

ASYMPTOTIC ANALYSIS OF A MULTIPHASE DRYING MODEL MOTIVATED BY COFFEE BEAN ROASTING *

NABIL T. FADAI[†], COLIN P. PLEASE[†], AND ROBERT A. VAN GORDER[†]

Abstract. Recent modelling of coffee bean roasting suggests that in the early stages of roasting, within each coffee bean, there are two emergent regions: a dried outer region and a saturated interior region. The two regions are separated by a transition layer (or, drying front). In this paper, we consider the asymptotic analysis of a multiphase model of this roasting process which was recently put forth and studied numerically, in order to gain a better understanding of its salient features. The model consists of a PDE system governing the thermal, moisture, and gas pressure profiles throughout the interior of the bean. Obtaining asymptotic expansions for these quantities in relevant limits of the physical parameters, we are able to determine the qualitative behaviour of the outer and interior regions, as well as the dynamics of the drying front. Although a number of simplifications and scaling are used, we take care not to discard aspects of the model which are fundamental to the roasting process. Indeed, we find that for all of the asymptotic limits considered, our approximate solutions faithfully reproduce the qualitative features evident from numerical simulations of the full model. From these asymptotic results we have a better qualitative understanding of the drying front (which is hard to resolve precisely in numerical simulations), and hence of the various mechanisms at play as heating, evaporation, and pressure changes result in a roasted bean. This qualitative understanding of solutions to the multiphase model is essential if one is to create more involved models that incorporate chemical reactions and solid mechanics effects.

Key words. multiphase model, coffee bean roasting, Stefan problem, asymptotic analysis, drying front

AMS subject classifications. 80A22, 80M35, 74N20, 82C26

1. Introduction. As one of the most valuable commodities in the world [1], the coffee industry relies on fundamental research to improve the techniques and processes relating to its products. In particular, in this paper, we will focus on the roasting process of coffee beans. Most of the literature concerning the roasting of coffee beans present experimental data (see e.g. [2, 3, 4]), and use regression analysis and simple empirical models to interpret the results. Recently, the literature has included a more in-depth discussion concerning the mathematical modelling of the roasting of coffee beans (see e.g. [5, 6]). While other aspects of coffee processing have been examined from a mathematical perspective (e.g. [7]), mathematical models to describe the roasting of coffee beans have, with the exception of a few studies, been largely unexplored.

In [5], a system of partial differential equations (PDEs) modelling the transport of moisture and heat throughout a coffee bean were derived and studied. This model uses the concept of “mass diffusivity” to describe the transport of moisture in the coffee bean that was originally derived in [8], which applies to lower-temperature evaporation. The ideas in [5] served as excellent motivation for the authors in [6] to derive a mathematical model from first principles using conservation equations. In [6], the concept of multiphase flow and water evaporation were included, and the resulting multiphase model (referred to as Model 2 in [6]) incorporated the production of carbon dioxide gas, latent heat due to evaporation within the coffee bean, and the changing porosity of the bean. The use of multiphase modelling was previously applied in a variety of food heating problems [9, 10, 11, 12, 13, 14] (of particular relevance was the bread baking model of Zhang et al. [15]), and is a natural framework to model the coffee bean roasting process. Some simplifications were made to this full model (in particular, neglecting carbon dioxide production) in order to allow some preliminary understanding of the model behaviour. By examining the numerical solution of this multiphase model, a “drying front” that propagates through to the center of the bean.

Mathematical models describing drying have been explored previously (see e.g. [16, 17, 18]), which relate the drying of wood, bricks, and other materials. However, in these models, the crucial parameter regime being explored is when water vapour produced from evaporation can easily permeate through the material and reach the external environment. Due to the rigid, impermeable cellulose structure within a coffee bean, the water vapour created in a coffee bean’s biological cells cannot be easily released into the roasting environment. In consequence, the ratio between evaporation dynamics and vapour transport is very large, which motivates us to explore the leading-order dynamics of coffee bean roasting using asymptotic analysis. Hence, the model presented here should be appropriate to any drying problem where these physical

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[†]Mathematical Institute, University of Oxford, Andrew Wiles Building, Radcliffe Observatory Quarter, Woodstock Road, Oxford, OX2 6GG, United Kingdom (Robert.VanGorder@maths.ox.ac.uk)

49 phenomena are relevant.

50 As we have seen previously in [6], numerical results suggest that there are three main regions within a
 51 coffee bean as it is roasted. The first main region (which we refer to as Region i) is where the vapour pressure
 52 of water aligns with the steam table pressure. The second main region (which we refer to as Region ii) is when
 53 the moisture content of the bean is negligible. Between these two regions, we expect a thin transition layer,
 54 or “drying front”, in which the moisture content is rapidly evaporated away. Issues surrounding numerical
 55 resolution make it difficult to resolve the dynamics near the drying front. In light of these observations, we
 56 are motivated to extend the numerical results shown in [6] via asymptotic methods in order to understand
 57 the qualitative features of the multiphase model, and in particular, the interplay between the transition layer
 58 and the two larger regions.

59 In the present paper, we begin our discussion of the asymptotics of the full multiphase model in Sec. 2.
 60 Motivated by the numerical results seen in [6], we determine an approximate form of the drying front. We
 61 then obtain the leading-order asymptotics in Regions i and ii, as well as within the drying front. Despite
 62 several simplifications, we are able to obtain reasonable agreement between the asymptotic approximations
 63 and the numerical solution of the multiphase model described in [6], and are confident that the asymptotics
 64 capture the qualitative dynamics of the problem. In order to obtain more explicit results, and motivated by
 65 the fact that the coffee bean is at roasting temperature throughout most of the roasting process, in Sec. 3 we
 66 fix the temperature at the roasting temperature. Under this assumption, the vapour pressure and moisture
 67 content remain constant to leading order in Region i, while the the dynamics within Region ii reduce to a
 68 Stefan problem [19], to leading order. By considering the case of a large Stefan number limit, we determine
 69 a leading-order expression for the drying front for various geometries. As we will only focus on symmetric
 70 geometries (e.g. planar and spherical geometries), we can obtain explicit expressions for the drying front
 71 where there is only one spatial variable. This single spatial variable will be denoted as r in all geometries
 72 considered. We focus on the planar and spherical geometries in particular as it is reasonable to represent a
 73 coffee bean either as a sphere, or as a slab of porous material “curled up” into the shape of a bean. In [6]
 74 and in Secs. 2-3, the evaporation is modelled by Langmuir’s evaporation equation [20], and this may not
 75 be the most accurate way to represent the evaporation of water in a coffee bean. Therefore, in Sec. 4, we
 76 consider a more general evaporation rate for the multiphase model. While the dynamics inside the drying
 77 front, where evaporation dominates water transport, will vary, we determine that the qualitative behaviour
 78 in Regions i and ii remains unchanged to this larger class of evaporation rates, suggesting that the explicit
 79 choice of evaporation equation is not pivotal to a qualitative understanding of coffee bean roasting. Finally,
 80 in Sec. 5, we provide a summary and discussion of the results.

81 **2. Asymptotics of the Multiphase Model with Variable Temperature.** The full multiphase
 82 model that we will analyse is described by the PDEs in symmetric geometries (i.e. using a single spatial
 83 variable r)

84 (1)
$$\frac{\partial S}{\partial t} = -\frac{1}{\epsilon^2} I_v,$$

85 (2)
$$\frac{\partial}{\partial t} \left[\frac{(1 + \mathcal{T})P(1 - \sigma S)}{1 + \mathcal{T}T} \right] = -\frac{1}{\delta} \frac{\partial S}{\partial t} + \nabla \cdot \left[\frac{(1 + \mathcal{T})P \nabla P}{1 + \mathcal{T}T} \right],$$

86 (3)
$$\frac{\partial T}{\partial t} + \mathcal{A}_1 \frac{\partial}{\partial t} [S(1 + \mathcal{T}T)] = \mathcal{A}_2 \frac{\partial S}{\partial t} + \mathcal{A}_3 \nabla \cdot [(1 + \mathcal{A}_4 S) \nabla T],$$

88 with the symmetry conditions at the centre of the bean (i.e. $r = 0$)

89 (4)
$$\nabla T \cdot \mathbf{n} = 0, \quad \nabla P \cdot \mathbf{n} = 0,$$

90 the boundary conditions at the surface of the bean (i.e. $r = 1$)

91 (5)
$$\nabla T \cdot \mathbf{n} = \nu \left(\frac{1 - \sigma S}{1 - \sigma} \right) \left(\frac{1 + \mathcal{A}_4}{1 + \mathcal{A}_4 S} \right) (1 - T),$$

92

93 (6)
$$P = \begin{cases} P_{ST}(T), & T < T_a, \\ P_a, & T \geq T_a, \end{cases}$$

94 and the initial conditions

95 (7)
$$S(r, 0) = 1, \quad T(r, 0) = 0, \quad P(r, 0) = P_{ST}(0).$$

96 Here, the evaporation rate I_v and steam table pressure $P_{ST}(T)$ are given by

97 (8)
$$I_v = S(1 - \sigma S)(P_{ST} - P)\sqrt{\frac{1 + \mathcal{F}}{1 + \mathcal{F}T}} \quad \text{and} \quad P_{ST}(T) = \exp\left(\frac{\beta(T - 1)}{1 + \mathcal{F}T}\right).$$

98 A complete derivation of this model from the “simplified” multiphase model presented in [6] can be found in
 99 Appendix A. Here, S is the saturation (the volume fraction of water divided by the total volume of water
 100 and gas), I_v is the evaporation rate of water, P is the partial pressure of water vapour, T is the normalized
 101 temperature, and p_{ST} is the steam table pressure. Our three PDEs describe conservation of mass in water
 102 and vapour, as well as conservation of energy. The only transport mechanism considered for water is via
 103 evaporation, whereas in the gas phase, water vapour is transported either via evaporation or via Darcy flow.
 104 Finally, we assume that heat is transported via conduction in all three phases within the bean, but via
 105 convection at the surface of the bean. A key feature of this model is ϵ , which can be interpreted as a rescaled
 106 ratio between Darcy-driven vapour transport and evaporation. One can interpret δ as a rescaled density ratio
 107 of water vapour to water, and σ represents the initial water-to-void volume ratio. The boundary condition
 108 (6) is slightly modified from that in [6] and is described in Appendix A. Here, P_a is the ambient vapour
 109 pressure in the roasting chamber. We will also make the assumption that the change in boundary conditions
 110 for P only occurs at one critical time, namely, t^* . We define t^* as when the time when the evaporation
 111 temperature T_a is achieved at the surface of the bean, i.e. as the solution to the equation

112 (9)
$$T(1, t^*) = T_a := P_{ST}^{-1}(P_a).$$

113 This critical time will be used not only to signal the transition from one asymptotic region to another, but
 114 also to determine the form of the boundary conditions to be used.

115 We divide our system of PDEs into three regions in order to understand the approximate dynamics that
 116 occur in each region of the coffee bean. Using parameter values shown in [6], a typical value of $\epsilon \approx 1.54 \times 10^{-4}$
 117 suggests that we should consider the limit of $\epsilon \rightarrow 0^+$. Therefore, we will assume that $\epsilon \ll 1$, δ is either $O(1)$
 118 or $\ll 1$, depending on which region we are examining, and all other parameters are $O(1)$. In this limit,
 119 we can see from (1) that if time and space remain unscaled, $I_v = 0$ will be our leading-order equation, and
 120 from (8), this can occur in one of three ways. The first is if the vapour pressure is in equilibrium with its
 121 steam table pressure, i.e. $P = P_{ST}$. As the initial data is consistent with the vapour pressure in equilibrium
 122 with its steam table pressure, this will be the first case we will observe (which will be referred to as Region
 123 i). Secondly, $I_v = 0$ can be achieved by setting $S = 0$. This corresponds to where there is no more water
 124 to evaporate off, and will be denoted as Region ii. A final case where $I_v = 0$ is when $S = \sigma^{-1}$; however,
 125 this corresponds to when the coffee bean is completely saturated with water, which we will discard as an
 126 extraneous case.

127 We will also consider a narrow “drying front” that connects the two physically relevant asymptotic regions
 128 where $I_v = 0$ (Regions i and ii). This drying front, which is centred about $r = R(t)$, propagates from the
 129 surface of the bean towards the center of the bean and is where the moisture content S quickly goes from 1
 130 to 0. In this drying front around $R(t)$, we find that the temperature is spatially uniform, but will vary as
 131 time progresses. The temperature profile within the drying front is denoted as $T^*(t)$. A schematic diagram
 132 of these three regions is shown in Figure 1, including the time t^* at which evaporation first occurs at the
 133 surface of the bean.

134 We will now discuss the leading-order asymptotics of the three main regions in the limit of $\epsilon \rightarrow 0^+$
 135 and, where applicable, the additional limit of $\delta \rightarrow 0^+$. In Region i, where $P = P_{ST}$ at leading-order, our
 136 leading-order equations will reduce to a system of two PDEs for S and T . However, we will only be able
 137 to obtain analytic results if we further expand the leading-order solution in ϵ with an asymptotic series in
 138 δ . As we are only concerned with the leading-order asymptotic behaviour in Region i, we will not worry
 139 about the relative magnitude between ϵ and δ , where we might observe cross-terms at higher orders. With
 140 this additional simplification, we determine that S is constant at leading-order, and T can be described by
 141 the heat equation. We can then determine a leading-order approximation of t^* , which is the solution of a

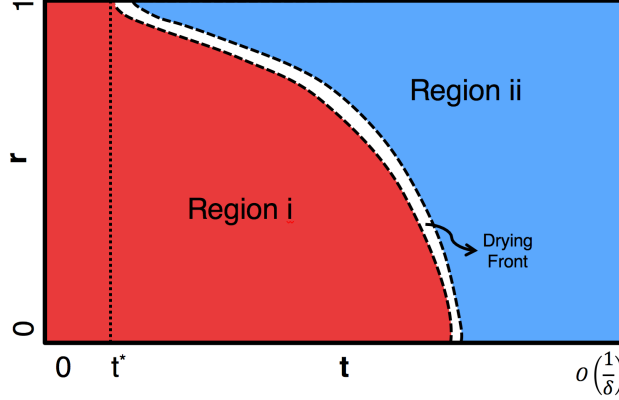


FIG. 1. A summary of where the different regions are as the bean dries. Region i is when the vapour pressure is in equilibrium, Region ii is the dry region, and the dashed lines indicate the narrow transition layer between the regions, which begins at time t^* , defined in (9).

transcendental equation. However, we note that for $t \geq t^*$, Region i is bounded in r between the drying front $R(t)$ and the centre of the bean, and we cannot find any analytic results via similarity solutions or separation of variables.

In order for Region ii to exist, we must have a transition layer in which S changes from 1 to 0. We expect this “drying front” to progress to the center of the bean. In this thin region, we will determine the leading-order ODE that governs the dynamics of S . In order to match with Region i, we also must have that T and P do not vary in space at leading-order; they will, however, vary in time. In consequence, our leading-order solution for T is denoted by $T^*(t)$, and for P to match with Region i, its leading-order solution is $P_{ST}(T^*(t))$. We will then show that the first order correction terms to T and P can be expressed in terms of the leading-order solution of S in the transition layer. While greater care needs to be taken when the transition layer is near the surface and centre of the bean, we will assume that the same dynamics in the rest of the transition layer apply at these endpoints. It is also important to note that we will assume that $\delta = O(1)$ in the transition layer.

Finally, we examine Region ii, where the evaporation has stopped due to lack of water. From our higher-order matching from exiting the transition layer, as well as the coupled PDE system for T and P , this gives us a Stefan problem in Region ii to not only determine T and P , but also $R(t)$ and $T^*(t)$. By assuming once again that $\delta \ll 1$, we obtain via a further asymptotic expansion the leading-order behaviour of T^* , R , P , and T in Cartesian and spherical geometries.

2.1. Asymptotics of Region i. In Region i, we have, from (1), $I_v = 0$ to leading order in the limit of $\epsilon \rightarrow 0^+$, implying that that $P = P_{ST}(T)$. Consider the asymptotic series valid as $\epsilon \rightarrow 0^+$,

$$(10) \quad S = S_0(r, t) + \epsilon S_1(r, t) + O(\epsilon^2), \quad T = T_0(r, t) + \epsilon T_1(r, t) + O(\epsilon^2), \quad P = P_{ST}(r, t) + \epsilon P_1(r, t) + O(\epsilon^2).$$

Substituting these asymptotic expansions into (2) and (3) gives us to lowest order

$$(11) \quad \frac{\partial S_0}{\partial t} \left(\frac{1}{\delta} - \sigma P_{ST}(T_0) \Lambda(T_0) \right) + (1 - \sigma S_0) \frac{\partial}{\partial t} [P_{ST}(T_0) \Lambda(T_0)] = \nabla \cdot [P_{ST}(T_0) \Lambda(T_0) \nabla P_{ST}(T_0)],$$

$$(12) \quad \frac{\partial S_0}{\partial t} (\mathcal{A}_1(1 + \mathcal{F}T_0) - \mathcal{A}_2) + \frac{\partial T_0}{\partial t} [1 + \mathcal{A}_1 \mathcal{F}S_0] = \mathcal{A}_3 \nabla \cdot [(1 + \mathcal{A}_4 S_0) \nabla T_0],$$

where $\Lambda(T_0) = \frac{1}{1 + \mathcal{F}T_0}$. As we cannot solve this system analytically, we now suppose that $\delta \ll 1$ and write an asymptotic series in powers of δ for S_0 and T_0 valid in the limit $\delta \rightarrow 0^+$ as

$$(13) \quad S_0 = \tilde{S}_0(r, t) + \delta \tilde{S}_1(r, t) + O(\delta^2), \quad T_0 = \tilde{T}_0(r, t) + \delta \tilde{T}_1(r, t) + O(\delta^2).$$

Substituting these asymptotic expansions into (11) gives us, to leading order, that $\frac{\partial \tilde{S}_0}{\partial t} = 0$. Therefore, the moisture content of the bean stays at its initial value, i.e. $\tilde{S}_0 = 1$. To lowest order, (12) then gives us

$$(14) \quad \frac{\partial \tilde{T}_0}{\partial t} = \mathcal{K} \nabla^2 \tilde{T}_0, \quad \text{where} \quad \mathcal{K} = \frac{\mathcal{A}_3(1 + \mathcal{A}_4)}{1 + \mathcal{A}_1 \mathcal{F}}.$$

173 Equation (14) has a time-dependent boundary condition, which depends on if evaporation has begun at the
 174 surface of the bean. This can be stated as

$$175 \quad (15) \quad \nabla \cdot \tilde{T}_0 \Big|_{r=1} = \nu \left(1 - \tilde{T}_0 \Big|_{r=1} \right), \quad t < t^*,$$

$$176 \quad (16) \quad \tilde{T}_0 \Big|_{r=R(t)} = T^*(t), \quad t \geq t^*.$$

178 Additionally, we will continue to impose the symmetry condition $\nabla \tilde{T}_0 \cdot \mathbf{n} = 0$ at $r = 0$, as well as the initial
 179 data $\tilde{T}_0(r, 0) = 0$. We are able to solve the PDE for $t < t^*$, and in particular, determine a leading-order
 180 approximation for t^* . By solving (14) in spherical co-ordinates, we obtain that

$$181 \quad (17) \quad \tilde{T}_0(r, t) = 1 - \sum_{n=1}^{\infty} \frac{c_n}{r} \sin(\mu_n r) \exp(-\mu_n^2 \mathcal{K} t).$$

182 where the eigenvalues μ_n satisfy the transcendental equation $\mu_n \cot(\mu_n) = 1 - \nu$ and the constants c_n have
 183 the form

$$184 \quad (18) \quad c_n = \begin{cases} \frac{2\nu \cos \mu_n}{\mu_n (\sin^2 \mu_n - \nu)}, & \nu \neq 1, \\ \frac{8(-1)^n}{\pi^2 (1+2n)^2}, & \nu = 1. \end{cases}$$

185 To determine t^* in spherical co-ordinates, denoted as t_{Sph}^* , we impose, from (9), that $\tilde{T}_0(1, t_{\text{Sph}}^*) = T_a$. When
 186 $\nu \neq 1$, this is equivalent to writing

$$187 \quad (19) \quad \sum_{n=1}^{\infty} \left(\frac{\cos^2 \mu_n}{\sin^2 \mu_n - \nu} \right) \exp(-\mu_n^2 \mathcal{K} t_{\text{Sph}}^*) = \frac{(1 - T_a)(1 - \nu)}{2\nu},$$

188 or when $\nu = 1$,

$$189 \quad (20) \quad \sum_{n=1}^{\infty} \frac{\exp\left(-\frac{\mathcal{K}\pi^2}{4}(1+2n)^2 t_{\text{Sph}}^*\right)}{(1+2n)^2} = \frac{\pi^2(1 - T_a)}{8}.$$

190 Using parameter values shown in [6], yielding $\mathcal{K} \approx 2.25$, $\nu \approx 0.585$, $T_a \approx 0.519$, this gives us that $t_{\text{Sph}}^* \approx 0.173$,
 191 or about 45.9 seconds in dimensional units.

192 Similarly, we can determine t^* in Cartesian co-ordinates, denoted as t_{Cart}^* , by determining that the
 193 solution of (14) in Cartesian co-ordinates, with a Neumann boundary condition at $r = 0$, is

$$194 \quad (21) \quad \tilde{T}_0(r, t) = 1 - \sum_{n=1}^{\infty} d_n \cos(\lambda_n r) \exp(-\lambda_n^2 \mathcal{K} t),$$

195 where

$$196 \quad (22) \quad \lambda_n \tan \lambda_n = \nu \quad \text{and} \quad d_n = \frac{2\nu \sin \lambda_n}{\lambda_n (\nu + \sin^2 \lambda_n)}.$$

197 This in turn allows us to determine t_{Cart}^* via the transcendental equation

$$198 \quad (23) \quad \sum_{n=1}^{\infty} \frac{\sin^2 \lambda_n}{\nu + \sin^2 \lambda_n} \exp(-\lambda_n^2 \mathcal{K} t_{\text{Cart}}^*) = \frac{1 - T_a}{2},$$

199 which, using parameter values stated above, gives us that $t_{\text{Cart}}^* \approx 0.494$, or about 131 seconds in dimensional
 200 units.

201 **2.2. Asymptotics of the Transition Layer.** In order to understand how S varies from 1 to 0, we
 202 must examine the transition layer in the $\epsilon \rightarrow 0^+$ limit, with all other parameters (including δ) being $O(1)$.
 203 We expect that this transition layer will happen when r is close to the ‘‘drying front’’ $R(t)$, so we introduce
 204 the scaling $r = R(t) + \epsilon \hat{r}$. Once again, we can expand P , T , and S as asymptotic series as $\epsilon \rightarrow 0^+$,

$$205 \quad (24) \quad S = S_0(\hat{r}, t) + \epsilon \hat{S}_1(\hat{r}, t) + O(\epsilon^2), \quad P = P_0(\hat{r}, t) + \epsilon P_1(\hat{r}, t) + O(\epsilon^2), \quad T = T_0(\hat{r}, t) + \epsilon T_1(\hat{r}, t) + O(\epsilon^2).$$

206 We will first show that $T_0(\hat{r}, t) \equiv T^*(t)$ and $P_0(\hat{r}, t) \equiv P^*(t) := P_{ST}(T^*(t))$. Do this, we note that, in order
 207 to match our transition layer into Region i, we must have that

$$208 \quad (25) \quad P_0|_{\hat{r} \rightarrow -\infty} \rightarrow P^*(t) \quad \text{and} \quad T_0|_{\hat{r} \rightarrow -\infty} \rightarrow T^*(t).$$

209 By substituting (24) into (2) and (3), we obtain at $O(\epsilon^{-2})$

$$210 \quad (26) \quad \frac{\partial}{\partial \hat{r}} \left[\frac{P_0 \frac{\partial P_0}{\partial \hat{r}}}{1 + \mathcal{T} T_0} \right] = 0, \quad \frac{\partial}{\partial \hat{r}} \left[\frac{\partial T_0}{\partial \hat{r}} (1 + \mathcal{A}_4 S_0) \right] = 0.$$

211 We note that these equations hold in any geometry at leading order, provided that we are sufficiently far
 212 away from any geometry-induced singularities that could produce additional derivative terms at $O(\epsilon^{-2})$,
 213 e.g. if $R(t) = O(\epsilon)$ in spherical co-ordinates. Integrating (26) and imposing (25) implies that $T_0(\hat{r}, t) \equiv T^*(t)$
 214 and $P_0(\hat{r}, t) \equiv P^*(t)$. To determine the leading-order behaviour for S , we note that using (24) in (8) and
 215 expanding gives

$$216 \quad (27) \quad P_{ST} = P^* \left(1 + \epsilon \frac{\beta(1 + \mathcal{T})}{(1 + \mathcal{T} T^*)^2} T_1 \right) + O(\epsilon^2),$$

$$I_v = -\epsilon \left(P_1 - \frac{\beta(1 + \mathcal{T})}{(1 + \mathcal{T} T^*)^2} T_1 P^* \right) S_0 (1 - \sigma S_0) \sqrt{\frac{1 + \mathcal{T}}{1 + \mathcal{T} T^*}} + O(\epsilon^2).$$

217 Using these along with (25), we obtain, at $O(\epsilon^{-1})$, that (1)-(3) give

$$218 \quad (28) \quad -R'(t) \frac{\partial S_0}{\partial \hat{r}} = \Psi(P_1, T_1) S_0 (1 - \sigma S_0),$$

$$219 \quad (29) \quad \sigma P^* R'(t) \frac{\partial S_0}{\partial \hat{r}} = -\frac{1}{\delta} \Psi(P_1, T_1) S_0 (1 - \sigma S_0) \left(\frac{1 + \mathcal{T} T^*}{1 + \mathcal{T}} \right) + P^* \frac{\partial^2 P_1}{\partial \hat{r}^2},$$

$$220 \quad (30) \quad -\mathcal{A}_1 (1 + \mathcal{T} T^*) R'(t) \frac{\partial S_0}{\partial \hat{r}} = \mathcal{A}_2 \Psi(P_1, T_1) S_0 (1 - \sigma S_0) + \mathcal{A}_3 \frac{\partial}{\partial \hat{r}} \left[(1 + \mathcal{A}_4 S_0) \frac{\partial T_1}{\partial \hat{r}} \right],$$

221
 222 where

$$223 \quad (31) \quad \Psi(P_1, T_1) := \sqrt{\frac{1 + \mathcal{T}}{1 + \mathcal{T} T^*}} \left(P_1 - \frac{\beta(1 + \mathcal{T})}{(1 + \mathcal{T} T^*)^2} T_1 P^* \right).$$

224 Finally, the matching conditions with Regions i and ii are

$$225 \quad (32) \quad S_0 \rightarrow 1, \quad P_1 \rightarrow 0, \quad \text{and} \quad T_1 \rightarrow 0 \quad \text{as} \quad \hat{r} \rightarrow -\infty,$$

$$226 \quad (33) \quad S_0 \rightarrow 0 \quad \text{as} \quad \hat{r} \rightarrow +\infty,$$

$$227 \quad (34) \quad \frac{\partial P_1}{\partial \hat{r}} \Big|_{\hat{r} \rightarrow +\infty} = \frac{\partial P}{\partial r} \Big|_{r \rightarrow R(t)},$$

$$228 \quad (35) \quad \frac{\partial T_1}{\partial \hat{r}} \Big|_{\hat{r} \rightarrow +\infty} = \frac{\partial T}{\partial r} \Big|_{r \rightarrow R(t)}.$$

230 In interpreting (32)-(35), we note that the limits where $r \rightarrow R(t)$ are matching conditions for Regions i and
 231 ii, whereas the limits where $\hat{r} \rightarrow \pm\infty$ refer to matching conditions for the transition layer.

232 We will now show that in the transition layer, the terms P_1 and T_1 can both be expressed in terms of
 233 S_0 alone. Firstly, by eliminating the terms with $\Psi(P_1, T_1)$ in (28) and (29), we obtain

$$234 \quad (36) \quad P^* \frac{\partial^2 P_1}{\partial \hat{r}^2} = \left[\sigma P^* - \frac{1}{\delta} \left(\frac{1 + \mathcal{T} T^*}{1 + \mathcal{T}} \right) \right] R'(t) \frac{\partial S_0}{\partial \hat{r}}.$$

235 Integrating this and imposing the matching conditions (32) yields

$$236 \quad (37) \quad \frac{\partial P_1}{\partial \hat{r}} = \left[\frac{1}{\delta} \left(\frac{1 + \mathcal{F}T^*}{(1 + \mathcal{F})P^*} \right) - \sigma \right] R'(t)(1 - S_0).$$

237 Similarly, eliminating terms with $\Psi(P_1, T_1)$ in (28) and (30) gives us

$$238 \quad (38) \quad R'(t) [\mathcal{A}_2 - \mathcal{A}_1(1 + \mathcal{F}T^*)] \frac{\partial S_0}{\partial \hat{r}} = \mathcal{A}_3 \frac{\partial}{\partial \hat{r}} \left[(1 + \mathcal{A}_4 S_0) \frac{\partial T_1}{\partial \hat{r}} \right].$$

239 Integrating and imposing the matching conditions (32) yields, after some rearranging,

$$240 \quad (39) \quad \frac{\partial T_1}{\partial \hat{r}} = -\frac{1}{\mathcal{A}_3} R'(t) [\mathcal{A}_2 - \mathcal{A}_1(1 + \mathcal{F}T^*)] \left(\frac{1 - S_0}{1 + \mathcal{A}_4 S_0} \right).$$

241 Finally, by rearranging (28) to isolate S_0 , we obtain

$$242 \quad (40) \quad \frac{\frac{\partial S_0}{\partial \hat{r}}}{S_0(1 - \sigma S_0)} = -\frac{\Psi(P_1, T_1)}{R'(t)}.$$

243 In order to write a single ODE for S_0 , we differentiate (40) with respect to \hat{r} , as well as substitute in (37)
244 and (39), to give us

$$245 \quad (41) \quad \frac{\partial^2 S_0}{\partial \hat{r}^2} - \left(\frac{\partial S_0}{\partial \hat{r}} \right)^2 \frac{1 - 2\sigma S_0}{S_0(1 - \sigma S_0)} + S_0(1 - \sigma S_0)(1 - S_0)\Upsilon(S_0) = 0,$$

246 where we define

$$247 \quad (42) \quad \Upsilon(S_0) := \sqrt{\frac{1 + \mathcal{F}}{1 + \mathcal{F}T^*}} \left[\frac{1}{\delta} \left(\frac{1 + \mathcal{F}T^*}{(1 + \mathcal{F})P^*} \right) - \sigma - \left(\frac{\beta(1 + \mathcal{F})P^*}{\mathcal{A}_3(1 + \mathcal{F}T^*)^2} \right) \left(\frac{\mathcal{A}_2 - \mathcal{A}_1(1 + \mathcal{F}T^*)}{1 + \mathcal{A}_4 S_0} \right) \right].$$

248 We note that, aside from the denominator $1 + \mathcal{A}_4 S_0$, the components of the function $\Upsilon(S_0)$ are independent
249 in \hat{r} . Let us assume that $S_0(\hat{r})$ is strictly monotone in \hat{r} . By taking $f(\hat{r}) = S_0(\hat{r})$ and $g(\hat{r}) = \frac{\partial S_0}{\partial \hat{r}}$, we can
250 transform (41) into the system of first-order ODEs

$$251 \quad (43) \quad \begin{cases} \frac{\partial f}{\partial \hat{r}} = g, \\ \frac{\partial g}{\partial \hat{r}} = g^2 \frac{1 - 2\sigma f}{f(1 - \sigma f)} - f(1 - \sigma f)(1 - f)\Upsilon(f), \end{cases}$$

252 and dividing the second equation of (43) by the first equation gives us

$$253 \quad (44) \quad \frac{dg}{df} = g \frac{1 - 2\sigma f}{f(1 - \sigma f)} - \frac{f(1 - \sigma f)(1 - f)\Upsilon(f)}{g}.$$

254 By monotonicity of $S_0(\hat{r})$, the function $\frac{dg}{df}$ is well-defined. We identify equation (44) as a Bernoulli-like ODE;
255 letting $w = g^2$, (44) becomes

$$256 \quad (45) \quad \frac{dw}{df} = 2w \frac{1 - 2\sigma f}{f(1 - \sigma f)} - 2f(1 - \sigma f)(1 - f)\Upsilon(f).$$

257 Our ODE system has now become a linear first-order ODE for $w(f)$. Multiplying both sides of (45) by the
258 integrating factor $f^{-2}(1 - \sigma f)^{-2}$ and imposing the matching conditions (33) and (32) gives us

$$259 \quad (46) \quad g(f) = -f(1 - \sigma f) \sqrt{2 \int_f^1 \frac{(1 - \chi)\Upsilon(\chi)}{\chi(1 - \sigma\chi)} d\chi}.$$

260 Here, we pick $g(f) = -\sqrt{w(f)}$ to agree with f being a strictly monotone decreasing function (so that S_0
261 transitions from 1 to 0). Returning to our original variables gives us the first-order non-linear autonomous
262 ODE for $S_0(\hat{r})$:

$$263 \quad (47) \quad \frac{\partial S_0}{\partial \hat{r}} = -S_0(1 - \sigma S_0) \sqrt{2 \int_{S_0}^1 \frac{(1 - \chi)\Upsilon(\chi)}{\chi(1 - \sigma\chi)} d\chi}.$$

264 Hence, we conclude that P and T do not drastically change within the transition layer. Additionally, the
265 $O(\epsilon)$ perturbations P_1 and T_1 can be related to S_0 , which is the solution of a first-order ODE in \hat{r} .

266 **2.3. Asymptotics of Region ii.** While the leading-order dynamics of S, T , and P have been deter-
 267 mined in the transition layer, we still do not have an explicit form for $R(t)$ and $T^*(t)$. To find these, we now
 268 examine Region ii, where zero water is present. From (1), we have that $S = 0$ at $O(\epsilon^{-2})$. However, this in
 269 turn causes a cascading effect in the asymptotic expansion in ϵ of (1), and we conclude that $S = o(\epsilon^n)$ for
 270 all natural numbers n . Assuming the asymptotic series as $\epsilon \rightarrow 0^+$

$$271 \quad (48) \quad T = T_0(r, t) + \epsilon T_1(r, t) + O(\epsilon^2), \quad P = P_0(r, t) + \epsilon P_1(r, t) + O(\epsilon^2),$$

272 and incorporating these substitutions into (2) and (3), we obtain

$$273 \quad (49) \quad \frac{\partial}{\partial t} \left[\frac{P_0}{1 + \mathcal{F}T_0} \right] = \nabla \cdot \left[\frac{P_0 \nabla P_0}{1 + \mathcal{F}T_0} \right],$$

$$274 \quad (50) \quad \frac{\partial T_0}{\partial t} = \mathcal{A}_3 \nabla^2 T_0.$$

276 For our boundary conditions in Region ii, we have the matching conditions, and these imply that (33)-(35)

$$277 \quad (51) \quad T_0|_{r \rightarrow R(t)} \rightarrow T^*(t), \quad P_0|_{r \rightarrow R(t)} \rightarrow P^*(t),$$

$$278 \quad (52) \quad \frac{\partial P_0}{\partial r} \Big|_{r \rightarrow R(t)} = \frac{\partial P_1}{\partial \hat{r}} \Big|_{\hat{r} \rightarrow +\infty} \rightarrow \left[\frac{1}{\delta} \left(\frac{1 + \mathcal{F}T^*}{(1 + \mathcal{F})P^*} \right) - \sigma \right] R'(t),$$

$$279 \quad (53) \quad \frac{\partial T_0}{\partial r} \Big|_{r \rightarrow R(t)} = \frac{\partial T_1}{\partial \hat{r}} \Big|_{\hat{r} \rightarrow +\infty} \rightarrow -\frac{1}{\mathcal{A}_3} [\mathcal{A}_2 - \mathcal{A}_1(1 + \mathcal{F}T^*)] R'(t).$$

281 In interpreting (52) and (53), we note that the limits where $r \rightarrow R(t)$ are for Region ii asymptotic expansions,
 282 whereas the limits where $\hat{r} \rightarrow +\infty$ refer to the transition layer asymptotic expansions.

283 We must also give an initial condition for $R(t)$, i.e. where the drying front begins. As the drying front
 284 starts from the surface of the bean and at the threshold temperature for evaporation, our initial conditions
 285 can be described as $R(t^*) = 1$, $T^*(t^*) = T_a$. Finally, our solutions must also continue to agree with the
 286 external boundary conditions of the system, namely,

$$287 \quad (54) \quad \frac{\partial T_0}{\partial r} \Big|_{r=1} = \nu \left(\frac{1 + \mathcal{A}_4}{1 - \sigma} \right) [1 - T_0|_{r=1}] \quad \text{and} \quad P_0|_{r=1} = P_a.$$

288 Therefore, our leading-order problem exhibits a coupled system of two Stefan-like problems. Motivated by
 289 the large Stefan-number approximation, we again assume that $\delta \ll 1$. By rescaling time with $\tau = \delta(t - t^*)$,
 290 we can examine the asymptotic series

$$291 \quad (55) \quad T_0 = \tilde{T}_0(r, \tau) + \delta \tilde{T}_1(r, \tau) + O(\delta^2), \quad P_0 = \tilde{P}_0(r, \tau) + \delta \tilde{P}_1(r, \tau) + O(\delta^2)$$

292 as $\delta \rightarrow 0^+$. In consequence, our leading-order Region ii problem (49)-(54) becomes

$$293 \quad (56) \quad \nabla^2 \tilde{T}_0 = 0,$$

$$294 \quad (57) \quad \nabla \cdot \left(\frac{P_0 \nabla \tilde{P}_0}{1 + \mathcal{F}\tilde{T}_0} \right) = 0,$$

$$295 \quad (58) \quad \tilde{T}_0|_{r \rightarrow R(\tau)} \rightarrow T^*(\tau),$$

$$296 \quad (59) \quad \tilde{P}_0|_{r \rightarrow R(\tau)} \rightarrow P^*(\tau),$$

$$297 \quad (60) \quad \frac{\partial \tilde{P}_0}{\partial r} \Big|_{r \rightarrow R(\tau)} \rightarrow \left(\frac{1 + \mathcal{F}T^*}{(1 + \mathcal{F})P^*} \right) R'(\tau),$$

$$298 \quad (61) \quad \frac{\partial \tilde{T}_0}{\partial r} \Big|_{r \rightarrow R(\tau)} \rightarrow 0,$$

$$299 \quad (62) \quad \frac{\partial \tilde{T}_0}{\partial r} \Big|_{r=1} = \nu \left(\frac{1 + \mathcal{A}_4}{1 - \sigma} \right) [1 - \tilde{T}_0|_{r=1}],$$

$$300 \quad (63) \quad \tilde{P}_0|_{r=1} = P_a,$$

$$301 \quad (64) \quad R(0) = 1,$$

$$302 \quad (65) \quad T^*(0) = T_a.$$

304 By solving (56) with boundary conditions (58) and (61), this implies that $\tilde{T}_0 \equiv T^*(\tau)$. However, by applying
 305 (62) to this solution, this forces $T^*(\tau) \equiv 1$. In consequence, this reduces our coupled Stefan problem into a
 306 Stefan problem for pressure alone, i.e.

$$307 \quad (66) \quad \nabla \cdot (\tilde{P}_0 \nabla \tilde{P}_0) = 0,$$

$$308 \quad (67) \quad \tilde{P}_0|_{r \rightarrow R(\tau)} \rightarrow 1,$$

$$309 \quad (68) \quad \tilde{P}_0|_{r=1} = P_a,$$

$$310 \quad (69) \quad \frac{\partial \tilde{P}_0}{\partial r} \Big|_{r \rightarrow R(\tau)} \rightarrow R'(\tau),$$

$$311 \quad (70) \quad R(0) = 1.$$

313 However, this solution cannot satisfy the initial condition (65) for $T^*(t)$. For this to be resolved, we would
 314 have to consider the full problem in t rather than τ . As we have a Robin boundary condition in T on the
 315 surface of the bean, a similarity solution not possible in any geometry, and therefore, an analytic solution
 316 for (49)-(54) is not readily available.

317 **2.3.1. Determining $R(t)$ in Cartesian Co-ordinates with $T^* \equiv 1$.** In the limiting case where
 318 $\delta \ll 1$, i.e. $T^* \equiv 1$, we can solve (66) with boundary conditions (67) and (68), provided that we neglect any
 319 short-time discrepancies between the initial condition $T^*(0) = T_a$ and $T^* \equiv 1$. Solving this PDE system
 320 gives us

$$321 \quad (71) \quad \tilde{P}_0(r, \tau) = \sqrt{1 - (1 - P_a^2) \left(\frac{r - R(\tau)}{1 - R(\tau)} \right)}.$$

322 Now, our Stefan condition (69) gives us the ODE

$$323 \quad (72) \quad \frac{dR}{d\tau} = -\frac{1 - P_a^2}{2(1 - R)}.$$

324 Based on the initial condition from (70), our drying front in Cartesian co-ordinates based on leading-order
 325 asymptotics, $R_{\text{Cart}}(\tau)$, is

$$326 \quad (73) \quad R_{\text{Cart}}(\tau) = 1 - \sqrt{(1 - P_a^2)\tau}.$$

327 By returning to the original timescale of Region ii, we determine that \tilde{P}_0 can be fully expressed in Cartesian
 328 co-ordinates as

$$329 \quad (74) \quad \tilde{P}_0^{\text{Cart}}(r, t) = \sqrt{P_a^2 + (1 - r) \sqrt{\frac{1 - P_a^2}{\delta(t - t_{\text{Cart}}^*)}}}.$$

330 Finally, we determine from (73) that the time to completely dry a bean based on leading-order asymptotics
 331 is

$$332 \quad (75) \quad t_{\text{Cart}}^{\text{dry}} = t_{\text{Cart}}^* + \frac{1}{\delta(1 - P_a^2)}.$$

333 Using parameter values shown in [6], as well as typical values $P_a = 0.0879$, $\delta = 0.1011$, $\sigma = 0.0842$, and
 334 $t_{\text{Cart}}^* \approx 0.494$, we compute that $t_{\text{Cart}}^{\text{dry}} \approx 10.46$, or about 2768 seconds in dimensional units.

335 **2.3.2. Determining $R(t)$ in Spherical Co-ordinates with $T^* \equiv 1$.** In the limiting case where
 336 $\delta \ll 1$, i.e. $T^* \equiv 1$, we have that in spherical co-ordinates, by solving (66) with boundary conditions (67)
 337 and (68), that

$$338 \quad (76) \quad \tilde{P}_0(r, \tau) = \sqrt{1 - \left(\frac{1 - P_a^2}{r} \right) \left(\frac{r - R(\tau)}{1 - R(\tau)} \right)},$$

339 and our Stefan condition (69) gives us the ODE

$$340 \quad (77) \quad \frac{dR}{d\tau} = -\frac{1 - P_a^2}{2R(1 - R)}.$$

341 We use our initial condition (70) to give us, in implicit form, that the inverse function of the drying front in
 342 spherical co-ordinates, $\tau_{\text{Sph}}(R)$, satisfies the equation

$$343 \quad (78) \quad \tau_{\text{Sph}}(R) = \frac{1 - R^2(3 - 2R)}{3(1 - P_a^2)}.$$

344 We can invert (78) and solve $R_{\text{Sph}}(\tau)$ in the correct domain and range and obtain

$$345 \quad (79) \quad R_{\text{Sph}}(\tau) = \frac{1}{2} \left(1 - \frac{\exp\left(\frac{2\pi i}{3}\right)}{\Xi(3(1 - P_a^2)\tau)} - \exp\left(\frac{-2\pi i}{3}\right) \Xi(3(1 - P_a^2)\tau) \right),$$

346 where

$$347 \quad (80) \quad \Xi(\chi) = \sqrt[3]{2\sqrt{\chi(\chi - 1)} - 2\chi + 1}$$

348 and $\Xi(\chi)$ uses the principal branch of the cube root. Now that we have determined $R(\tau)$ in spherical co-
 349 ordinates, we can return to our original timescale of the problem and obtain that our leading-order asymptotic
 350 approximation for P is

$$351 \quad (81) \quad \tilde{P}_0^{\text{Sph}}(r, t) = \sqrt{1 - \left(\frac{1 - P_a^2}{r}\right) \left(1 - \frac{2(1 - r)}{1 + \frac{\exp\left(\frac{2\pi i}{3}\right)}{\Xi(3\delta(1 - P_a^2)(t - t_{\text{Sph}}^*)} + \exp\left(\frac{-2\pi i}{3}\right) \Xi(3\delta(1 - P_a^2)(t - t_{\text{Sph}}^*))} \right)}.$$

352 To determine the time where the bean becomes fully dry, we substitute $R = 0$ into (78) to obtain, in our
 353 original timescale, that

$$354 \quad (82) \quad t_{\text{Sph}}^{\text{dry}} = t_{\text{Sph}}^* + \frac{1}{3\delta(1 - P_a^2)}.$$

355 Therefore, to leading order, the time for a spherical coffee bean to dry out completely is $t_{\text{Sph}}^{\text{dry}} \approx 3.495$, or
 356 about 925 seconds in dimensional units. Figure 2(a) shows a comparison between the Cartesian and spherical
 357 asymptotic approximations of $R(t)$.

358 **2.4. Comparison of Asymptotic Approximations with Numerical Results.** We now compare
 359 these asymptotic approximations with the numerical solution of the PDE system (1)-(3). In particular, the
 360 main result that we wish to consider from the asymptotics is the approximate form of the drying front $R(t)$.
 361 As we can see in Figure 2(b), the general shape of the dimensional drying front $R(t)$ agrees reasonably well
 362 with the dimensional drying front seen in the numerical solution, especially as $R(t) \rightarrow 0$. However, we also
 363 see that the drying time in the numerical solution is larger than the predicted $t_{\text{Sph}}^{\text{dry}}$ from asymptotic results.
 364 This is to be expected, as the asymptotic results used were for when the Stefan number $\frac{1}{\delta} \rightarrow +\infty$. Therefore,
 365 for a smaller (but still large) Stefan number, we expect the drying time to be longer. Additionally, these
 366 approximations for the drying front $R(t)$ only hold for the critical assumption $T^* \equiv 1$. Because $T^*(t)$ will
 367 be less than unity, this will cause the drying front to be slower than the asymptotic approximation, which
 368 can explain why the numerical solution takes longer to dry out the entire bean.

369 **3. Asymptotics of the Multiphase Model with Constant Temperature.** In Section 2, we have
 370 given an analysis of the leading-order equations governing Region i, Region ii, and the transition layer.
 371 However, many of the leading-order equations cannot be solved unless a second asymptotic limit (in δ) is
 372 taken. Note that in this limit, the temperature within Region ii is, at leading-order, at roasting temperature.
 373 As the thermal timescale of the multiphase model is much smaller than the vapour diffusive timescale,

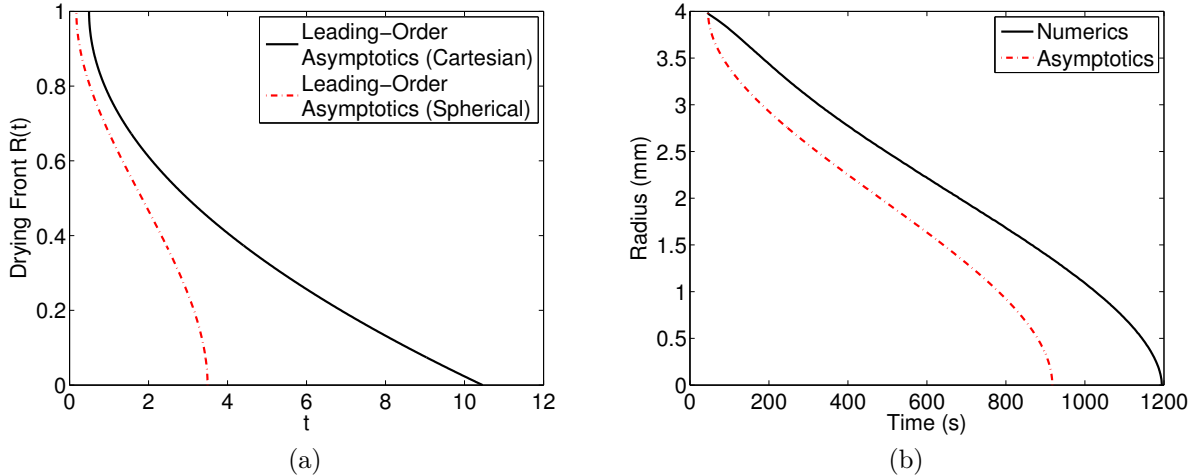


FIG. 2. (a) Comparison of the drying fronts $R_{Cart}(t)$ and $R_{Sph}(t)$. (b) Comparison of the dimensional drying front $R_{Sph}(t)$, shown in dash-dot red, against the numerical solution of the (dimensional) “simplified” multiphase model from [6] in spherical co-ordinates, shown in black.

374 it seems reasonable to examine a simplified model where the coffee bean is held at roasting temperature
 375 throughout.

376 In this section, we will now impose that $T \equiv 1$ throughout the bean, which reduces the multiphase model
 377 to a PDE system in two variables. In consequence, this means that evaporation starts at the beginning of
 378 roasting rather than after a threshold amount of time (i.e. $t^* = 0$) and our system of PDEs (1)-(2) become

$$379 \quad (83) \quad \frac{\partial S}{\partial t} = -\frac{1}{\epsilon^2}(1-P)S(1-\sigma S),$$

380

$$381 \quad (84) \quad \frac{\partial}{\partial t} [(1-\sigma S)P] = \frac{1}{\delta \epsilon^2}(1-P)S(1-\sigma S) + \nabla \cdot (P\nabla P),$$

382 with boundary conditions

$$383 \quad (85) \quad P|_{r=1} = P_a, \quad \frac{\partial P}{\partial r}|_{r=0} = 0,$$

384 and initial conditions

$$385 \quad (86) \quad S(r, 0) = 1, \quad P(r, 0) = 1.$$

386 Formally, we will consider the asymptotics of this system in the limit as $\epsilon \rightarrow 0^+$; δ will either be $\ll 1$ or
 387 $O(1)$, and all other parameters are assumed to be $O(1)$.

388 **3.1. Asymptotics in Region i.** In Region i, we have that $P = 1$ to leading order. This automatically
 389 satisfies the internal Neuman boundary condition $P_r|_{r=0} = 0$. Therefore, by substituting (83) into (84), with
 390 $P \sim 1$, we have that, to leading order,

$$391 \quad (87) \quad \frac{\partial}{\partial t} [(1-\sigma S)] = -\frac{1}{\delta} \frac{\partial S}{\partial t}.$$

392 For this to happen requires that $\frac{\partial S}{\partial t} = 0$. Thus, S is held at its initial value, i.e. $S \equiv 1$, and P, S are constant
 393 to leading order in Region i. It is important to note that, since we assume that Region i is never in contact
 394 with the surface of the bean, the boundary condition at $r = 1$ does not apply.

395 **3.2. Asymptotics of the Transition Layer.** As is done in Section 2, we introduce the scaling $r =$
 396 $R(t) + \epsilon \hat{r}$ to examine the behaviour as S transitions from 1 to 0. Again, we can expand P and S as asymptotic
 397 series as $\epsilon \rightarrow 0$:

$$398 \quad (88) \quad S = S_0(\hat{r}, t) + \epsilon S_1(\hat{r}, t) + O(\epsilon^2), \quad P = 1 + \epsilon P_1(\hat{r}, t) + O(\epsilon^2).$$

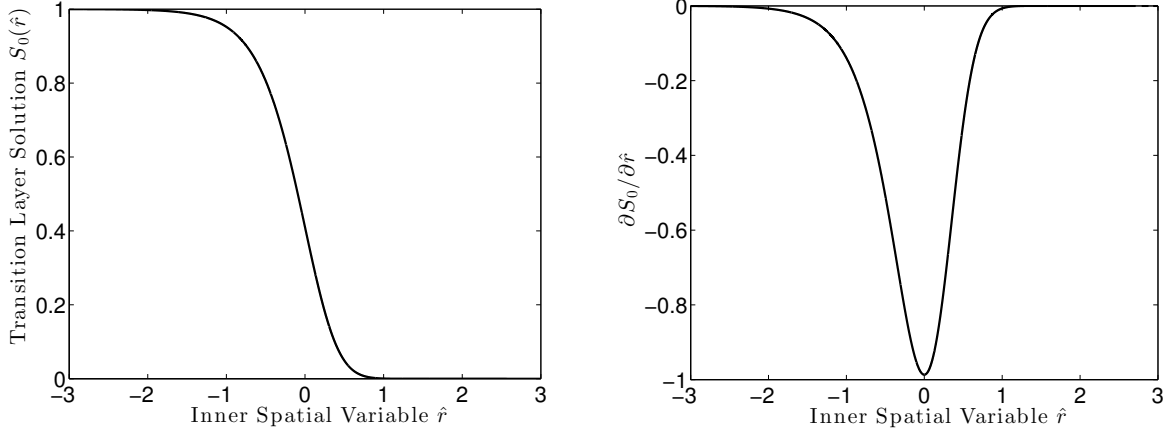


FIG. 3. Numerical solution of the ODE (90). The left panel shows the solution $S_0(\hat{r})$, and the right panel shows its spatial derivative $\frac{\partial S_0}{\partial \hat{r}}$. For uniqueness, we pick a constant of integration so that $S_0(\hat{r})$ has an inflection point at $r = 0$.

399 Noting that temperature is now constant (implying that $T_1 \equiv 0$ and $T^* \equiv 1$), our equation to $\Upsilon(S_0)$ shown
400 in (42) reduces to $\Upsilon(S_0) \equiv \frac{1}{\delta} - \sigma$. From (37), this gives us

$$401 \quad (89) \quad \frac{\partial P_1}{\partial \hat{r}} = \left(\frac{1}{\delta} - \sigma \right) R'(t)(1 - S_0),$$

402 and from (47), gives us the first-order non-linear autonomous ODE:

$$403 \quad (90) \quad \frac{\partial S_0}{\partial \hat{r}} = -S_0(1 - \sigma S_0) \sqrt{2 \left(\frac{1}{\delta} - \sigma \right) \left[\frac{1 - \sigma}{\sigma} \log \left(\frac{1 - \sigma}{1 - \sigma S_0} \right) + \log \left(\frac{1}{S_0} \right) \right]}.$$

404 It is important to note a few key points about the ODE (90). Firstly, it is not explicitly solvable. Secondly,
405 due to translational invariance, we require an additional constraint for uniqueness. This can be achieved by
406 assuming the unique inflection point of S_0 occurs at $r = 0$. With this additional constraint, we numerically
407 solve (90) and plot the results in Figure 3.

408 As we cannot obtain a closed-form solution to (90), we consider the asymptotic behaviour as $S_0(\hat{r})$
409 approaches either endpoint of the transition layer, i.e. as $\hat{r} \rightarrow \pm\infty$. While what is done here is not a formal
410 asymptotic analysis, we can still use the results to find an approximation valid for all \hat{r} . To do this, we
411 introduce the one-to-one transformation

$$412 \quad (91) \quad \Phi(S_0) = \log \left(\frac{S_0}{1 - \sigma S_0} \right) \iff S_0(\Phi) = \frac{1}{\sigma + \exp(-\Phi)},$$

413 which allows us to transform (90) into

$$414 \quad (92) \quad \frac{\partial \Phi}{\partial \hat{r}} = -\sqrt{2 \left(\frac{1}{\delta\sigma} - 1 \right) (\log [(1 - \sigma)^{1-\sigma} (\sigma \exp(\Phi) + 1)] - \sigma\Phi)}.$$

415 This transform from S_0 to Φ will allow us to compute a more accurate approximation as $\hat{r} \pm \infty$. We first
416 examine the case when $\Phi \rightarrow -\infty$, which corresponds to the asymptotic behaviour of $S_0 \rightarrow 0^+$, and yields

$$417 \quad (93) \quad \frac{\partial \Phi}{\partial \hat{r}} \sim -\sqrt{2 \left(\frac{1}{\delta\sigma} - 1 \right) ((1 - \sigma) \log(1 - \sigma) - \sigma\Phi)} \text{ as } \Phi \rightarrow -\infty.$$

418 By separating variables and integrating, we obtain that as $\hat{r} \rightarrow +\infty$,

$$419 \quad (94) \quad \Phi(\hat{r}) \sim \left(\frac{1 - \sigma}{\sigma} \right) \log(1 - \sigma) - \frac{1}{2} \left(\frac{1}{\delta} - \sigma \right) (\hat{r} - C_1)^2,$$

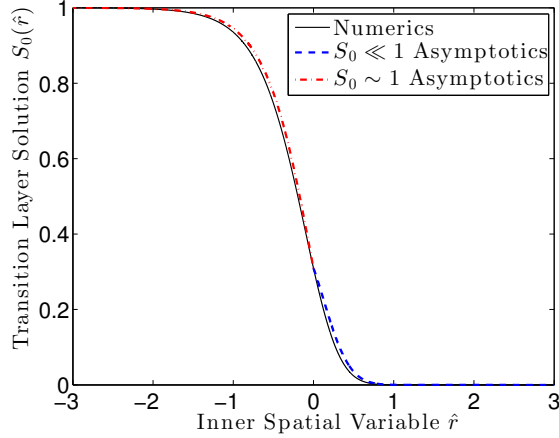


FIG. 4. Comparison of the numerical solution $S_0(\hat{r})^{Num}$ with the approximate solutions for $S_0 \ll 1$, shown in (95), and for $S_0 \sim 1$, shown in (98). Constants in the approximate solutions are chosen so that continuity at the origin is held, as well as that the absolute error between the numerical solution and the approximate solutions is minimised.

420 where C_1 is a constant of integration. Finally, we can use (91) which tells us that

421 (95)
$$S_0(\hat{r}) \sim \frac{1}{\sigma + (1 - \sigma)^{-\frac{1-\sigma}{\sigma}} \exp\left(\frac{1-\delta\sigma}{2\delta}(\hat{r} - C_1)^2\right)} \quad \text{as } \hat{r} \rightarrow +\infty.$$

422 We note that this asymptotic approximation is only valid for when $\Phi \leq \left(\frac{1-\sigma}{\sigma}\right) \log(1-\sigma)$, which is equivalent
423 to stating that $\hat{r} \geq C_1$.

424 To construct an asymptotic approximation for (90) when $S_0 \sim 1$, i.e. when $\Phi \sim \log\left(\frac{1}{1-\sigma}\right)$, we note that

425 (96)
$$\frac{\partial\Phi}{\partial\hat{r}} \sim (\log(1-\sigma) + \Phi) \sqrt{\left(\frac{1}{\delta} - \sigma\right)(1-\sigma)}.$$

426 By separating variables and integrating, we see that

427 (97)
$$\Phi \sim -\log(1-\sigma) - C_2 \exp\left(\hat{r} \sqrt{\left(\frac{1}{\delta} - \sigma\right)(1-\sigma)}\right) \quad \text{as } \hat{r} \rightarrow -\infty,$$

428 where C_2 is a constant of integration. Transforming back to S_0 using (91), this tells us that

429 (98)
$$S_0(\hat{r}) \sim \frac{1}{\sigma + (1 - \sigma) \exp\left(C_2 \exp\left(\hat{r} \sqrt{\left(\frac{1}{\delta} - \sigma\right)(1-\sigma)}\right)\right)} \quad \text{as } \hat{r} \rightarrow -\infty.$$

430 As this asymptotic expansion only has a singularity when $\Phi = -\log(1-\sigma)$, i.e. when $S_0 = 1$, this asymptotic
431 approximation to the solution is valid for all \hat{r} .

432 Finally, in order to compare these asymptotic approximations to the numerical results shown in Figure 3,
433 we wish to “patch” these two approximations together in a way that minimises the absolute error between the
434 approximations and the numerical solution while still being valid for all \hat{r} . We can solve this minimisation
435 problem numerically and obtain that, using parameter values shown in [6] as well as the typical values
436 $\delta = 0.102$ and $\sigma = 0.0842$, the smallest maximum absolute error of ≈ 0.03092 is achieved when $C_1 \approx -0.1973$,
437 and $C_2 \approx 1.235$. Additionally, in order for the asymptotic approximations to agree with the numerical
438 solution $S_0(\hat{r})^{Num}$ at the origin, this implies that $S_0(0)^{Num} \approx 0.3092$. A comparison between the two
439 asymptotic approximations and the numerical solution is shown in Figure 4, which shows good agreement
440 for all \hat{r} .

441 **3.3. Asymptotics in Region ii.** In Region ii, we have, from (83), that $S = 0$ at $O(\epsilon^{-2})$, which again
 442 causes a cascading effect in the asymptotic expansion of (83). Therefore, we conclude that $S = o(\epsilon^n)$ for all
 443 natural numbers n . Additionally, by substituting the asymptotic series $P \sim P_0(r, t) + \epsilon P_1(r, t) + O(\epsilon^2)$, (84)
 444 becomes, at leading order,

$$445 \quad (99) \quad \frac{\partial P_0}{\partial t} = \nabla \cdot (P_0 \nabla P_0).$$

446 From (89), as well as noting that our evaporation start time $t^* = 0$ in this section, our boundary and initial
 447 conditions become

$$448 \quad (100) \quad P_0 \Big|_{r=1} = P_a, \quad P_0 \Big|_{r \rightarrow R(t)} \rightarrow 1, \quad \frac{\partial P_0}{\partial r} \Big|_{r \rightarrow R(t)} \rightarrow \left(\frac{1}{\delta} - \sigma \right) R'(t), \quad R(0) = 1.$$

449 One could interpret the PDE system (99)-(100) as a Stefan problem, with $\frac{1}{\delta} - \sigma$ acting as the Stefan constant.
 450 This problem has a similarity solution in Cartesian co-ordinates, as will be shown in Section 3.3.1, although
 451 it cannot be explicitly solved. However, we can also examine the physically relevant large Stefan-number
 452 limit by letting $\delta \rightarrow 0^+$, as was done in Section 2.3. By rescaling time with $\tau = \delta t$ and considering the
 453 asymptotic series $P_0 \sim \tilde{P}_0(r, \tau) + \delta \tilde{P}_1(r, \tau) + O(\delta^2)$, (99)-(100) become

$$454 \quad (101) \quad \nabla \cdot (\tilde{P}_0 \nabla \tilde{P}_0) = 0, \quad \tilde{P}_0 \Big|_{r=1} = P_a, \quad \tilde{P}_0 \Big|_{r \rightarrow R(\tau)} \rightarrow 1, \quad \frac{\partial \tilde{P}_0}{\partial r} \Big|_{r \rightarrow R(\tau)} \rightarrow R'(\tau), \quad R(0) = 1.$$

455 Solving (101) like in Section 2, we determine that in Cartesian co-ordinates,

$$456 \quad (102) \quad \tilde{P}_0(r, t) = \sqrt{P_a^2 + (1-r) \sqrt{\frac{1-P_a^2}{\delta t}}}, \quad R(t) = 1 - \sqrt{(1-P_a^2)\delta t},$$

457 and in spherical co-ordinates,

$$458 \quad (103) \quad \tilde{P}_0(r, t) = \sqrt{1 - \left(\frac{1-P_a^2}{r} \right) \left[1 - \frac{2(1-r)}{1 + \frac{\exp(\frac{2\pi i}{3})}{\Xi(3(1-P_a^2)\delta t)} + \frac{\Xi(3(1-P_a^2)\delta t)}{\exp(\frac{2\pi i}{3})}} \right]},$$

$$R(t) = \frac{1}{2} \left(1 - \frac{\exp(\frac{2\pi i}{3})}{\Xi(3(1-P_a^2)\delta t)} - \frac{\Xi(3(1-P_a^2)\delta t)}{\exp(\frac{2\pi i}{3})} \right),$$

459 where $\Xi(\chi) = \sqrt[3]{2\sqrt{\chi(\chi-1)} - 2\chi + 1}$.

460 **3.3.1. Determining $R(t)$ in Cartesian Co-ordinates using Similarity Solutions.** One might
 461 consider using a similarity solution to solve the system (99)-(100) without the assumption that $\delta \ll 1$. To do
 462 this, we let $P_0 = h(\eta)$, where $\eta = \frac{1-r}{\sqrt{t}}$. Substituting this transformation into (99) gives us, using Cartesian
 463 co-ordinates,

$$464 \quad (104) \quad (h(\eta)h'(\eta))' + \frac{\eta}{2}h'(\eta) = 0,$$

465 and (100) becomes

$$466 \quad (105) \quad h(0) = P_a, \quad h(\lambda) = 1, \quad h'(\lambda) = \frac{\lambda(1-\delta\sigma)}{2\delta}.$$

467 Here, $\eta = \lambda$ corresponds to the moving boundary $R(t)$. Thus, our drying front based on the Cartesian
 468 similarity solution, defined as $R_{SS}(t)$, is therefore $1 - \lambda\sqrt{t}$. We note that our choice of η allows us to
 469 automatically satisfy the initial condition $R(0) = 1$. Thus, we can determine from this equation when the
 470 bean will be completely dry, i.e. when $R_{SS}(t) = 0$. This gives us $t_{SS}^{\text{dry}} = \frac{1}{\lambda^2}$. As (104) is not explicitly solvable,
 471 it is necessary to numerically solve this boundary value problem in order to determine λ . Using the shooting
 472 method, with the typical values $P_a = 0.0879$, $\delta = 0.1011$, and $\sigma = 0.0842$, we find that $\lambda \approx 0.3152$, implying
 473 that $t_{SS}^{\text{dry}} \approx 10.06$, or about 2664 seconds in dimensional units. With less than a 1% relative error to t_{SS}^{dry} , we
 474 conclude that $t_{\text{Cart}}^{\text{dry}} \approx 9.964$, as described in (75), is a very good approximation to the drying time computed
 475 from the similarity solution. Figure 5(a) shows a comparison of the drying front $R_{SS}(t)$ with drying fronts
 476 determined previously via asymptotic methods, namely, $R_{\text{Cart}}(t)$ given in (73), and $R_{\text{Sph}}(t)$, given in (79).

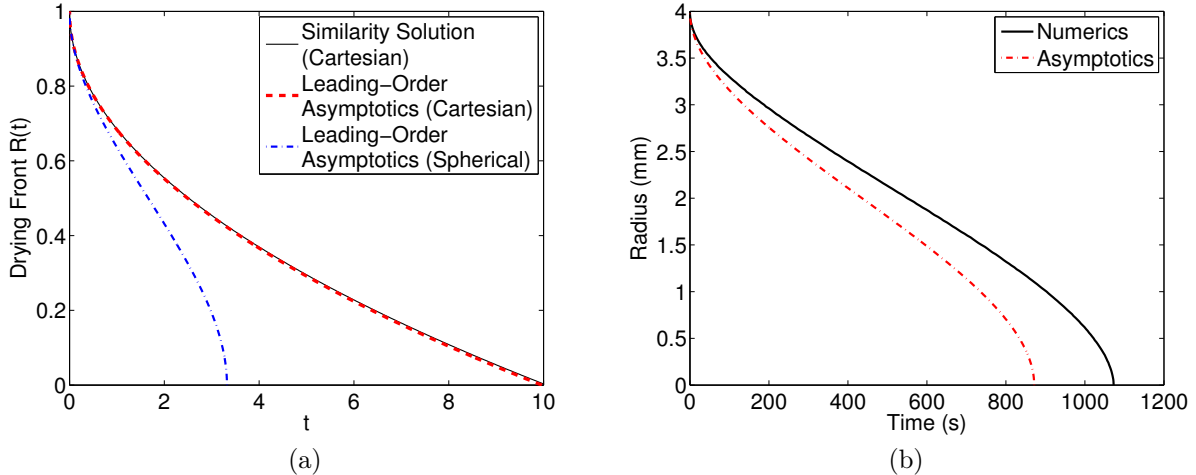


FIG. 5. (a) Comparison of the drying fronts $R_{SS}(t)$, $R_{Cart}(t)$, and $R_{Sph}(t)$ in the constant-temperature regime. (b) Comparison of the dimensional drying front $R_{Sph}(t)$, shown in dash-dot red, against the numerical solution of the (dimensional) “simplified” multiphase model from [6] in spherical co-ordinates with $T \equiv 1$, shown in black.

477 **3.4. Comparison of Asymptotic Approximations with Numerical Results.** Comparing these
 478 asymptotic approximations with the numerical solutions of (83)-(84), we can see in Figure 5(b), the general
 479 shape of the dimensional drying front $R(t)$ agrees well with the dimensional drying front seen in the numerical
 480 solution. Because we no longer have the difference between the approximation $T^* \equiv 1$ and the initial condition
 481 $T^*(t^*) = T_a < 1$ as we did in Section 2, it is expected that the drying front $R(t)$ determined via asymptotics
 482 has a better fit to the numerics. We see, like in Section 2, that the drying time in the numerical solution
 483 is larger than t_{Sph}^{dry} , which was determined from asymptotic results shown in (82). However, this is to be
 484 expected; a large (but finite) Stefan number would cause the drying time to be longer than the time produced
 485 by the limit $\delta \rightarrow 0^+$.

486 **4. Asymptotics of the Multiphase Model for more general evaporation rates with Constant**
 487 **Temperature.** In the constant temperature approximation, we have assumed that the evaporation rate I_v
 488 has taken the form $I_v(S, P) = (1 - P)S(1 - \sigma S)$. It is quite possible that this Langmuir’s evaporation rate
 489 [20] may not be the best way to model water evaporation in a roasting coffee bean. We therefore briefly
 490 examine a larger class of evaporation rates in order to highlight the differences a revised model would present.
 491 We will now consider a general class of evaporation rates, in which I_v can be written as

492 (106)
$$I_v(S, P) = -\epsilon F(S)G\left(\frac{P-1}{\epsilon}\right),$$

493 such that F, G are continuous functions independent of ϵ and satisfy the conditions

494 (107)
$$F(0) = 0,$$

495 (108)
$$F(S) > 0 \text{ for all } S \in (0, 1),$$

496 (109)
$$G(0) = 0,$$

497 (110)
$$G'(0) = \lambda > 0.$$

499 Physically speaking, (107) implies that evaporation does not occur with zero water content, and (108)
 500 indicates that evaporation will not stop if a bean is partially saturated. Additionally, (109) tells us that no
 501 evaporation occurs when the vapour pressure is at steam table pressure. However, (110) means that a small
 502 decrease in vapour pressure from the steam table pressure will cause evaporation (rather than condensation)
 503 to occur. To relate back to Langmuir’s evaporation rate, this would be the case where $F(S) = S(1 - \sigma S)$
 504 and $G(\psi) = \psi$. Using this general form, the leading-order solutions in Regions i and ii remain the same, as
 505 Region i still follows from (109) and Region ii follows from (107). However, in the transition layer, we can
 506 show the dynamics for more general classes of evaporation rates. Using the same asymptotic series expansion

507 as shown in Section 2.2, with the additional simplifications of $T \equiv 1$ and $T_1 \equiv 0$, our leading-order solution
 508 for (83) becomes

$$509 \quad (111) \quad -R'(t) \frac{\partial S_0}{\partial \hat{r}} = F(S_0)G(P_1),$$

510 and our leading-order solution for (84) becomes

$$511 \quad (112) \quad \delta \left[\frac{\partial^2 P_1}{\partial \hat{r}^2} - \sigma R'(t) \frac{\partial S_0}{\partial \hat{r}} \right] = F(S_0)G(P_1).$$

512 By equating these two expressions, we determine that, since the matching conditions (32)-(34) remain the
 513 same, (89) must also continue to hold:

$$514 \quad (113) \quad \frac{\partial P_1}{\partial \hat{r}} = R'(t) \left(\frac{1}{\delta} - \sigma \right) (1 - S_0).$$

515 Thus, the Stefan condition (113) for Region ii is the same, implying that the asymptotics for Regions i and
 516 ii are identical for all general evaporation rates of the form stated previously. Now, by dividing (113) by
 517 (111), we obtain the following ODE for P_1 in terms of S_0 :

$$518 \quad (114) \quad \frac{dP_1}{dS_0} = - \frac{R'(t)^2 \left(\frac{1}{\delta} - \sigma \right) (1 - S_0)}{F(S_0)G(P_1)}.$$

519 By separating variables, we can integrate (114) and impose the matching conditions (32) once again and
 520 obtain

$$521 \quad (115) \quad \int_0^{P_1} G(\psi) d\psi = (R'(t))^2 \left(\frac{1}{\delta} - \sigma \right) \int_0^{1-S_0} \frac{\chi}{F(1-\chi)} d\chi.$$

522 By defining

$$523 \quad (116) \quad \mathcal{F}(X) = \int_0^X \frac{\chi}{F(1-\chi)} d\chi,$$

524 we have that

$$525 \quad (117) \quad S_0 = 1 - \mathcal{F}^{-1} \left(\frac{\int_0^{P_1} G(\psi) d\psi}{(R'(t))^2 \left(\frac{1}{\delta} - \sigma \right)} \right).$$

526 Substituting (117) into (113), and noting that the function \mathcal{F}^{-1} is only a function of the variable P_1 , we
 527 can separate the subsequent ODE and obtain that

$$528 \quad (118) \quad \mathcal{G}(P_1) := \int^{P_1} \frac{d\tilde{P}}{\mathcal{F}^{-1} \left(\frac{\int_0^{\tilde{P}} G(\psi) d\psi}{(R'(t))^2 \left(\frac{1}{\delta} - \sigma \right)} \right)} = R'(t) \left(\frac{1}{\delta} - \sigma \right) (\hat{r} - C),$$

529 where C is chosen so that the matching condition (32) is satisfied. Thus, by a final inversion of the function
 530 \mathcal{G} , as well as substituting back into (117), our solutions in the transition layer are

$$531 \quad (119) \quad P_1(\hat{r}, t) = \mathcal{G}^{-1} \left(R'(t) \left(\frac{1}{\delta} - \sigma \right) (\hat{r} - C) \right), \quad S_0(\hat{r}, t) = 1 - \mathcal{F}^{-1} \left(\frac{\int_0^{P_1(\hat{r})} G(\psi) d\psi}{(R'(t))^2 \left(\frac{1}{\delta} - \sigma \right)} \right).$$

532 In general, the functions \mathcal{F} , \mathcal{F}^{-1} , \mathcal{G} , and \mathcal{G}^{-1} are not easy to determine. However, the main purpose
 533 of examining the transition layer is to examine the leading-order behaviour of S . One way to do this is
 534 to determine the leading-order ODE for S by approximating $G(\psi)$ near $\psi = 0$. Since $P \sim 1 + \epsilon P_1$ in the

535 transition layer, we can make the approximation that $G(P_1) \sim \lambda P_1$. Rearranging (111) and differentiating
 536 the expression with respect to \hat{r} , as well as substitute in (113), this gives us the ODE

$$537 \quad (120) \quad \frac{\partial^2 S_0}{\partial \hat{r}^2} - \left(\frac{\partial S_0}{\partial \hat{r}} \right)^2 \frac{F'(S_0)}{F(S_0)} + \lambda \left(\frac{1}{\delta} - \sigma \right) (1 - S_0) F(S_0) = 0.$$

538 If we allow $u = S_0$ and $w = \left(\frac{\partial S_0}{\partial \hat{r}} \right)^2$, we obtain, in similar nature by methods shown in Section 2.2, that

$$539 \quad (121) \quad \frac{dw}{du} - 2w \frac{F'(u)}{F(u)} = -2\lambda \left(\frac{1}{\delta} - \sigma \right) (1 - u) F(u).$$

540 Multiplying (121) by the integrating factor $[F(u)]^{-2}$ as well as ensuring that $w = 0$ when $u = 1$, gives us

$$541 \quad (122) \quad w = 2\lambda \left(\frac{1}{\delta} - \sigma \right) F(u)^2 \int_u^1 \frac{1 - \chi}{F(\chi)} d\chi.$$

542 Thus, if we return to our original variables of the ODE, and choose the negative branch of the square root
 543 so $S_0(\hat{r})$ transitions from 1 to 0 as \hat{r} increases, we have

$$544 \quad (123) \quad \frac{\partial S_0}{\partial \hat{r}} = -F(S_0) \sqrt{2\lambda \left(\frac{1}{\delta} - \sigma \right) \int_{S_0}^1 \frac{1 - \chi}{F(\chi)} d\chi}.$$

545 Motivated by the form of this ODE, we now consider a different evaporation rate that yields an explicit
 546 solution for both (119) and (123). Suppose that $F(S) = S^2(1 - S)$ and $G(\psi) = \psi$, i.e. $I_v = S^2(1 - S)(1 - P)$.
 547 This implies, after integrating inside the square root, that (123) becomes

$$548 \quad (124) \quad \frac{\partial S_0}{\partial \hat{r}} = -\sqrt{2 \left(\frac{1}{\delta} - \sigma \right) [S_0(1 - S_0)]^{\frac{3}{2}}}.$$

549 This ODE can be explicitly solved by separating variables, and its solution is

$$550 \quad (125) \quad S_0(\hat{r}) = \frac{1}{2} \left[1 - \frac{\hat{r} - C}{\sqrt{(\hat{r} - C)^2 + \frac{8\delta}{1 - \delta\sigma}}} \right],$$

551 where C is an arbitrary constant. This also tells us, from (113), that

$$552 \quad (126) \quad P_1(\hat{r}, t) = \frac{\left(\frac{1}{\delta} - \sigma \right) R'(t)}{2} \left[\hat{r} - C + \sqrt{(\hat{r} - C)^2 + \frac{8\delta}{1 - \delta\sigma}} \right].$$

553 These results can be verified by using (119) and noting that for this choice of $F(S)$ and $G(\psi)$,

$$554 \quad (127) \quad \begin{aligned} \mathcal{F}(X) &= \frac{X}{1 - X}, & \mathcal{F}^{-1}(X) &= \frac{X}{1 + X}, \\ \mathcal{G}(X) &= X - \frac{2(R'(t))^2 \left(\frac{1}{\delta} - \sigma \right)}{X}, & \mathcal{G}^{-1}(X) &= \frac{1}{2} \left[X + \sqrt{X^2 + 8(R'(t))^2 \left(\frac{1}{\delta} - \sigma \right)} \right]. \end{aligned}$$

555 Therefore, for more general evaporation rates (106) that satisfying conditions (107)-(110), we have now shown
 556 that the asymptotic behaviour in Regions i and ii is the same as in the specific earlier case. Additionally,
 557 we have determined a general solution for the moisture content and vapour pressure in the transition layer,
 558 defined in terms of inverse functions of integrals. These results allow us to conclude that while Langmuir's
 559 evaporation rate may not be the best model to describe the evaporation of water in roasting coffee beans,
 560 the differences are only observed in the thin transition layer.

561 **5. Discussion.** In this paper, we have extended results of the “simplified” form of the multiphase model
562 presented in [6] via asymptotic methods, in order to better understand the qualitative features of the coffee
563 bean roasting process. Motivated by previous numerical results, we considered the limit $\epsilon \rightarrow 0^+$, representing
564 the situation where vapour transport rate by Darcy flow is much smaller than the evaporation rate. The
565 asymptotic analysis showed that the solution could be divided into two main regions and a transition layer.
566 The entire bean was in the first region until a time t^* , where a thin transition layer appears at the surface of
567 the bean. This transition layer then propagated into the bean creating a second main region between it and
568 the surface of the bean. This asymptotic limit is different from what has been studied previous in drying
569 models, since the rigid cellulose structure of the solid coffee bean creates a large build-up of vapour pressure
570 before in order to drive the vapour to the external environment. The analysis shows that a narrow drying
571 front, represented by the transition layer, is crucial to the drying process in this limit.

572 In the first region, the vapour pressure is in equilibrium with the steam table pressure and the moisture
573 content of the bean remains at its initial value, with heat flow governed by the heat equation. In the
574 thin transition region, the moisture content changes rapidly from its initial value to a small value. Here,
575 evaporation dominates and the temperature and vapour pressure remain spatially uniform. Finally, in the
576 second main region, there is almost no water and therefore no evaporation. The problem in this second region
577 consists of diffusion equations for the heat and vapour flow with coupling through the matching conditions,
578 similar to a Stefan problem, at the transition layer.

579 Numerical simulations suggest that the externally applied roasting temperature is attained globally
580 fairly quickly; hence, the case where temperature is fixed at roasting temperature was considered. This also
581 allowed the coupled Stefan problem to be reduced to a single Stefan problem, which could then be solved
582 via similarity solutions or large Stefan number asymptotics. The leading order expressions are shown to
583 agree well with the dynamics of the drying front found from numerical simulations, under both spherical
584 and planar geometries.

585 Motivated from the fact that Langmuir’s equation, described in [20], may not be the best representation
586 of water evaporation from a coffee bean a more general class of evaporation rates was also examined. By
587 continuing to assume a constant roasting temperature, it was shown that the explicit form of the evapora-
588 tion rate only affects the dynamics within the transition layer: in particular, how the moisture content S
589 transitions from its initial moisture to zero.

590 Despite several simplifications made in obtaining asymptotic solutions in each of the regions of the coffee
591 bean, a reasonable agreement between the asymptotic approximations and the numerical solution of the
592 multiphase model as described in [6] has been obtained. This suggests that the asymptotics found here
593 accurately capture the qualitative behaviour of the coffee bean roasting process, and provide an acceptable
594 compromise between a simpler heat transfer model (such as those presented in [5]) and more complicated
595 multiphase models. The asymptotic results presented in this paper can be extended in order to determine the
596 asymptotic dynamics of related heat and mass transfer models. The complete multiphase model described
597 in [6] incorporates variable porosity, and by using similar methods to those shown here, one might determine
598 the leading-order behaviour of the multiphase model with variable porosity. Similarly, one might use the
599 general asymptotic results for the multiphase model discussed here to guide the development of relevant
600 solid mechanics models, which take into account the structural properties of the coffee bean and allow for
601 variations in coffee quality due to structural deformations which may occur during heating and roasting.
602 Asymptotic results may also guide in the development of more complicated models involving many more
603 chemical reactions, as well as in understanding taste and aromatic properties of the final product.

604 **Appendix A. Derivation of the Multiphase Model.** Here, we will derive the multiphase model
605 discussed in this paper, which is motivated from the model presented in [6]. The “simplified” multiphase
606 model from [6] is

$$607 \quad (128) \quad \phi \frac{\partial S}{\partial t} = -I_v,$$

$$609 \quad (129) \quad \phi \frac{\partial}{\partial t} \left(\frac{(1-S)p_v}{1+\mathcal{F}T} \right) = \frac{1}{\alpha_2} I_v + \mathcal{D}_3 \nabla \cdot \left(\frac{p_v \nabla p_v}{1+\mathcal{F}T} \right),$$

$$611 \quad (130) \quad \alpha_1 \mathcal{E}_1 (1-\phi) \mathcal{F} \frac{\partial T}{\partial t} + \phi \frac{\partial}{\partial t} (S(1+\mathcal{F}T)) = -\gamma I_v + \mathcal{F} ((\zeta_1(1-\phi) + \zeta_3\phi) \nabla^2 T + (\zeta_2 - \zeta_3)\phi \nabla \cdot (S\nabla T)).$$

612 with the boundary conditions at the surface of the bean

$$613 \quad (131) \quad \nabla T \cdot \mathbf{n} = (1 - T) \frac{\text{Nu}_v \zeta_3 \phi (1 - S)}{\zeta_1 (1 - \phi) + \zeta_2 \phi S + \zeta_3 \phi (1 - S)}, \quad p_v = 1 + \mathcal{T} \quad \text{at } r = 1,$$

614 the symmetry conditions at the centre of the bean

$$615 \quad (132) \quad \nabla T \cdot \mathbf{n} = 0, \quad \nabla P \cdot \mathbf{n} = 0 \quad \text{at } r = 0,$$

616 the initial conditions

$$617 \quad (133) \quad S(r, 0) = S_0, \quad T(r, 0) = 0, \quad P(r, 0) = p_{ST}(0),$$

618 along with

$$619 \quad (134) \quad I_v = \phi^2 \frac{(p_{ST} - p_v) S (1 - S)}{\sqrt{1 + \mathcal{T} T}} \quad \text{and} \quad p_{ST}(T) = B_1 \exp\left(\frac{B_2 \mathcal{T} T}{1 + \mathcal{T} T}\right).$$

620 Here, ϕ is the porosity (the ratio of the total volume the gas and liquid phases occupy to the total repre-
621 sentative volume), S is the saturation (the volume fraction of water divided by the total volume of water
622 and gas), I_v is the evaporation rate of water, p_v is the partial pressure of water vapour, T is the normalized
623 temperature, p_{ST} is the steam table pressure, $\mathcal{T} = (T_\infty - T_0)/T_0$, while \mathcal{D}_3 , α_1 , α_2 , γ , ζ_1 , ζ_2 , ζ_3 are dimen-
624 sionless groups involving physical parameters. We note that in [6], the initial saturation value was defined
625 as S_0 . Since this will mean something different in the analysis of this paper, we have changed the initial
626 saturation to be $S_0 \equiv \sigma$. Using the scaling

$$627 \quad (135) \quad S = \sigma \hat{S}, \quad T = \hat{T}, \quad p_v = p_{ST}(1) \hat{P}, \quad t = \frac{\phi}{\mathcal{D}_3 p_{ST}(1)} \hat{t},$$

628 and defining the parameters

$$629 \quad (136) \quad \epsilon = \frac{\sqrt{\mathcal{D}_3 \sqrt{1 + \mathcal{T}}}}{\phi}, \quad \delta = \frac{p_{ST}(1) \alpha_2}{\sigma (1 + \mathcal{T})}, \quad \beta = \frac{B_2 \mathcal{T}}{1 + \mathcal{T}},$$

$$630 \quad (137) \quad \mathcal{A}_1 = \frac{\sigma \phi}{\alpha_1 \mathcal{L}_1 (1 - \phi) \mathcal{T}}, \quad \mathcal{A}_2 = \gamma \mathcal{A}_1, \quad \mathcal{A}_3 = \frac{\phi (\zeta_1 (1 - \phi) + \zeta_3 \phi)}{\mathcal{D}_3 p_{ST}(1) \alpha_1 \mathcal{L}_1 (1 - \phi)}, \quad \mathcal{A}_4 = \frac{\phi \sigma (\zeta_2 - \zeta_3)}{\zeta_1 (1 - \phi) + \zeta_3 \phi},$$

632 the model of [6] is put into the form (dropping hats)

$$633 \quad (138) \quad \frac{\partial S}{\partial t} = -\frac{1}{\epsilon^2} I_v,$$

$$634 \quad (139) \quad \frac{\partial}{\partial t} \left[\frac{(1 + \mathcal{T}) P (1 - \sigma S)}{1 + \mathcal{T} T} \right] = -\frac{1}{\delta} \frac{\partial S}{\partial t} + \nabla \cdot \left[\frac{(1 + \mathcal{T}) P \nabla P}{1 + \mathcal{T} T} \right],$$

$$635 \quad (140) \quad \frac{\partial T}{\partial t} + \mathcal{A}_1 \frac{\partial}{\partial t} [S(1 + \mathcal{T} T)] = \mathcal{A}_2 \frac{\partial S}{\partial t} + \mathcal{A}_3 \nabla \cdot [(1 + \mathcal{A}_4 S) \nabla T].$$

637 Here, the rescaled evaporation rate I_v and the rescaled steam table pressure $P_{ST}(T)$ are given by

$$638 \quad (141) \quad I_v = S(1 - \sigma S)(P_{ST} - P) \sqrt{\frac{1 + \mathcal{T}}{1 + \mathcal{T} T}} \quad \text{and} \quad P_{ST}(T) = \exp\left(\frac{\beta(T - 1)}{1 + \mathcal{T} T}\right).$$

639 The boundary conditions we impose on the PDE system (1)-(3) are the symmetry conditions at the centre
640 of the bean

$$641 \quad (142) \quad \nabla T \cdot \mathbf{n} = 0, \quad \nabla P \cdot \mathbf{n} = 0 \quad \text{at } r = 0,$$

642 as well as the heat transfer condition

$$643 \quad (143) \quad \nabla T \cdot \mathbf{n} = \nu \left(\frac{1 - \sigma S}{1 - \sigma} \right) \left(\frac{1 + \mathcal{A}_4}{1 + \mathcal{A}_4 S} \right) (1 - T) \quad \text{at } r = 1,$$

644 where

$$645 \quad (144) \quad \nu = \frac{\text{Nu}_v \zeta_3 \phi (1 - \sigma)}{(\zeta_1 (1 - \phi) + \zeta_3 \phi) (1 + \mathcal{A}_4)}.$$

646 Previously, the model introduced in [6] imposes a Dirichlet condition in P at the surface of the bean. We
647 will instead impose a different boundary condition for P in order to prevent condensation from occurring at
648 the surface of the bean. This can be achieved by imposing that P is aligned with the steam table pressure
649 for temperatures below the evaporating temperature, i.e.

$$650 \quad (145) \quad P|_{r=1} = \begin{cases} P_{ST}(T), & T < T_a, \\ P_a, & T \geq T_a. \end{cases}$$

651 Here, $P_a := \frac{1+\mathcal{J}}{p_{ST}(1)}$ and $T_a := P_{ST}^{-1}(P_a)$. We will also make the assumption that the change in boundary
652 conditions for P only occurs at one critical time, namely, t^* . We define t^* as when the time when the
653 evaporation temperature T_a is achieved at the surface of the bean, i.e. as the solution to the equation
654 $T(1, t^*) = T_a$. Finally, we impose the initial conditions corresponding to uniform initial moisture content,
655 room temperature, and equilibrium steam table pressure, i.e.

$$656 \quad (146) \quad S(r, 0) = 1, \quad T(r, 0) = 0, \quad P(r, 0) = P_{ST}(0).$$

657

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