigate its application to the estimation of such models in future work.

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## 7. Online appendices

### 7.1. The NK model of Fernández-Villaverde et al. (2012)

Households choose consumption  $C_t$  and labour supply  $L_t$  to maximise their utility, given the wage  $W_t$ , the rate of inflation  $\Pi_t$  and the nominal interest rate  $R_t$ . This leads to the FOCs:

$$\frac{1}{C_t} = \mathbb{E}_t \frac{\beta_{t+1} R_t}{C_{t+1} \Pi_{t+1}}, \quad \psi L_t^\vartheta C_t = W_t.$$

Firms choose prices to maximise profits. This leads them to set a relative price  $\Pi_t^*$  satisfying:

$$\begin{aligned} \text{MC}_t &= \frac{W_t}{A_t}, \quad \varepsilon x_{1,t} = (\varepsilon - 1)x_{2,t}, \\ x_{1,t} &= \frac{\text{MC}_t Y_t}{C_t} + \theta \mathbb{E}_t \beta_{t+1} \Pi_{t+1}^\varepsilon x_{1,t+1}, \quad x_{2,t} = \Pi_t^* \left( \frac{Y_t}{C_t} + \theta \mathbb{E}_t \beta_{t+1} \frac{\Pi_{t+1}^{\varepsilon-1}}{\Pi_{t+1}^*} x_{2,t+1} \right), \end{aligned}$$

where  $A_t$  is productivity,  $\text{MC}_t$  is marginal costs and  $Y_t$  is output.

The central bank follows a standard Taylor rule, with monetary policy shock  $M_t$ , and government spending is a stochastic fraction  $S_{g,t}$  of total output:

$$R_t = \max \left\{ 1, R^{1-\rho_r} R_{t-1}^{\rho_r} \left[ \left( \frac{\Pi_t}{\Pi} \right)^{\phi_\pi} \left( \frac{Y_t}{Y} \right)^{\phi_y} \right]^{1-\rho_r} M_t \right\}, \quad G_t = S_{g,t} Y_t.$$

Inflation and price dispersion evolve according to:

$$1 = \theta \Pi_t^{\varepsilon-1} + (1 - \theta) (\Pi_t^*)^{1-\varepsilon}, \quad v_t = \theta \Pi_t^\varepsilon + (1 - \theta) (\Pi_t^*)^{-\varepsilon}.$$

Finally, market clearing in goods and labour markets imply:

$$Y_t = C_t + G_t, \quad Y_t = \frac{A_t L_t}{v_t}.$$

The stochastic processes  $\beta_t$ ,  $A_t$ ,  $M_t$  and  $S_{g,t}$  are all log AR(1).

### 7.2. Our modified version of the two-country model of Clarida et al. (2002)

Let  $\tilde{y}_t$ ,  $\pi_t$  and  $i_t$  represent the domestic output gap, inflation rate and interest rate respectively, and let  $\tilde{y}_t^*$ ,  $\pi_t^*$  and  $i_t^*$  denote their foreign equivalents. Let  $v_t$  and  $v_t^*$  be home and foreign demand shocks. When both countries are symmetric, the linearised model is described by the following four equations, and another four in which the roles of home and foreign country are swapped:

$$\begin{aligned} \tilde{y}_t &= \mathbb{E}_t \tilde{y}_{t+1} - \sigma_0^{-1} (i_t - \mathbb{E}_t \pi_{t+1} - \kappa_0 \mathbb{E}_t \Delta \tilde{y}_{t+1}^* + \kappa_0 (1 - \rho_v) v_t - \bar{i}), \\ \pi_t &= \Phi (\theta \beta \mathbb{E}_t \pi_{t+1} + \tau \pi_{t-1}) + \lambda \tilde{y}_t, \\ i_t &= \max \{ 0, \bar{i} + (1 - \phi_i) (\phi_\pi \pi_t + \phi_y \tilde{y}_t) + \phi_i (i_{t-1} - \bar{i}) \}, \\ v_t &= \rho_v v_{t-1} + \varepsilon_t^v. \end{aligned}$$

These represent the Euler equation, the NK Phillips curve, the Taylor rule and the evolution of the exogenous shock, respectively.

The parameters used in the equations above are the following functions of the structural parameters:

$$\kappa_0 = \frac{\sigma - 1}{2}, \quad \sigma_0 = \sigma - \kappa_0, \quad \kappa = \sigma - \kappa_0 + \varphi, \quad \lambda = \frac{(1 - \theta)(1 - \beta\theta)}{\theta} \kappa, \quad \bar{i} = \frac{1}{\beta}$$

$$\Phi = \frac{1}{\theta + \tau(1 - \theta(1 - \beta))},$$

where  $\sigma := 1/3$  is the inverse elasticity of intertemporal substitution in consumption,<sup>26</sup>  $\varphi := 1$  is the inverse Frisch elasticity of labour supply,  $\beta := 0.99$  is the discount factor,  $\theta := 3/4$  is the probability a firm's price must remain fixed and  $\tau := 1/10$  is the fraction of backwards looking price setters. We also set  $\phi_\pi := 1.5$ ,  $\phi_y := 0.125$  and  $\phi_i := 0.8$  in the central bank's reaction function and the shocks' persistence as  $\rho_v := 0.7$ .

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<sup>26</sup> The model implicitly assumes a unitary substitution elasticity between domestic and foreign goods. Hence, for values of  $\sigma$  smaller than one, any increase in foreign production is accompanied by an increase in domestic production, since the negative effect due to the implied real appreciation is dominated by the fall in the real interest rate stemming from an increase in domestic consumer price inflation. With  $\sigma = 1$  (i.e. log utility) both effects balance each other out, since in that case  $\kappa_0 = 0$ , meaning there is no transmission at all. We could derive similar results for lower substitution elasticities and higher values of  $\sigma$ , but we maintain this parameterisation to keep the model close to that of Clarida et al. (2002).